The WKB Method
(Wentzel, Kramers, Brillouin)

Consider the 1-D Helmholtz eqn

$$\frac{d^2}{dx^2} y + k_0^2 y = 0$$

with k_0 a constant. This eqn has two exponential solutions ~~the~~ for $k_0 \neq 0$,

$$y \sim e^{\pm i k_0 x}$$

corresponding to oscillatory behavior for k_0 real. In a real physical system k_0 will be a function of the local parameters (temperature, density .....).

<u>example</u> sound wave

Small amplitude sound waves satisfy the wave eqn

$$\frac{\partial^2}{\partial t^2} \tilde{p} - c_s^2 \frac{\partial^2}{\partial x^2} \tilde{p} = 0$$

where $\tilde{p}$ is the pressure perturbation of the wave and $c_s^2 \sim T$ with T the ambient temperature of the gas.

A wave launched with frequency ω_0 has

$$\tilde{p} \sim e^{-i\omega_0 t}$$

and satisfies the equation

$$c_s^2 \frac{\partial^2}{\partial x^2}\tilde{p} + \omega_0^2 \tilde{p} = 0$$

or

$$\frac{\partial^2}{\partial x^2}\tilde{p} + k_0^2 \tilde{p} = 0$$

with $k_0^2 = \omega_0^2/c_s^2$. The wavevector k_0 depends on the local temperature, which might vary weakly with x.

Suppose that the parameters of a system vary weakly with x so that

$$\frac{\partial^2}{\partial x^2} y + k_0^2 g(x)\, y = 0$$

where $g \sim 1$. If the variation of g with x is weak, do we expect the oscillatory solution to change significantly?

Under what conditions do we expect the solution to be nearly unchanged?

Take the scale length of g to be L

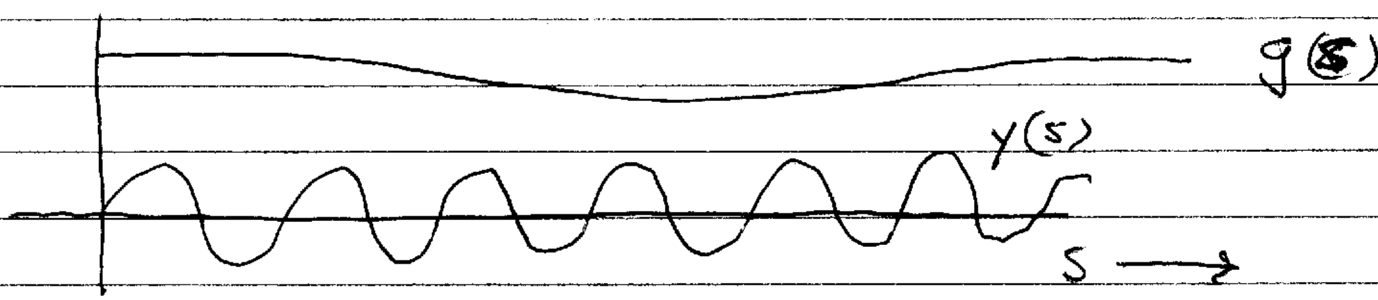
and define a new variable $s = x/L$.
Now

$$\frac{d^2}{ds^2} y + k_0^2 L^2 g(s) y = 0$$

where $dg/ds \sim 1$. Now $k_0 L$ is the only parameter in the problem.

$\Rightarrow$ we expect that when $k_0 L \gg 1$, the oscillatory behavior to be largely unchanged

$\Rightarrow$ the local solution oscillates many times before the local wavevector changes significantly.



$\Rightarrow$ the WKB model is a formalism for treating a system with $k_0 L \gg 1$.

WKB Formulation

Consider $\quad \dfrac{d^2 y}{dx^2} + k_0^2 g(x)\, y = 0$

where g is analytic and $dg/dx \sim 1$. and $k_0 \gg 1$. We look for a solution of the form

$$y \sim e^{i S(x)}$$

where

$$S(x) = \sum_{m=0}^{\infty} k_0^r \frac{S_m(x)}{k_0^m}$$

$$y' = i S' y$$

$$y'' = i S'' y - S'^2 y$$

$$i S'' y - S'^2 y + k_0^2 g y = 0$$

$$\Rightarrow i S'' - S'^2 + k_0^2 g = 0 \qquad (1)$$

Evaluate this eqn. at each order in k_0 and equate the coefficients of each order

$\Rightarrow$ principle of asymptotic balance

To lowest order: $\quad S = k_0^r S_0$

and $\quad i k_0^r S_0'' - k_0^{2r} S_0'^2 + k_0^2 g = 0$

Note that if $g = 1$, the solution is

$$S = k_0 x = k_0^r S_0 \implies r = 1$$

$$S_0 = x$$
$$S_0' = 1$$
$$S_0'' = 0$$

The deviation from this solution is because of $g(x)$. Since $dg/dx \sim 1$ ~~whe~~ we take $d/dx \sim 1$ when acting on S

$\implies$ the ordering of terms is given ~~by~~ controlled by the power of k_0 with $k_0 \gg 1$

$$\underbrace{i k_0^r S_0''}_{\underbrace{k_0^r}_{\text{small}}} - \underbrace{k_0^{2r} S_0'^2}_{k_0^{2r}} + \underbrace{k_0^2 g}_{k_0^2} = 0$$

$$\implies r = 1$$

$$S_0'^2 = g \implies S_0' = \pm\sqrt{g}$$

$$S_0 = \pm \int^x dx' \sqrt{g(x')}$$

and

$$S = k_0 \sum_{m=0}^{\infty} \frac{1}{k_0^m} S_m$$

$\implies$ insert into (1).

The complete equation for S is given by

$$k_0 i \sum_{m=0}^{\infty} \frac{1}{k_0^m} S_m'' - k_0^2 \left(\sum_{m=0}^{\infty} \frac{1}{k_0^m} S_m' \right)^2 + k_0^2 g = 0$$

$$i \sum_{m=0}^{\infty} \frac{S_m''}{k_0^{m+1}} - \left(\sum_{m=0}^{\infty} \frac{S_m'}{k_0^m} \right)^2 + g = 0$$

<u>zero order</u> : $k_0^0 \Rightarrow S_0'^2 = g$

as before

<u>first order</u> : k_0^{-1}

$$i S_0'' - 2 S_0' S_1' = 0$$

$$\Rightarrow S_1' = \frac{i S_0''}{2 S_0'} = \frac{i}{2} \frac{d}{dx} \ln(S_0')$$

$$S_1 = \frac{i}{2} \ln(S_0') = \frac{i}{2} \ln(\pm \sqrt{g})$$

<u>second order</u> : k_0^{-2}

$$i S_1'' - 2 S_0' S_2' - S_1'^2 = 0$$

$\Rightarrow$ solve for S_2

Thus, retaining S_0 and S_1,

$$y \sim e^{\pm i k_0 S_0} \, e^{i k_0 \frac{S_1}{k_0}}$$

$$= e^{\pm i k_0 \int^x dx' \sqrt{g}} \, e^{i \left(\frac{i}{2}\right) \ln\left(\pm \sqrt{g}\right)}$$

$$= \frac{1}{\left(\pm \sqrt{g}\right)^{1/2}} \, e^{\pm i k_0 \int^x dx' \sqrt{g}}$$

The general solution is a linear combination of these two solutions. It is simpler to write the answer in terms of the local wavevector

$$k(x) = k = k_0 \sqrt{g(x)}$$

$$y \sim \frac{1}{k^{1/2}} \, e^{\pm i \int^x dx' \, k(x')}$$

For a constant k this reduces to the original solution

$$y \sim e^{\pm i k x}$$

When k varies in space, for k real

the exponential of $\int^x dx' k(x')$ yields the phase of the wave, while the $k^{-1/2}$ yields the variation in amplitude.

$\Rightarrow$ The $k^{-1/2}$ arises from the variation in the momentum flux

$$P_k \sim k|y|^2 \sim k y^* y \sim k\, k^{-1/2} k^{-1/2} \sim 1$$

Thus, as k varies the momentum flux of the wave is preserved.

$\Rightarrow$ a reminder that the calculation is based on the assumption that

$$kL \gg 1$$

$\Rightarrow$ the variation of the wavelength of the wave is short compared with the scale L over which is varies.

$\Rightarrow$ when L is comparable to k^{-1}, the medium causes reflection

$\Rightarrow$ WKB does not describe reflection

$\Rightarrow$ the forward and backward waves are independent solutions

<u>Example</u> The Airy equation

$$y'' - xy = 0$$

We previously solved this equation using the series representation. We now want to solve the equation approximately using the WKB method.

Where is the large parameter? The local wave vector is given by

$$k = (-x)^{1/2}. \quad \Rightarrow \text{oscillatory for } x < 0.$$

The scale length L is defined by the rate of variation of k or k^2

$$\frac{1}{L} \sim \left| \frac{1}{k^2} \frac{d}{dx} k^2 \right| \sim \left| \frac{1}{-x} \frac{d}{dx}(-x) \right| \sim \frac{1}{|x|}$$

$$kL \sim (-x)^{3/2} \Rightarrow |x|^{3/2} \gg 1$$

The WKB solution is valid for $|x| \gg 1$.

$$\Rightarrow \text{not valid at } x = 0.$$

WKB solution:

$$y \sim \frac{1}{(-x)^{1/4}} e^{\pm i \int^{x} dx' (-x')^{1/2}}$$

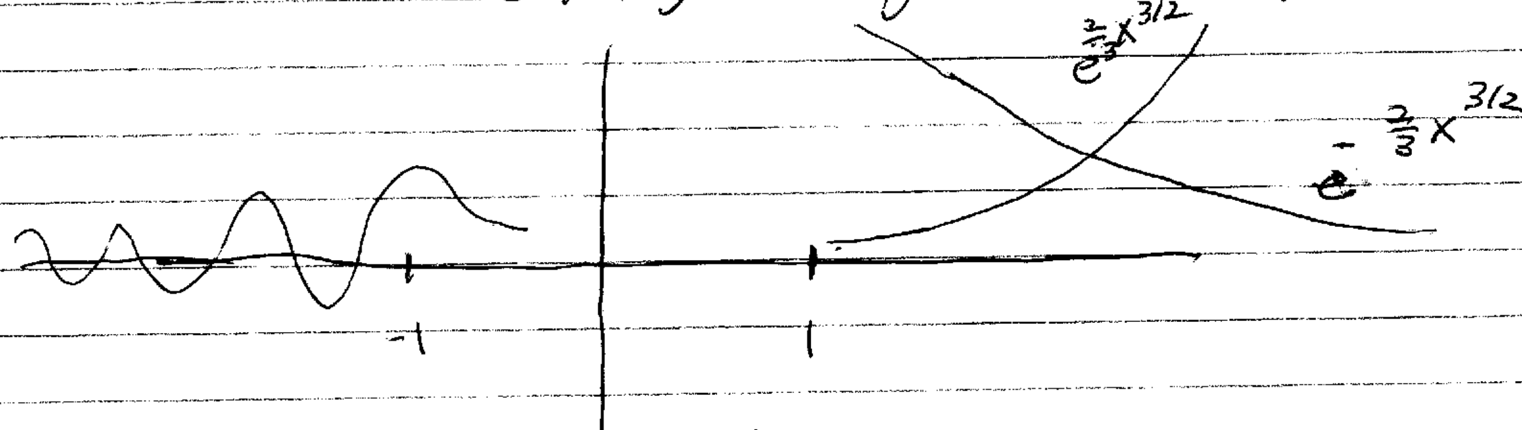
$\underline{x > 0}$

$$y \sim \frac{1}{x^{1/4}} e^{\pm \frac{2}{3} x^{3/2}}$$

$\underline{x < 0}$

$$y \sim \frac{1}{(-x)^{1/4}} e^{\pm i \frac{2}{3} (-x)^{3/2}}$$

$\Rightarrow$ these solutions are the same as those obtained from the saddle point technique from the integral representation



The WKB solutions fail as $x \to 0$.

For $x < 0$, the wavelength increases as $-x$ becomes smaller.