Fourier Transforms and Integral Representations (Arfken Ch 20)

A useful technique for solving differential equations involves a linear transformation called the Fourier transform. We will first introduce this transformation then show how it can be used to solve differential equations and construct integral representations of solutions.

We define

$$F(k) = \int_{-\infty}^{\infty} dz\, f(z)\, e^{-ikz}$$

$F(k)$ is the Fourier transform of $f(z)$. We will now show that the inverse transform is given by

$$f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, F(k)\, e^{ikz}$$

Let

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, F(k)\, e^{ikz}$$

Substituting $F(k)$ from above,

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikz} \int_{-\infty}^{\infty} dz'\, f(z')\, e^{-ikz'}$$

$$= \int_{-\infty}^{\infty} dz'\, f(z')\, \delta(z-z')$$

where

$$\delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(z-z')} \qquad (1)$$

is the Dirac delta function. This function has the property that

$$\int_{-\infty}^{\infty} dz'\, \delta(z-z')\, f(z') = f(z) \qquad (2)$$

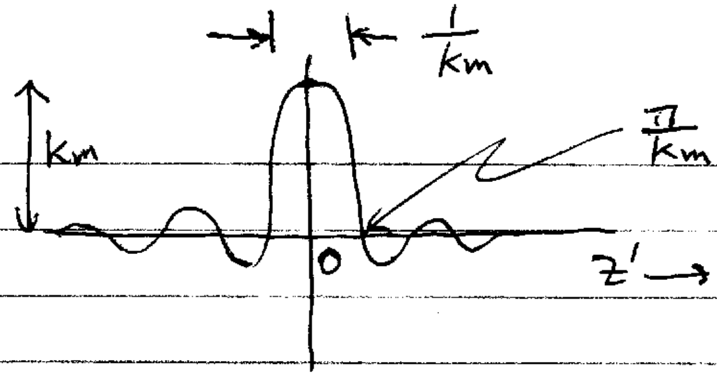
$\Rightarrow$ $\delta(z')$ has the property that its amplitude goes to ∞ at zero argument but is zero elsewhere such that

$$\int_{-\infty}^{\infty} dz'\, \delta(z') = 1 \qquad (3)$$

Can consider the δ function as

$$\delta(z') = \lim_{K_m \to \infty} \frac{1}{2\pi}\int_{-K_m}^{K_m} dk\, e^{ikz'}$$

$$= \lim_{K_m \to \infty} \frac{1}{\pi}\, \frac{\sin K_m z'}{z'}$$



$\Rightarrow$ height k_m and width $\frac{1}{k_m}$

$\Rightarrow$ area 1

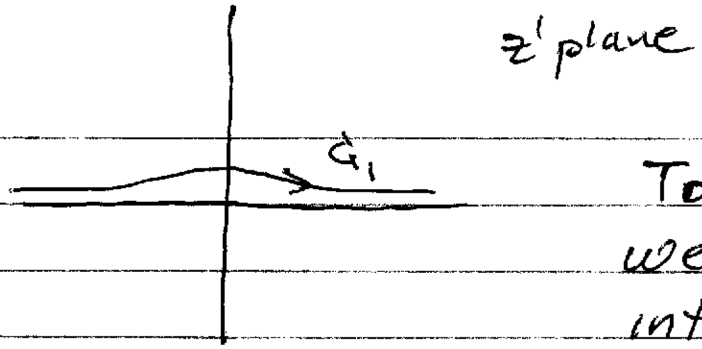
Want to prove that that the delta function in (1) has the property shown in (2).

To show this we divide I into two integrals
$I = I_1 + I_2$

$$I_1 = \frac{1}{2\pi} \int_{-\infty}^{0} dk\, e^{ikz} \int_{-\infty}^{\infty} dz'\, f(z')\, e^{-ikz'}$$

$$I_2 = \frac{1}{2\pi} \int_{0}^{\infty} dk\, e^{ikz} \int_{-\infty}^{\infty} dz'\, f(z')\, e^{-ikz'}$$

As long as $f(z)$ is analytic on ~~the~~ and around the real axis, we can move the integral along z' into the complex plane. In I_1 the integral can move up in the z' plane as shown by C_1,



To see that this is allowed we need to check that the integral remains bounded as $k \to -\infty$ in I_1. The behavior as $k \to \infty$ is controlled by

$$e^{-ikz'} = e^{-ik(x'+iy')} = e^{-ikx'}\,e^{ky'}$$

Since C_1 is in the UHP, $y'>0$ and as $k \to -\infty$ the exponential goes to zero so the contour over C_1 is allowed. Thus,

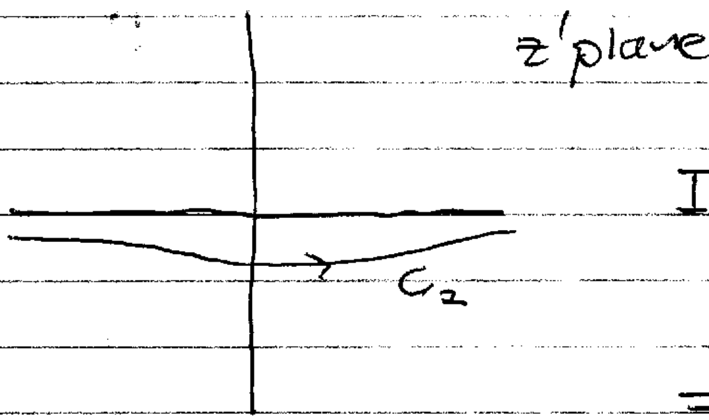
$$I_1 = \frac{1}{2\pi}\int_{C_1} dz'\, f(z') \int_{-\infty}^{0} dk\, e^{ik(z-z')}$$

$$= \frac{1}{2\pi}\int_{C_1} dz'\, f(z')\, \frac{e^{ik(z-z')}}{i\,(z-z')}\Bigg|_{-\infty}^{0}$$

$$= \frac{1}{2\pi i}\int_{C_1} dz'\, f(z')\, \frac{1}{z-z'}$$

since the endpoint at $k=-\infty$ is zero. Note that this endpoint would not be zero if z' were real.

Similar deform z' in I_2 as



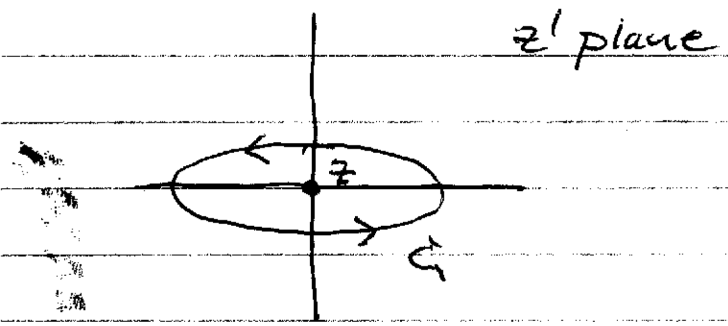
$$I_2 = \frac{1}{2\pi} \int_{C_2} dz' \, f(z') \int_0^{\infty} dk \, e^{ik(z-z')}$$

$$= \frac{1}{2\pi i} \int_{C_2} dz' \, \frac{f(z')}{(z-z')} (-1)$$

$$= \frac{1}{2\pi i} \int_{C_2} dz' \, \frac{f(z')}{z'-z} .$$

Combining I_1 and I_2 we are left with

$$I = \frac{1}{2\pi i} \int_{C} dz' \, \frac{f(z')}{z'-z}$$



$I = f(z)$ from

Cauchy's integral formula

$\Rightarrow$ this proves the the δ function has the property shown in (2).

Integral representation of the Airy function

The Airy equation is given by

$$\frac{d^2}{dx^2} f - x f = 0$$

We previously solved this equation by expanding f in a series around $x=0$ valid for all x and with a WKB solution valid for large $|x|$. The power series is of limited value for large $|x|$ because its behavior is difficult to determine.

We can construct an integral representation valid for all x by Fourier transforming the equation. Suppose

$$f(x) = \int_{C} dk\, e^{ikx} F(k)$$

where C will be specified. This is a generalized Fourier representation.

$\Longrightarrow$ the contour C is not limited to the real axis in the k plane.

Insert this form for $f(x)$ into the Airy eqn. The d^2/dx^2 acts on e^{ikx},

$$\int_C dk\, F(k)\left[-k^2 - x\right] e^{ikx} = 0$$

$\Rightarrow$ need to get rid of x

$$\Rightarrow \quad x e^{ikx} = \frac{1}{i}\frac{d}{dk} e^{ikx}$$

$$\int_C dk\, F(k)\left(-k^2 - \frac{1}{i}\frac{d}{dk}\right) e^{ikx} = 0$$

$\Rightarrow$ want $\frac{d}{dk}$ to act on $F(k)$ so integrate by parts

$$\int_C dk\, e^{ikx}\left(-k^2 + \frac{1}{i}\frac{d}{dk}\right) F(k) = 0$$

where have taken

$$F(k)e^{ikx}\Big| = 0 \text{ on the endpoints of } C.$$

Since this eqn is valid for all x, we must have

$$\frac{d}{dk}F(k) - ik^2 F(k) = 0.$$

$\Rightarrow$ we have reduced our second order equation in x to a first order equation in k.

In general, if an equation in x has a power x^p, this will yield an equation in k space of order

$$\frac{d^p}{dk^p}.$$

Thus, the order of the equation can be reduced if p is smaller than the original order of the equation in x.

Integrating the equation for $F(k)$

$$\Rightarrow \quad F(k) \sim e^{ik^3/3}$$

so that

$$f(x) = \int_C dk\, e^{ikx} e^{ik^3/3}$$

The integrand is an analytic function in the finite complex k plane

$\Rightarrow$ what about the endpoints?

$\Rightarrow$ require

$$e^{ikx} e^{ik^3/3} = 0 \quad \text{at the endpoints}$$

$\Rightarrow$ only endpoints are at $k \to \infty$

For large k, must have

$$e^{ik^3/3} \to 0$$

since $ik^3/3$ dominates ikx at large k.

$\Rightarrow$ need to find the ~~ang~~ phase angle of k such that $e^{ik^3/3} \to 0$

$\Rightarrow$ Let $k = se^{i\theta}$ with s real and positive.

$$e^{ik^3/3} = e^{i\frac{s^3}{3}e^{3i\theta}}$$

Must have $ie^{3i\theta} = -1$

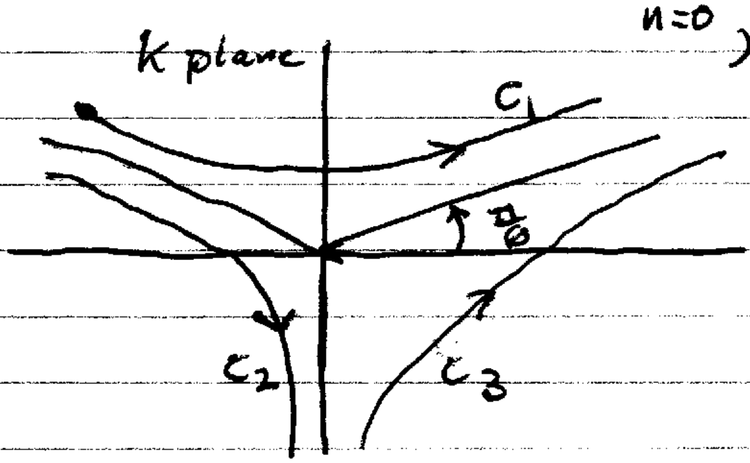
$$e^{i3\theta} = i = e^{i\frac{\pi}{2}} e^{\pm 2n\pi i}$$

$$3\theta = \frac{\pi}{2} \pm 2n\pi$$

$$\theta = \frac{\pi}{6} \pm \frac{2n\pi}{3}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$$

$n=0$, $n=1$, $n=-1$



Three possible contours in the k plane.
Three solutions?

No. Since the integrand is analytic,

$$C_1 = C_2 + C_3$$

so only two of the contours yield linearly independent solutions. For example,

$$f_1(x) = \int_{C_1} e^{ikx} e^{ik^3/3}\, dk$$

$$f_2(x) = \int_{C_2} e^{ikx} e^{ik^3/3}\, dk$$

$\Rightarrow$ valid for all x.

$\Rightarrow$ power series, can be obtained by expanding

$$e^{ikx} = \sum_{m=0}^{\infty} \frac{1}{m!} (ikx)^m$$

and carrying out the k space integrals.

$\Rightarrow$ Can also evaluate the integrals for large $|x|$ using the saddle point technique

Saddle point treatment for large $|x|$:

saddle points: $h(k) = ikx + i\frac{1}{3}k^3$

$$h' = 0 = ix + ik^2 \Rightarrow k^2 = -x$$

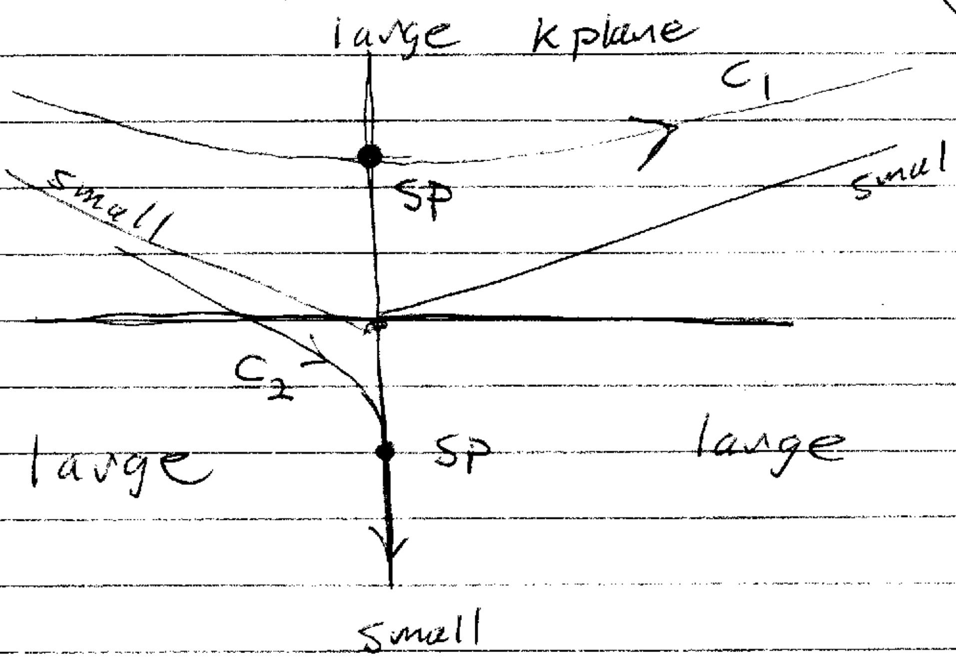
$$h'' = 2ik$$

$x > 0$

$k_{sp} = \pm i\, x^{1/2}$

$x < 0$

$k_{sp} = \pm(-x)^{1/2}$



Deform C_1 and C_2 through the SPs for $x > 0$

what is direction of the PSD through the SPs?

$x > 0$

Saddle Point 1: $k_{sp_1} = i\, x^{1/2}$

$$h(k) \simeq h(k_{sp_1}) + \frac{1}{2}h''(k - k_{sp_1})^2$$

$$= i\, i x^{1/2} x + i\frac{1}{3} i^3 x^{3/2} \cancel{(k - k_{sp_1})^2}$$

$$+ \frac{1}{2} 2i\, i x^{1/2} (k - k_{sp})^2$$

$$h(k) \simeq -\frac{2}{3}x^{3/2} - x^{1/2}(k-k_{sp_1})^2$$

$$e^h \simeq e^{-\frac{2}{3}x^{3/2}} e^{-x^{1/2} r^2 e^{2i\theta}}$$

with $k - k_{sp1} = re^{i\theta}$

For PSD through $k_{sp1} \Longrightarrow \theta = 0$

$$f_1 \simeq \int_{-\infty}^{\infty} dr\, e^{-(2/3)x^{3/2}} e^{-x^{1/2}r^2}$$

$s = r x^{1/4}$

$$= \frac{e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}} \int_{-\infty}^{\infty} ds\, e^{-s^2}$$

$$= \sqrt{\pi}\, \frac{e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}} \qquad \underline{x \gg 1}$$

$f_1 \to 0$ as $x \to \infty$

Saddle Point 2 for $x > 0$: $k_{sp_2} = -i x^{1/2}$

$\Rightarrow$ near sp 2

$$h \simeq i(-i x^{1/2})x + i\frac{1}{3}(-i x^{1/2})^3$$

$$+ \frac{1}{2} 2i(-i x^{1/2})(k - k_{sp_2})^2$$

$$h(k) \simeq \frac{2}{3} x^{3/2} + x^{1/2}\left(k - k_{sp2}\right)^2$$

Let $k - k_{sp} = r e^{i\theta}$

$$e^{h(k)} \simeq e^{\frac{2}{3}x^{3/2}} \, e^{x^{\frac{1}{2}} r^2 e^{2i\theta}}$$

$\Rightarrow$ PSD $e^{2i\theta} = -1 = e^{\pm i\pi}$

$\theta = \pm \frac{\pi}{2}$

$\Rightarrow \theta = -\frac{\pi}{2}$

$\Rightarrow$ match direction of C_2 through sp2

$\Rightarrow dk = dr\, e^{-i\frac{\pi}{2}}$

$$f_2 = e^{-i\frac{\pi}{2}} e^{\frac{2}{3}x^{3/2}} \int_{-\infty}^{\infty} dr\, e^{-x^{1/2} r^2}$$

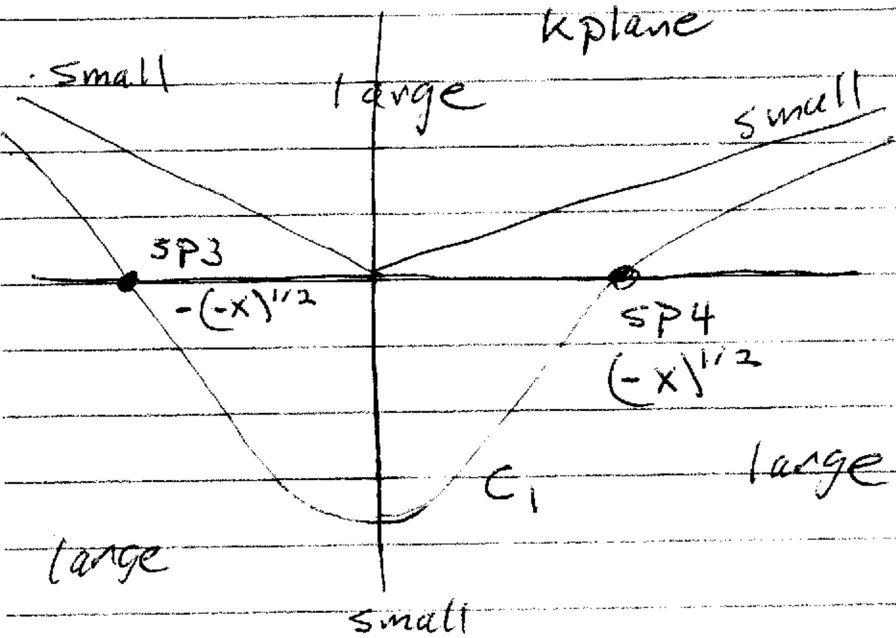
$$= -i \frac{e^{\frac{2}{3}x^{3/2}}}{x^{1/4}} \sqrt{\pi} \quad \text{for } x \gg 1$$

$\Rightarrow f_2 \to \infty$ as $x \to \infty$.

For large x, the solutions match the WKB solutions

$\Rightarrow$ coefficients are known

$X < 0, \; k_{sp} = \pm(-x)^{1/2}$



Deform C_1 through SP3 and SP4

Near SP3 : $k_{sp3} = -(-x)^{1/2}$

$$h(k) \simeq ix\left(-(-x)^{1/2}\right) + \frac{1}{3}i\left(-(-x)^{1/2}\right)^3$$
$$+ \frac{1}{2}2i\left[-(-x)^{1/2}\right]\left(k - k_{sp3}\right)^2$$

$$= \frac{2}{3}i(-x)^{3/2} - i(-x)^{1/2}\left(k - k_{sp3}\right)^2$$

Let $k - k_{sp3} = re^{i\theta}$

$$e^h \simeq e^{\frac{2}{3}i(-x)^{3/2}} e^{-i(-x)^{1/2}r^2e^{2i\theta}}$$

$$ie^{2i\theta} = 1 \qquad e^{i\frac{\pi}{2}}e^{2i\theta} = 1$$

$$\theta = -\frac{\pi}{4}$$

$$dk = dr\, e^{-i\frac{\pi}{4}}$$

Similar for SP4: $k_{sp4} = (-x)^{1/2}$

$$e^{h} \simeq e^{-\frac{2}{3}i(-x)^{3/2}} \, e^{-(-x)^{1/2} r^2}$$

with $\theta = \frac{\pi}{4}$

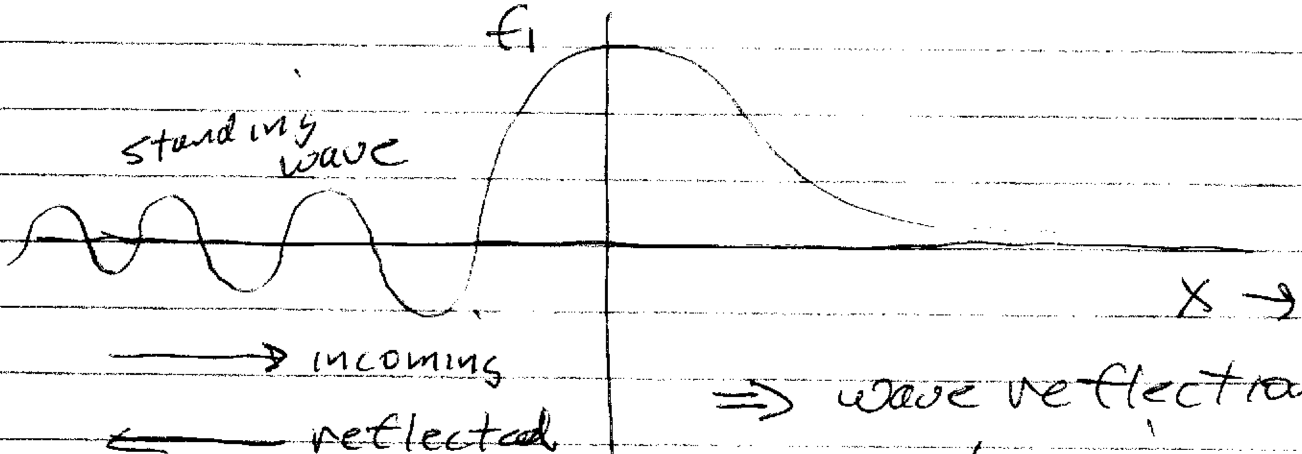
$dk = dr\, e^{i\frac{\pi}{4}}$

$$\Rightarrow f_1 = \left[e^{\frac{2}{3}i(-x)^{3/2}} e^{-i\frac{\pi}{4}} + e^{-i\frac{2}{3}(-x)^{3/2}} e^{i\frac{\pi}{4}} \right] \otimes \int_{-\infty}^{\infty} dr\, e^{-(-x)^{1/2} r^2}$$

$$f_1 = \frac{2\sqrt{\pi}}{(-x)^{1/4}} \cos\left[\frac{2}{3}(-x)^{3/2} - \frac{\pi}{4} \right]$$

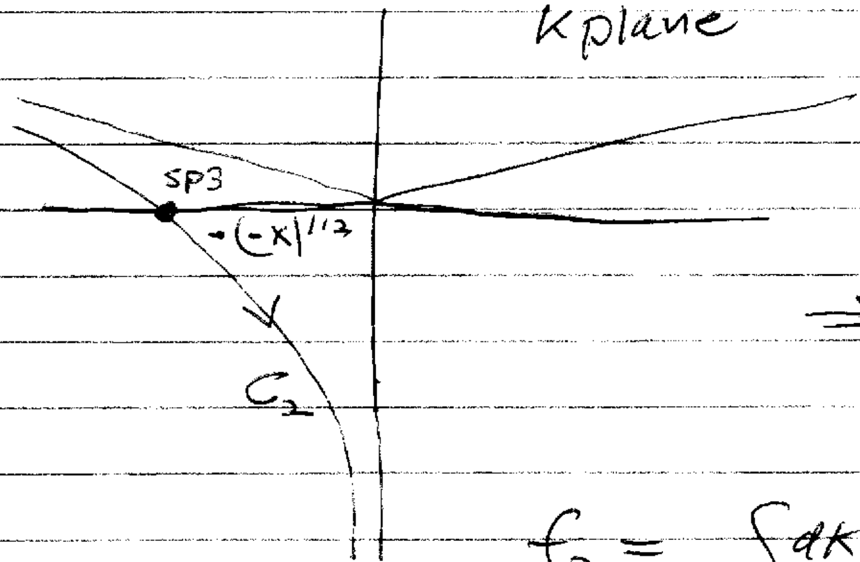
$\Rightarrow$ standing wave with equal right and left-going components

$\Rightarrow$ e.g. take time dependence as $\sim e^{-i\omega t}$



$\Rightarrow$ wave reflection at a turning point

What about f_2 ?



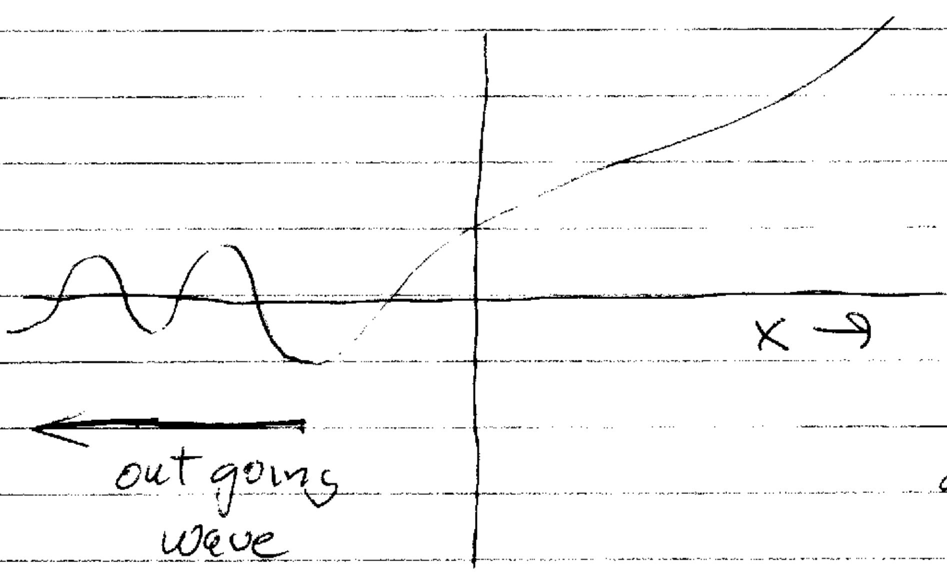
Deform C_2 through SP3

$\Rightarrow$ does not go through SP4

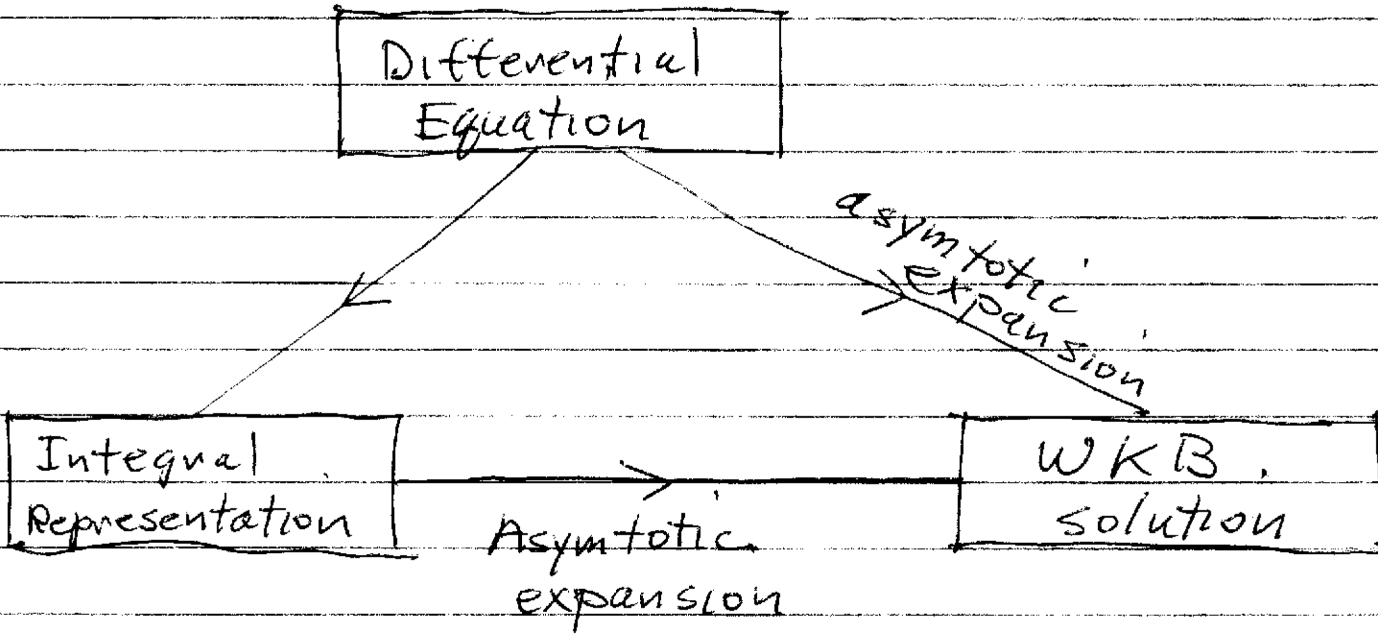
$$f_2 = \int_{SP3} dk\, e^{h(k)}$$

$$= \frac{e^{-i\frac{\pi}{4}}\, e^{\frac{2}{3} i (-x)^{3/2}} \sqrt{\pi}}{(-x)^{1/4}}$$

$\Rightarrow$ since integral through SP3 is the same as in f_1



Corresponds to a wave that is tunneling through a barrier from the right and propagates out to the left.



We have learned how to use asymtotic methods to calculate WKB solutions of equations and asymtotic methods to evaluate integrals. These techniques are related as shown above.