Laplace transforms

Fourier transforms are no defined when the functions diverge at $\pm\infty$. For example, in calculations of the stability of circuits, fluids, plasma, ..., a solution can have a time dependence

$$f(t) \sim e^{\gamma t}$$

where $Re(\gamma) > 0$. Or we are interested in an initial value problem where a fluctuation starts at a known amplitude and we want to find its time evolution.

We define the Laplace transform as

$$F(\omega) = \int_0^\infty dt\, f(t)\, e^{i\omega t}$$

where $Im(\omega) > 0$ and we require that $Im(\omega)$ is sufficiently positive so that

$$\lim_{t\to\infty} f(t)\, e^{i\omega t} \longrightarrow 0$$

example: Let $f(t) = e^{\Gamma t}\,\theta(t)$

where $\Theta(t) = 0$ for $t < 0$
$= 1$ for $t > 0$

$$F(\omega) = \int_0^\infty dt\, e^{i\omega t} e^{\Gamma t} = \frac{e^{(i\omega+\Gamma)t}}{i\omega+\Gamma}\Bigg|_0^\infty$$

$$= -\frac{1}{i\omega+\Gamma}$$

with $\mathrm{Im}\,\omega > \Gamma$.

Inverse transform

We can't use the previous results from the Fourier transform directly since $f(t)$ may be unbounded at ∞. Consider a function $f(t)$ that is zero for $t < 0$ and define

$$G(t) = f(t)\, e^{-\gamma t}$$

with γ such that

$$\lim_{t\to\infty} G = 0$$

Our previous results for the FT apply to G since $G \to 0$ at $\pm\infty$.

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{-ipt} \int_{-\infty}^{\infty} dt'\, G(t')\, e^{ipt'}$$

or

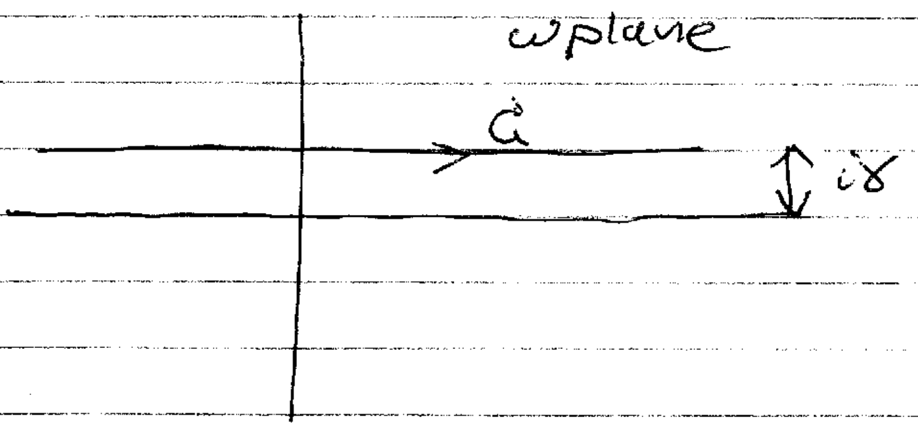
$$f(t)e^{-\gamma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{-ipt} \int_0^{\infty} dt'\, f(t')\, e^{-\gamma t'} e^{ipt'}$$

Let $\omega = p + i\gamma$

$$f(t) = \frac{1}{2\pi}\int_C d\omega\, e^{-i\omega t} \int_0^{\infty} dt'\, f(t')\, e^{i\omega t'}$$

$$f(t) = \frac{1}{2\pi}\int_C d\omega\, e^{-i\omega t}\, F(\omega)$$

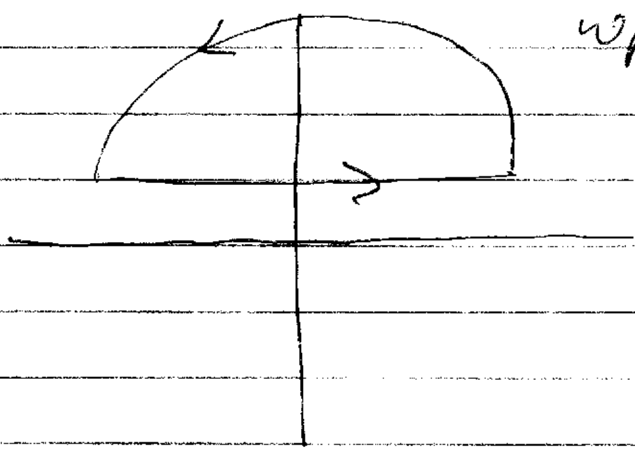
where C must lie above all the singularities of $F(\omega)$. Why?



For $t < 0$, want $f(t) = 0$. For $t < 0$ can close the contour C in the UHP where

$$e^{-i\omega t} \to 0$$

For $t < 0$



ω plane

If $F(\omega)$ has no singularities in the UHP,

$f(t) = 0$ for $t < 0$.

<u>example</u>:

$$\dot{y} - \nu y = 0 \quad \text{with } y(0) \text{ arbitrary}$$

$\dot{y} = \frac{dy}{dt}$

$$\Rightarrow y = y(0) e^{\nu t} \Rightarrow \text{growing solution}$$

Solve this using the Laplace transform

$$\int_0^\infty dt\, e^{i\omega t} \underbrace{(\dot{y}} - \nu y) = 0$$

integrate by parts

$$e^{i\omega t} y \Big|_0^\infty - \int_0^\infty dt\, i\omega\, y(t) - \nu\, Y(\omega) = 0$$

$$(-i\omega - \nu)\, Y(\omega) = y(0)$$

since $e^{i\omega t} y(t) = 0$ as $t \to \infty$.

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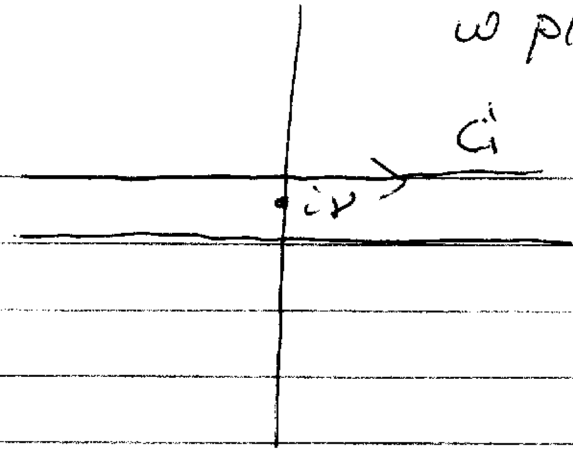
$$Y(\omega) = \frac{y(0)}{-i\omega - \nu} = \frac{y(0)}{-i(\omega - i\nu)}$$

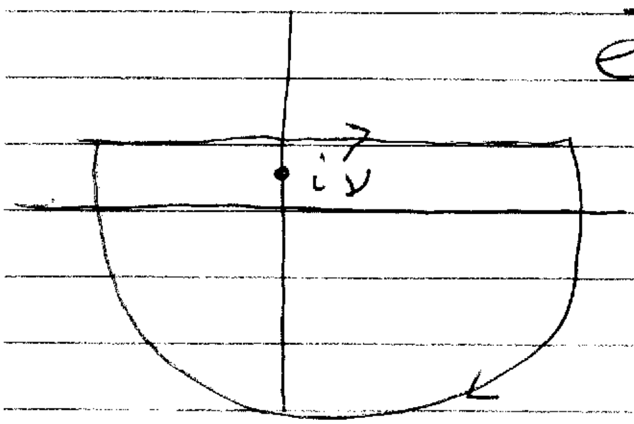


$$y(t) = \int_{C_i} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{y(0)}{-i(\omega - i\nu)}$$

~~$y(t) = 0$ for $t < 0$~~

For $t > 0$, close C_i in the LHP where

$e^{-i\omega t} \rightarrow 0 \Rightarrow$ Jordan's Lemma so ok.



$$f(t) = -2\pi i \frac{1}{2\pi} \frac{1}{(-i)} e^{-i(i\nu)t} y(0)$$

$$= y(0) e^{\nu t}$$

<u>example</u> One Dimensional Diffusion Equation

Consider a 1-D diffusion equation

$$\frac{\partial f}{\partial t} - D \frac{\partial^2}{\partial x^2} f = 0$$

where $f(x, 0)$ is specified

$t = 0$

Take the F.T with respect to x by operating with

$$\int dx\, e^{-ikx}$$

yields

$$\frac{\partial}{\partial t} F(k,t) - D \int dx\, e^{-ikx} \frac{\partial^2}{\partial x^2} f = 0$$

Integrating by parts in the x integral and assuming $f \to 0$ as $x \to \pm\infty$,

$$\frac{\partial}{\partial t} F(k,t) + D k^2 F(k,t) = 0$$

where

$$F(k,0) = \int_{-\infty}^{\infty} dx\, f(x,0)\, e^{-ikx}$$

Now take the LT of this equation. Operate with

$$\int_0^{\infty} dt\, e^{i\omega t}$$

$$\int_0^{\infty} dt\, e^{i\omega t} \underbrace{\frac{\partial F(k,t)}{\partial t}}_{\text{integrate by parts}} + k^2 D \hat{F}(k,\omega) = 0$$

where

$$\hat{F}(k,\omega) = \int_0^{\infty} dt\, e^{i\omega t} F(k,t)$$

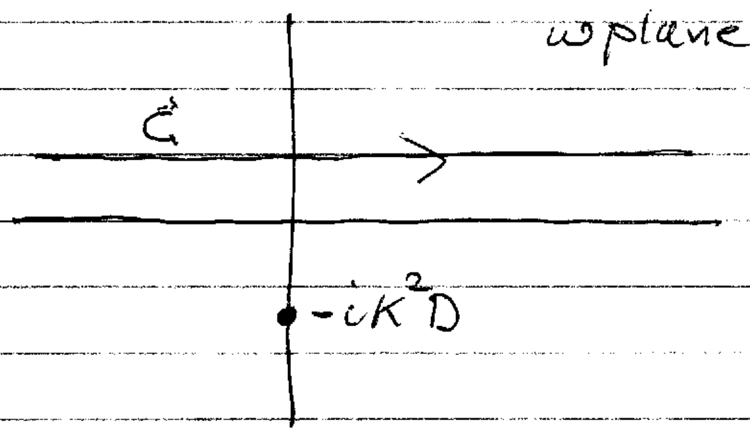
$$e^{i\omega t} F(k,t)\Big|_0^\infty - i\omega \hat{F}(k,\omega) + k^2 D \hat{F}(k,\omega) = 0$$

$$\hat{F}(k,\omega) = \frac{F(k,0)}{-i\omega + k^2 D}$$

Inverse Laplace transform

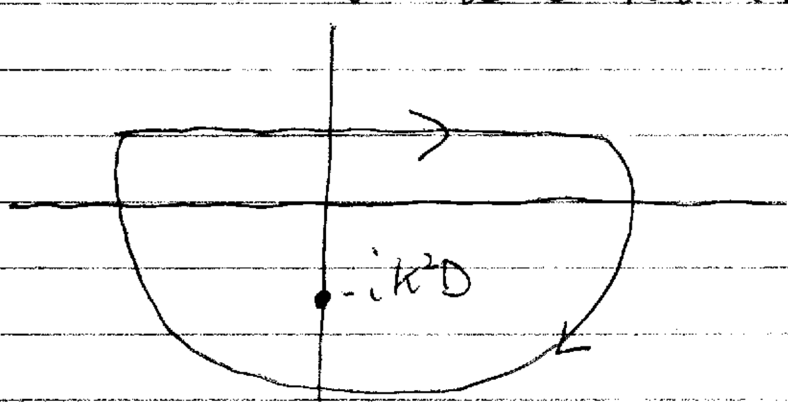
$$F(k,t) = \int_C \frac{d\omega}{2\pi} e^{-i\omega t} \frac{F(k,0)}{-i\omega + k^2 D_\perp}$$

Have a singularity at $\omega = -ik^2 D$



For $t < 0$ close in UHP and no singularities so $F = 0$

For $t > 0$ close C in LHP and calculate the residue at $\omega = -ik^2D$. Contour at ∞ does not contribute (Jordan's Lemma).



$$F(k,t) = -\frac{2\pi i}{2\pi} \frac{1}{-i} e^{-i(-ik^2D)t}$$

$$\otimes F(k,0)$$

$$F(k,t) = F(k,0)\, e^{-k^2 D t}$$

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, e^{-k^2 D t}\, F(k,0)$$

Substitute $F(k,0)$

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, e^{-k^2 D t} \int_{-\infty}^{\infty} dx'\, f(x',0)\, e^{-ikx'}$$

$$= \int_{-\infty}^{\infty} dx'\, \frac{1}{2\pi}\, f(x',0)\, \underbrace{\int_{-\infty}^{\infty} dk\, e^{ik(x-x')}\, e^{-k^2 D t}}_{I}$$

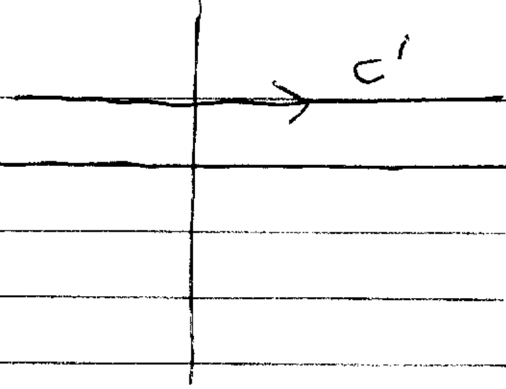
$$I = \int_{-\infty}^{\infty} dk\, e^{ik(x-x')}\, e^{-k^2 D t}$$

$$= \int_{-\infty}^{\infty} dk\, e^{-Dt\left[k^2 + 2ki\frac{(x-x')}{2Dt} - \frac{(x-x')^2}{4D^2t^2}\right]}\, e^{-\frac{(x-x')^2}{4Dt}}$$

$$= \int_{-\infty}^{\infty} dk\, e^{-Dt\left[k + i\frac{(x-x')}{2Dt}\right]^2}\, e^{-\frac{(x-x')^2}{4Dt}}$$

Let $\quad s = k + i\,\dfrac{(x-x')}{2Dt}$

s plane



$$I = \int_{c'} ds \, e^{-s^2 D t} \, e^{-\frac{(x-x')^2}{4Dt}}$$

Move c' to the real axis. Can we do this?

$\Rightarrow$ integrand is analytic

$\Rightarrow$ integrand goes to zero at ∞ near the real axis

$$I = e^{-\frac{(x-x')^2}{4Dt}} \int_{-\infty}^{\infty} ds \, e^{-s^2 D t}$$

Let $r^2 = s^2 Dt$

$$= e^{-\frac{(x-x')^2}{4Dt}} \frac{1}{(Dt)^{1/2}} \underbrace{\int_{-\infty}^{\infty} dr \, e^{-r^2}}_{\sqrt{\pi}}$$

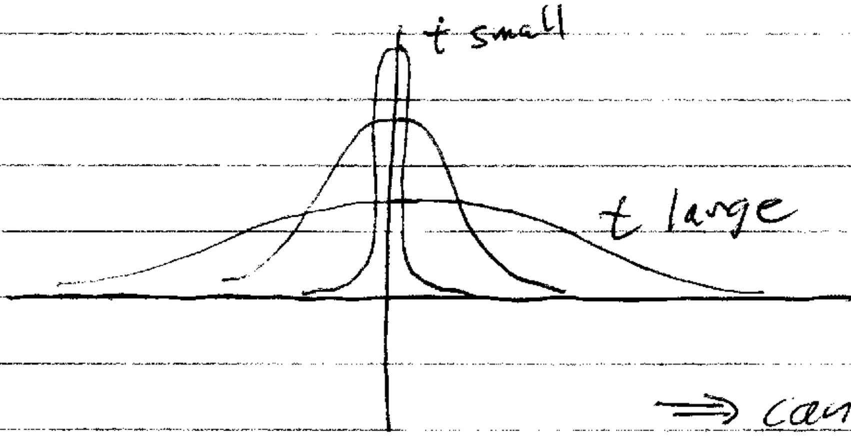
$$f(x,t) = \int_{-\infty}^{\infty} dx' \, f(x',0) \, \frac{e^{-\frac{(x-x')^2}{4Dt}}}{(4\pi D t)^{1/2}}$$

Suppose

$$f(x, t=0) = \delta(x)$$

$$f(x,t) = \frac{1}{(4\pi D t)^{1/2}} e^{-\frac{x^2}{4Dt}}$$

$\Rightarrow$ a Gaussian that spreads in time



The total area is preserved. How do you know this?

$\Rightarrow$ can integrate the solution

$\Rightarrow$ original equation

$$\frac{\partial}{\partial t} f - D \frac{\partial^2}{\partial x^2} f = 0$$

Integrate $\int_{-\infty}^{\infty} dx$

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} dx\, f(x) - D \underbrace{\int_{-\infty}^{\infty} dx \frac{\partial^2}{\partial x^2} f}_{\frac{\partial}{\partial x} f \Big|_{-\infty}^{\infty} = 0} = 0$$

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} dx\, f(x) = 0$$