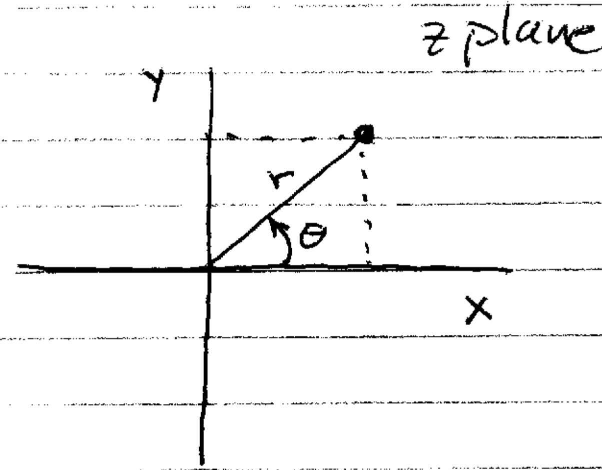
Functions of a complex variable
(Arfken Ch 11)

Basics



$$z = x + iy = re^{i\theta}$$
$$\equiv (x, y)$$
$$\mathrm{Arg}(z) \equiv \theta$$

Multiplication rules as in real numbers with $i^2 = -1$

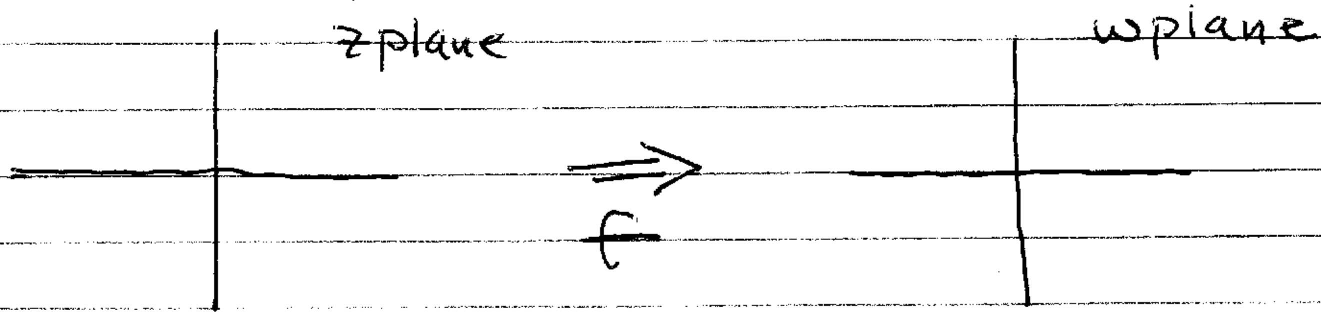
Complex conjugate $z^* \equiv x - iy = re^{-i\theta}$

$$zz^* = r^2 = |z|^2 \Rightarrow |z| = r$$

Maps

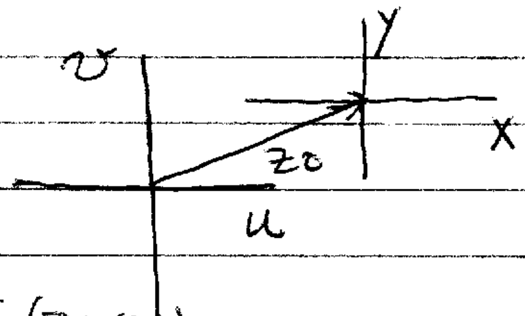
Let $w = f(z) = u(x,y) + i\,v(x,y)$

The function f defines a mapping between the complex z plane and the complex w plane



Examples of maps

(1) $w = z + z_0 \Rightarrow$ translation

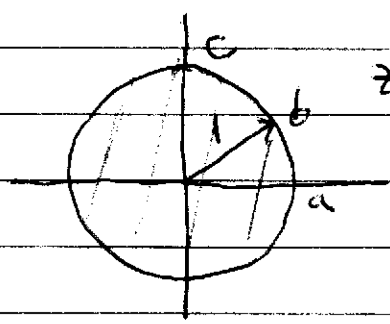


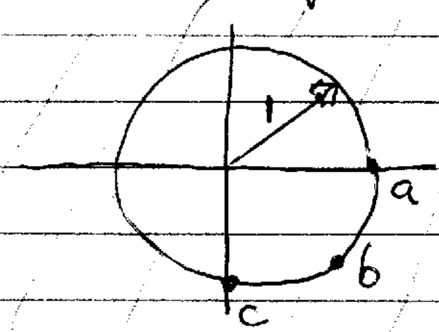
(2) $w = z_0 z = r r_0 e^{i(\theta+\theta_0)} \Rightarrow$ multiplication

$\Rightarrow$ rotation by θ_0

$\Rightarrow$ amplification by r_0

(3) Inversion $\quad w = \frac{1}{z} = \frac{1}{r} e^{-i\theta}$





Inside of unit circle is mapped to outside the unit circle in the w plane

horizontal line in z plane $\quad y = y_0$

$$w = u + iv = \frac{1}{x + iy_0} = \frac{x - iy_0}{x^2 + y_0^2}$$

$$u = \frac{x}{x^2 + y_0^2} \; ; \quad v = -\frac{y_0}{x^2 + y_0^2}$$

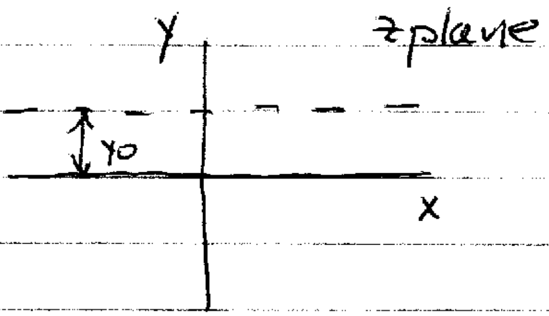
As x varies, traces curve in w plane

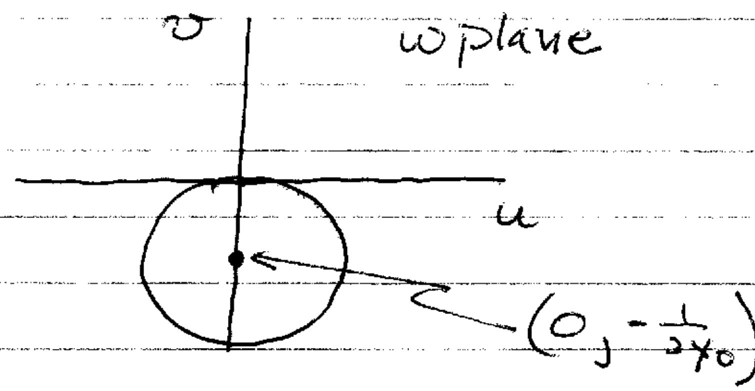
$\Rightarrow$ eliminate x $\Rightarrow$ $x^2 = -\frac{y_0}{v} - y_0^2$

$$u^2 = \frac{x^2}{(x^2+y_0^2)^2} = -\left(\frac{y_0}{v} + y_0^2\right)\frac{v^2}{y_0^2}$$

$$u^2 + v^2 + \frac{v}{y_0} = 0$$

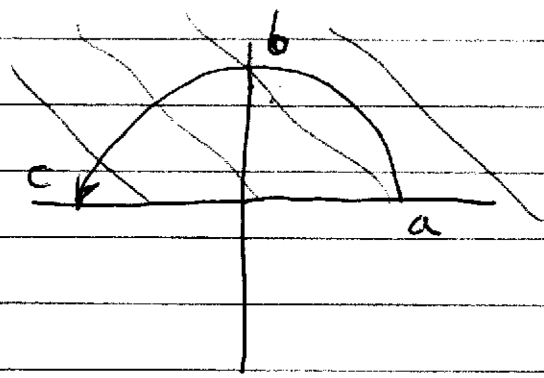
$$u^2 + \left(v + \frac{1}{2y_0}\right)^2 = \frac{1}{4y_0^2}$$

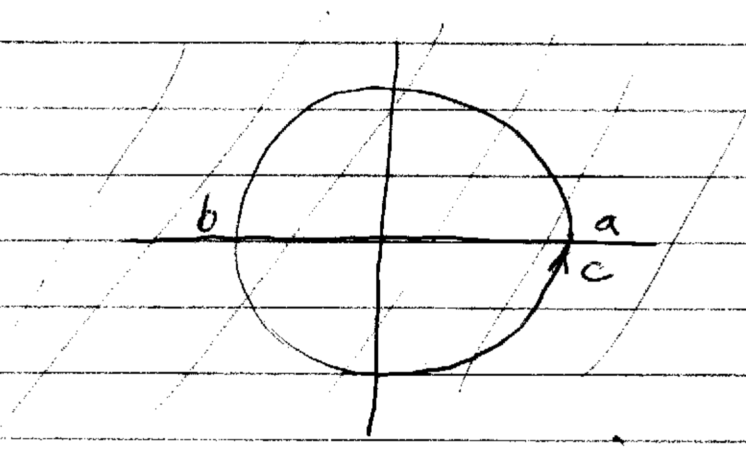




* These mappings were all one to one since the entire z plane mapped to the entire w plane.

4 $w = z^2$

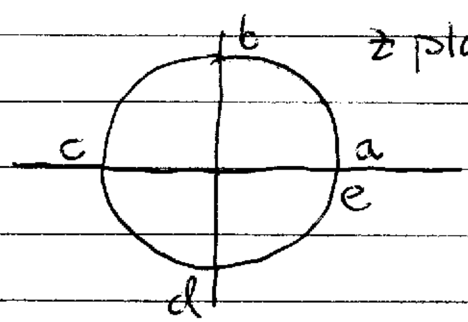


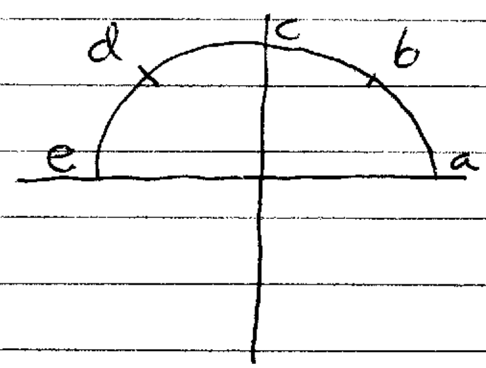


Upper half of z plane maps to entire w plane
Lower half of z plane maps to entire w plane

Any point z_0 and $-z_0$ map to the same point in the w plane.

5 $w = z^{1/2}$ (inverse of 4)





Note that $z = re^{i\theta}$ and $z = re^{i\theta + 2\pi i}$, the same point in the z plane correspond to

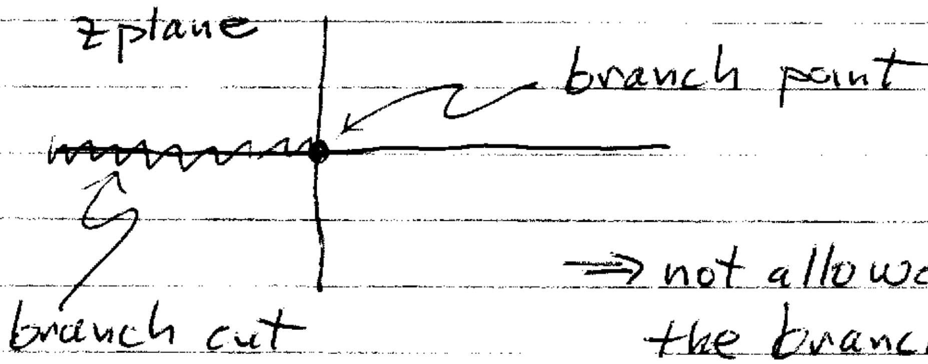
$$w = r^{1/2} e^{i\frac{\theta}{2}} \quad \text{and} \quad w = r^{1/2} e^{i\frac{\theta}{2}} e^{i\pi}$$
$$= -r^{1/2} e^{i\frac{\theta}{2}},$$

two different points in the w plane

$\Rightarrow$ $z^{1/2}$ is a multivalued function

$\Rightarrow$ means that the value of $z^{1/2}$ is not definite
$\Rightarrow$ this is not ok

Introduce a <u>branch cut</u> in the z plane



$\Rightarrow$ not allowed to cross the branch cut

$\Rightarrow$ Once the phase of z is defined along the real positive axis, then $\theta = \mathrm{Arg}(z)$ can only change by $\pm\pi$ over the entire z plane.

If $\theta = 0$ on the positive real axis,

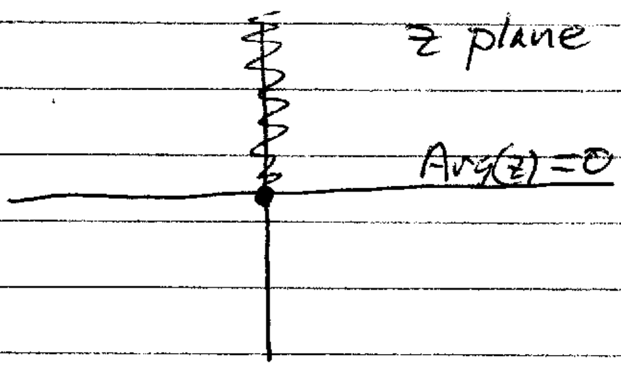
$$\theta \in (-\pi, \pi).$$

If $\theta = 2\pi$ on the positive real axis,

$$\theta \in (\pi, 3\pi).$$

$\Rightarrow$ $w = z^{1/2}$ is a single valued function in the cut z plane.

The <u>branch cut</u> can extend from the origin to infinity along any direction



$\Theta \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$

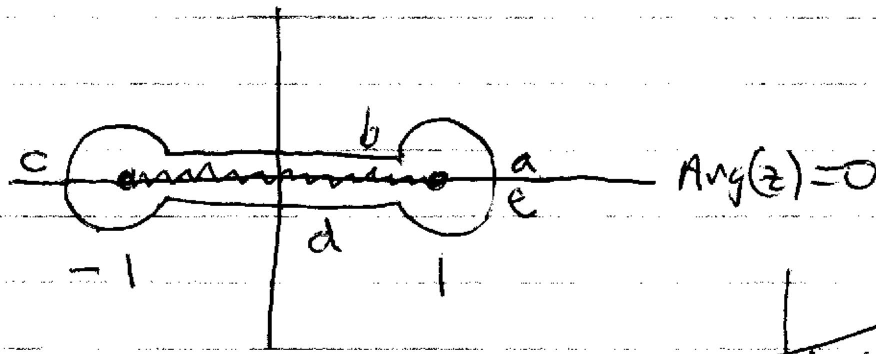
The <u>branch point</u> is the point of termination of the branch cut.

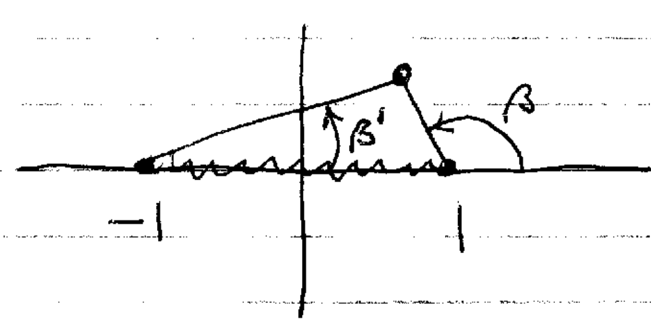
<u>Example</u> Define a branch cut to make $w = (z^2-1)^{1/2}$ a single valued function

Do we need a branch cut that extends to infinity?

$\Rightarrow$ For large z $w \simeq z$ $\Rightarrow$ single valued

$\Rightarrow$ branch cut does not have to extend to infinity





$z-1 = \alpha e^{i\beta}$

$z+1 = \alpha' e^{i\beta'}$

$$w = (z-1)^{1/2}(z+1)^{1/2}$$

At a : $\text{Arg}(z-1)^{1/2} = 0$, $\text{Arg}(z+1)^{\frac{1}{2}} = 0$

$\beta = 0$ $\qquad \beta' = 0$

$$\text{Arg}(z^2-1)^{\frac{1}{2}} = 0$$

At b : $\text{Arg}(z-1)^{1/2} = \text{Arg}(\alpha^{1/2} e^{i\frac{\beta}{2}}) = \frac{\pi}{2}$

since $\beta = \pi$

$\text{Arg}(z+1)^{1/2} = 0$ since $\beta' = 0$

$$\text{Arg}(z^2-1)^{1/2} = \frac{\pi}{2}$$

At c : $\text{Arg}(z-1)^{1/2} = \frac{\pi}{2}$

$\text{Arg}(z+1)^{1/2} = \pi/2$

$\text{Arg}(z^2-1)^{1/2} = \pi$

At d : $\text{Arg}(z-1)^{\frac{1}{2}} = \pi/2$ since $\beta = \pi$

$\text{Arg}(z+1)^{1/2} = \pi$ since $\beta' = 2\pi$

$\text{Arg}(z^2-1)^{1/2} = 3\pi/2$

At (e) : $\operatorname{Arg}(z-1)^{\frac{1}{2}} = \pi$ since $\beta = 2\pi$

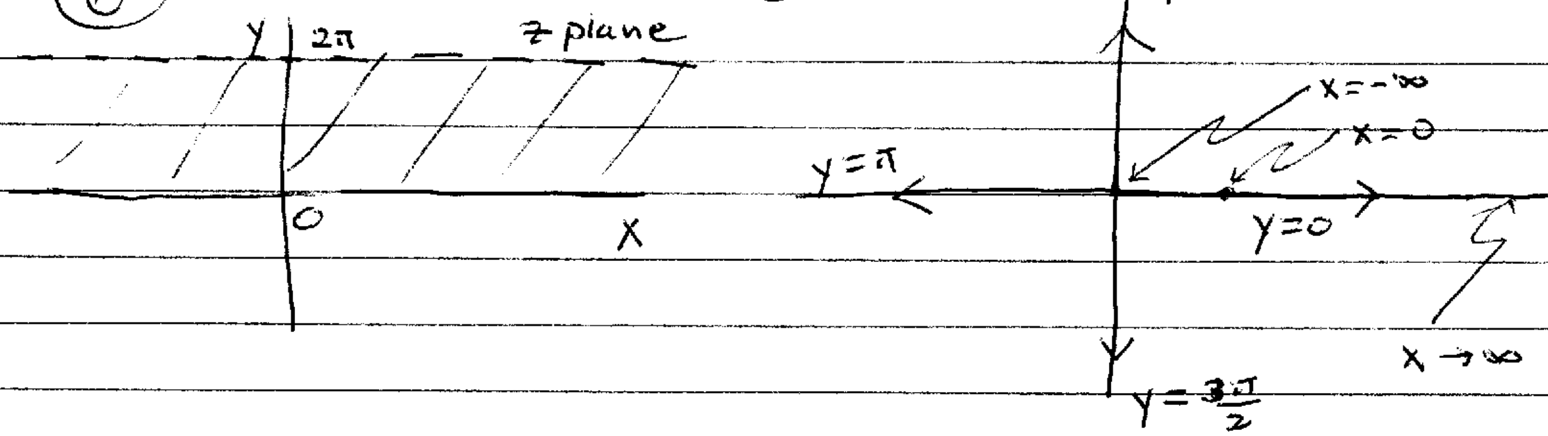
$\operatorname{Arg}(z+1)^{\frac{1}{2}} = \pi$ since $\beta' = 2\pi$

$\operatorname{Arg}(z^2-1)^{\frac{1}{2}} = 2\pi$

$$\Rightarrow e^{2\pi i} = 1 = e^{0}$$

$\Rightarrow (z^2-1)^{\frac{1}{2}}$ is single valued

(6) $w = e^{z} = e^{x} e^{iy}$



A horizontal line in the z plane maps to a radial line eminating from $w=0$ in the w plane

$$w = e^{x} e^{iy_0}$$

$$x \to -\infty \Rightarrow w = 0$$

$y \in (0, 2\pi)$ maps onto the entire w plane

Each successive interval of 2π maps to the entire w plane.

Is the mapping single valued ?

7 $w = \ln(z)$ (inverse of 6)

$w = \ln(re^{i\theta}) = \ln(r) + i\theta = u + iv$

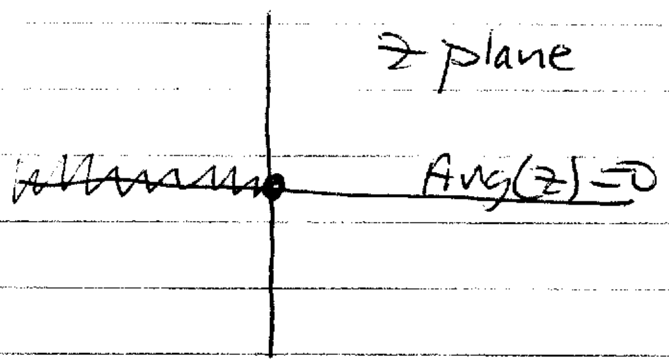
In the z plane $\theta \rightarrow \theta + 2n\pi$

$v \rightarrow v + 2n\pi$

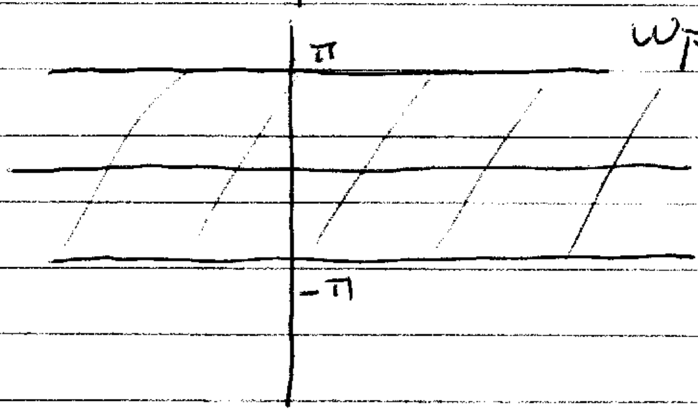
A single point in z maps to many points in w.

$\Rightarrow$ multivalued function

Define a branch cut in the z plane



$w = \ln(z)$ is a single valued function in the cut z plane.



mapping of the cut z plane.

Limit

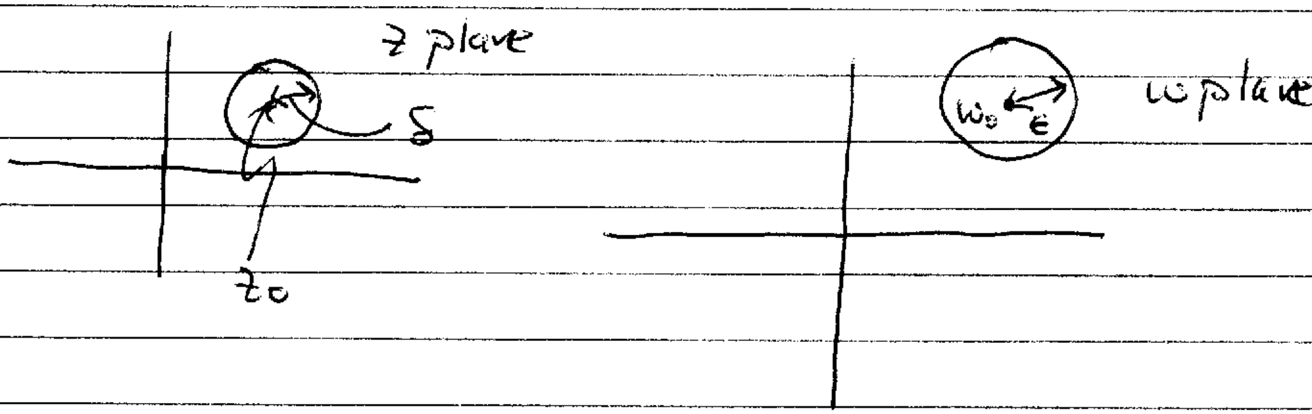
Consider a function $f(z) = w$ then

$$\lim_{z \to z_0} f(z) = w_0$$

means that as z approaches z_0 from any direction, $f(z)$ approaches w_0. By taking a smaller and smaller neighborhood of z around z_0, $f(z)$ can be made arbitrarily close to w_0.

Mathematically, for every positive ϵ there exists a positive δ such that

$$|f(z) - w_0| < \epsilon \quad \text{when} \quad |z - z_0| < \delta$$



A function is <u>continuous</u> at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Derivatives

$$f'(z) \equiv \frac{df}{dz} \equiv \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

From the definition of limit f' must be the same independent of the direction in the z plane in which $\Delta z \to 0$. That is, $\Delta z = \Delta x + i\Delta y$ and can take $\Delta x = 0$ and take $\lim_{\Delta y \to 0}$ or the reverse.

Cauchy-Riemann Conditions

Suppose $f'(z_0)$ exists

$$f(z_0) = w_0 = u(x_0, y_0) + i\,v(x_0, y_0)$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{u(x_0+\Delta x, y_0+\Delta y) + i\,v(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0) - i\,v(x_0, y_0)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{\frac{\partial u}{\partial x_0}\Delta x + \frac{\partial u}{\partial y_0}\Delta y + i\frac{\partial v}{\partial x_0}\Delta x + i\frac{\partial v}{\partial y_0}\Delta y}{\Delta x + i\Delta y}$$

$\Delta y = 0$:

$$f'(z_0) = \frac{\partial u}{\partial x_0} + i\frac{\partial v}{\partial x_0}$$

$\Delta x = 0$:

$$f'(z_0) = -i\frac{\partial u}{\partial y_0} + \frac{\partial v}{\partial y_0}$$

They must be equal

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x_0} &= \frac{\partial v}{\partial y_0} \\ \frac{\partial v}{\partial x_0} &= -\frac{\partial u}{\partial y_0} \end{aligned}}$$

Cauchy-Riemann Conditions

<u>Theorem</u>: If $f'(z)$ exists then the CR conditions are satisfied.

$\Rightarrow$ just proved this

<u>Theorem</u>: If the CR conditions are satisfied at a point z_0, then $f'(z_0)$ exists at that point.

proof:

$$f'(z_0) = \lim_{z \to z_0} \frac{\frac{\partial u}{\partial x_0}\Delta x + \frac{\partial u}{\partial y_0}\Delta y + i\frac{\partial v}{\partial x_0}\Delta x + i\frac{\partial v}{\partial y_0}\Delta y}{\Delta x + i\Delta y}$$

eliminate u derivatives using CR conditions

$$= \lim_{z \to z_0} \frac{\frac{\partial v}{\partial y_0}\Delta x - \frac{\partial v}{\partial x_0}\Delta y + i\frac{\partial v}{\partial x_0}\Delta x + \frac{\partial v}{\partial y_0}\Delta y}{\Delta x + i\Delta y}$$

$$= \lim_{z \to z_0} \frac{\frac{\partial v}{\partial y_0}(\Delta x + i\Delta y) + i\frac{\partial v}{\partial x_0}(\Delta x + i\Delta y)}{\Delta x + i\Delta y}$$

$$= \frac{\partial v}{\partial y_0} + i\frac{\partial v}{\partial x_0}$$ $\Rightarrow$ limit exists

<u>example</u> $f(z) = e^z = e^x(\cos y + i \sin y)$

$$u = e^x \cos y$$
$$v = e^x \sin y$$
$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial v}{\partial y} = e^x \cos y$$
$$\frac{\partial v}{\partial x} = e^x \sin y = -\frac{\partial u}{\partial y}$$
$$f'(z) = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y = e^z$$

$\Rightarrow$ as expected

<u>Note that</u> $f'(z)$ <u>does not exist</u> for arbitrary u, v even if they are continuous and their derivatives exist.

$\Rightarrow$ ~~usual rules for diff. products~~ apply

<u>Analytic function</u>

If $f'(z_0)$ exists at z_0 and everywhere in a neighborhood ($|z - z_0| < \epsilon$ for some ϵ), then $f(z)$ is <u>analytic</u> at z_0.

A function that is analytic everywhere in the complex plane is an <u>entire</u> function.

<u>example</u>

① $P_n(z) = a_0 + a_1 z + \cdots a_n z^n$ is an entire function

② $f(z) = |z|^2 = x^2 + y^2$

$u = x^2 + y^2$

$v = 0$

CR require $\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} = 0$

$\frac{\partial v}{\partial x} = 0 = -\frac{\partial u}{\partial y} = -2y$

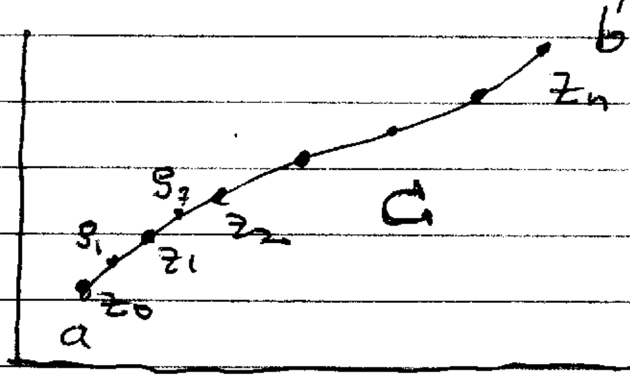
$\Rightarrow$ satisfied at $z = 0$

$\Rightarrow$ differentiable at $z = 0$

$\Rightarrow$ not analytic anywhere.

Contour integrals

Consider a contour in the complex z plane
Divide this contour into n segments
with n intermediate points $\xi_1, \xi_2 \cdots \xi_n$



where ξ_j is on C
between z_j and z_{j-1}

Let $\quad S_n = \sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1})$

If $\lim_{n\to\infty}$ S_n exists then

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1}) \equiv \int_{\substack{a \\ C}}^{b} dz\, f(z)$$

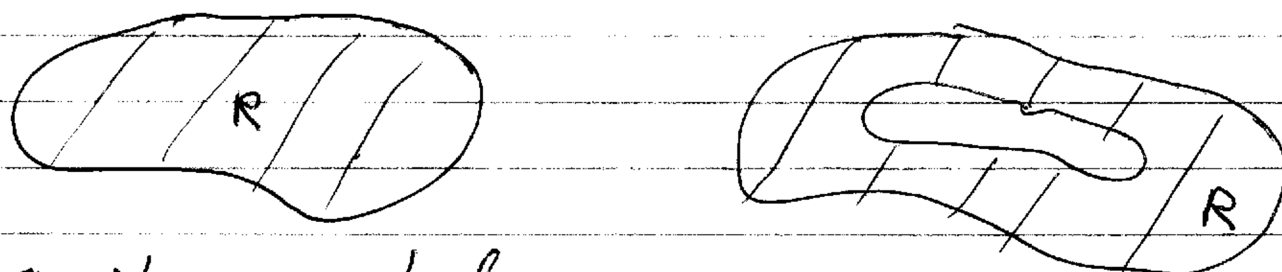
The integral is defined as the contour integral over C.

Cauchy - Goursat Theorem (C-G)

Theorem: If a function $f(z)$ is analytic throughout a simply connected region R, then for every closed contour C in R

$$\oint_C dz\, f(z) = 0$$

Note: simply connected - every closed curve in a domain contains only points in the domain



simply connected

not simply connected

(1) Less general proof: requires Stokes theorem so that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ be continuous

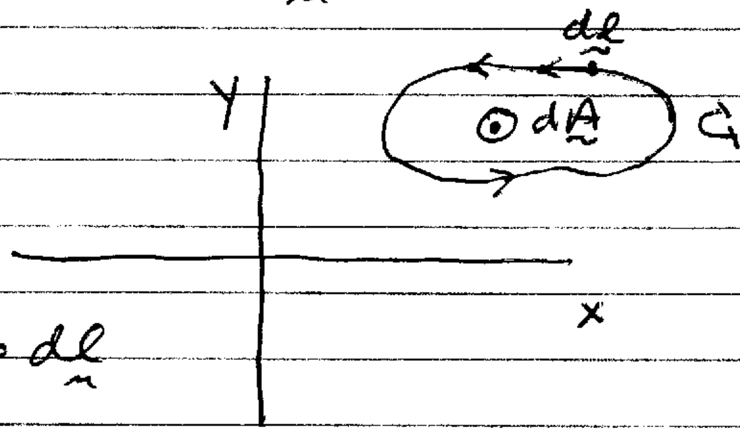
$$\oint_C dz\, f(z) = \oint_C (u+iv)(dx+idy)$$

$$= \oint_C \left[(u\,dx - v\,dy) + i(v\,dx + u\,dy) \right]$$

From Stokes theorem for real functions

$$\oint_C \underset{\sim}{B} \cdot d\underset{\sim}{\ell} = \int_A \nabla \times \underset{\sim}{B} \cdot d\underset{\sim}{A}$$

Note: $\underset{\sim}{B}$ is a vector



$$\oint_C (u\,dx - v\,dy) = \int_C \underset{\sim}{B} \cdot d\underset{\sim}{\ell}$$

$$\underset{\sim}{B} = u\hat{x} - v\hat{y}$$

$$\Downarrow$$

$$\int_A dA\,(\nabla \times \underset{\sim}{B})_z = \int dA\left(-\frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial x}\right)$$

$$= -\int dx\,dy\,\underbrace{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}_{0 \text{ from CR conditions}}$$

$$\oint_C (v\,dx + u\,dy) \Longrightarrow \oint_C dx\,dy\,\underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{0 \text{ from CR}}$$

$$\underset{\sim}{B} = v\hat{x} + u\hat{y}$$

$$(\nabla \times \underset{\sim}{B})_z = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

$$\Longrightarrow \boxed{\oint_C dz\, f(z) = 0}$$

② General Proof :

Prelude: Schwarz inequality

$$\left| \int_C dz\, f(z) \right| \leq \int_C |dz|\, |f(z)|$$

$$\int_C dz\, f(z) = \int_C f(z) \left(\frac{dx}{dt} + i \frac{dy}{dt} \right) dt$$

$\Rightarrow$ t is real and parameterizes the contour $\Rightarrow$ distance along contour

Let $G = f(z)\left(\frac{dx}{dt} + i \frac{dy}{dt} \right)$

$$\Rightarrow \int_C dt\, G \equiv r_0 e^{i\theta_0} \Rightarrow r_0 = \int dt\, G e^{-i\theta_0}$$

$$= \int dt\, \mathrm{Re}\left(G e^{-i\theta_0} \right)$$

$$\left| \int_C dz\, f(z) \right| \equiv r_0 = \int dt\, \mathrm{Re}\left(G e^{-i\theta_0} \right)$$

$$\leq \int dt\, \left| \mathrm{Re}\left(G e^{-i\theta_0} \right) \right|$$

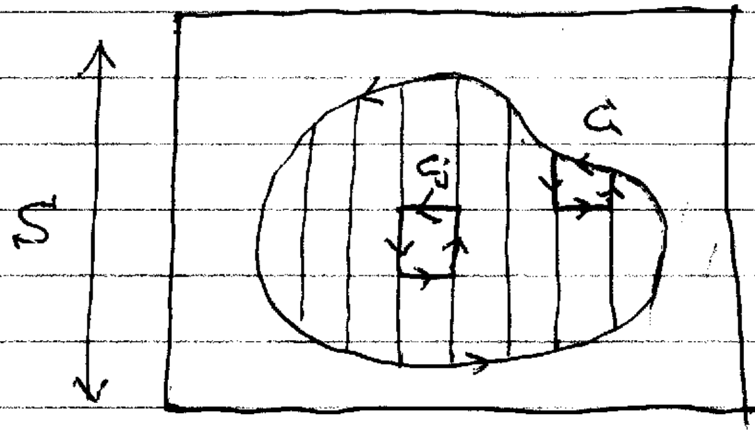
$$\leq \int dt\, \left| G e^{-i\theta_0} \right| = \int dt\, |G|$$

$\Rightarrow$ from inequalities of integrals of real functions

$$dt\, |G| = |f(z)| \left| \frac{dx}{dt} + i \frac{dy}{dt} \right| dt = |f(z)|\, |dz|$$

$\Rightarrow$ General proof of C-G theorem

$\Rightarrow$ divide region into squares and partial squares along boundary



Let z_j be an interior point to square

$$\oint_C f(z)\,dz = \sum_j \oint_{C_j} f(z)\,dz$$

From the derivative of f at z_j

$$\frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) = \delta_j(z)$$

with $|\delta_j(z)| < \epsilon$

For a point on the boundary C_j

$$f(z) = [\delta_j(z) + f'(z_j)](z - z_j) + f(z_j)$$

$$\oint_{C_j} dz\, f(z) = \oint_{C_j} dz \left\{ f(z_j) + [\delta_j(z) + f'(z_j)](z - z_j) \right\}$$

All integrals but that over $\delta_j(z)$ are zero by previous proof $\Rightarrow$ Stokes theorem ok for them

$$\left|\int_{C_j} dz\, f(z)\right| = \left|\oint_{C_j} dz\, \delta_j(z)(z-z_j)\right|$$

$$\leq \oint_{C_j} |\delta_j||z-z_j||dz| < \epsilon \oint_{C_j} |dz||z-z_j|$$

$$|z-z_j| \leq \sqrt{2}\, s_j$$

with s_j the length of the edge of the square $\Rightarrow$ want to bound the integral

a) Interior squares

$$\left|\oint_{C_j} dz\, f(z)\right| < \epsilon \sqrt{2}\, s_j (4 s_j) = \sqrt{2}\, 4 \epsilon A_j$$

$$A_j = s_j^2$$

b) Partial squares

$$\left|\oint_{C_j} dz\, f(z)\right| < \epsilon \sqrt{2}\, s_j [4 s_j + L_j]$$

L_j = arc length of edge square

$$< \sqrt{2}\, 4 A_j \epsilon + \sqrt{2}\, \epsilon\, s' L_j$$

$$\left|\oint_{C} dz\, f(z)\right| < \sqrt{2}\, 4 \epsilon \sum_j A_j + \sqrt{2}\, \epsilon\, s' \sum L_j$$

$$< \sqrt{2}\, 4 \epsilon\, s'^2 + \sqrt{2}\, \epsilon\, s' L$$

L = arc length of C

Since ϵ can be made arbitrarily small and δ^2 and δL are bounded

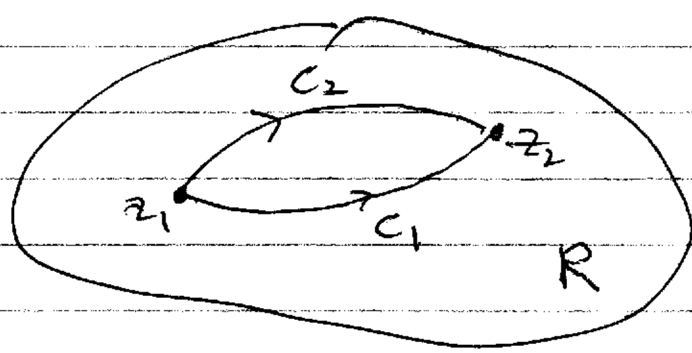
$$\oint_{C_1} dz\, f(z) = 0$$

Path Independence of Integrals

Consider a simply connected domain R and two points z_1, z_2 in R. Consider any two contours C_1 and C_2 that are entirely within R. If $f(z)$ is analytic within R,

$$\int_{z_1,\, C_1}^{z_2} dz\, f(z) = \int_{z_1,\, C_2}^{z_2} dz\, f(z)$$

$\Rightarrow$ the integral is independent of path

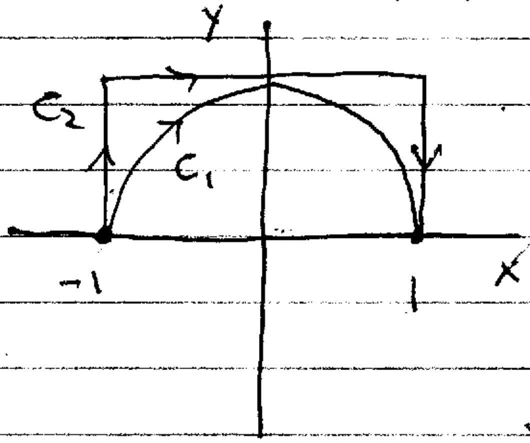


Since

$$\underbrace{\int_{C_1} dz\, f(z) - \int_{C_2} dz\, f(z)}_{\text{closed contour in R.}} = 0$$

example: Let $f(z) = \frac{1}{z}$

Consider integrals over C_1 and C_2. Show that they yield the same values



$$I_1 = \int_{C_1} dz \, \frac{1}{z}$$

$$z = e^{i\theta}, \quad dz = i\, d\theta\, e^{i\theta}$$

$$I_1 = \int_{\pi}^{0} \frac{i\, d\theta\, e^{i\theta}}{e^{i\theta}} = -i\pi$$

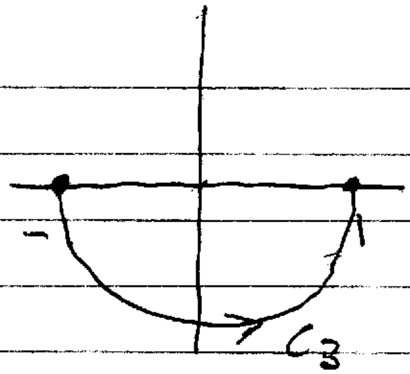
$$I_2 = \int_{C_2} \frac{dx + i\,dy}{x + iy} = \int_{C_2} \frac{(dx + i\,dy)(x - iy)}{x^2 + y^2}$$

$$= \int_{C_2} dx \frac{x}{x^2+y^2} + \int_{C_2} dy \frac{y}{x^2+y^2} + i\left[\int_{C_2} dy \frac{x}{x^2+y^2} - \int_{C_2} \frac{dx\, y}{x^2+y^2} \right]$$

$$= \underbrace{\int_{-1}^{1} dx \frac{x}{x^2+1}}_{\text{odd} \Rightarrow 0} + \underbrace{\int_{0}^{1} dy \frac{y}{1+y^2} + \int_{1}^{0} dy \frac{y}{1+y^2}}_{\text{cancel} \Rightarrow 0} + i\left[\quad \right]$$

$$+ i\left[\int_{0}^{1} dy \frac{(-1)}{1+y^2} + \int_{1}^{0} dy \frac{(1)}{1+y^2} - \int_{-1}^{1} dx \frac{1}{1+x^2} \right]$$

$$= -2i \int_{-1}^{1} \frac{dx}{1+x^2} = -2i \tan^{-1} x \Big|_{-1}^{1} = -\pi i$$

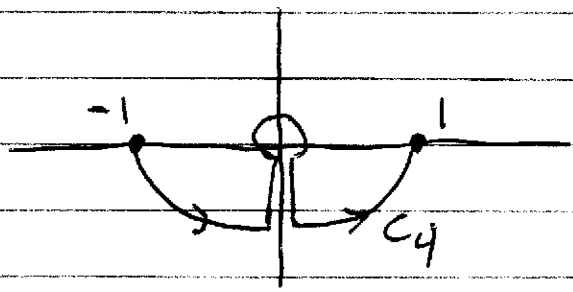


$$I_3 = \int_{C_3} dz \, \frac{1}{z} = i\pi$$

Why?

$f(z)$ is singular (not analytic at $z=0$).

$\Rightarrow$ can't move C_1 or C_2 to C_3 without crossing singularity

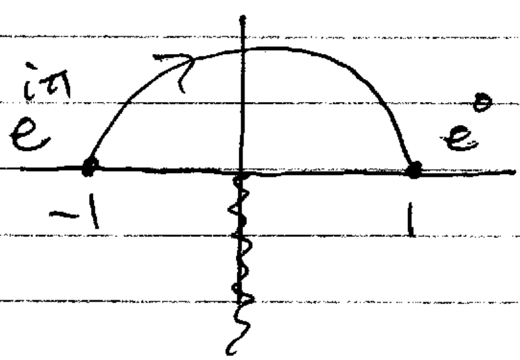


$$I_4 = -i\pi$$

Use indefinite integral

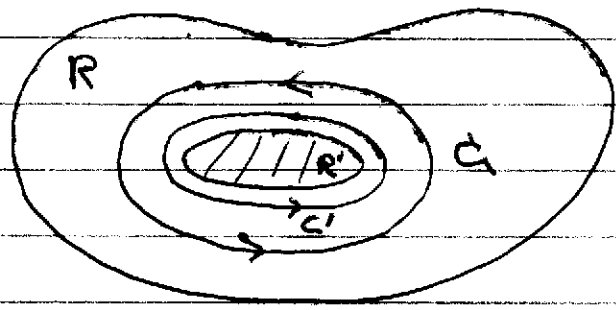
$$I = \int_{-1}^{1} dz \, \frac{1}{z} = \ln(z) \Big|_{e^{i\pi}}^{e^{0}} = -\ln(e^{i\pi})$$

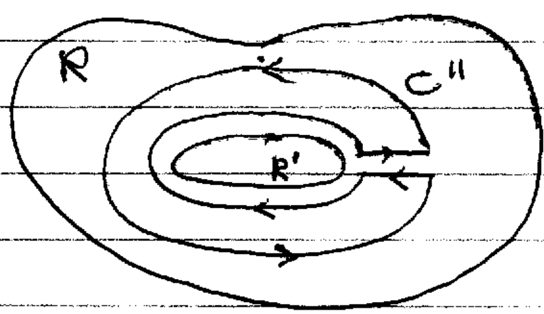
$$= -i\pi$$



Multiply Connected Regions

Suppose $f(z)$ is analytic everywhere in a simply connected region R except in a region R'





From CG Theorem

$$\oint_{C''} f(z)\,dz = 0$$

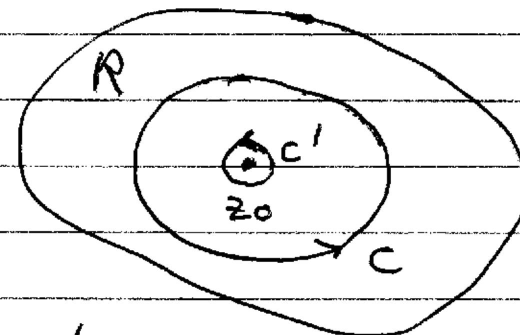
$$\Rightarrow \quad \oint_{C} dz\, f(z) - \oint_{C'} dz\, f(z) = 0$$

$\Rightarrow$ two oppositely directed contours cancel $\Leftrightarrow$ move close

$$\boxed{\oint_{C} dz\, f(z) = \oint_{C'} dz\, f(z)}$$

Cauchy's Integral Formula

$$\oint_{C} \frac{f(z)\,dz}{z-z_0} = 2\pi i f(z_0)$$



For f analytic in R.

Proof: $\frac{f(z)}{z-z_0}$ analytic everywhere but z_0

$$\oint_{C} \frac{f(z)\,dz}{z-z_0} = \oint_{C'} \frac{f(z)}{z-z_0}$$

$$z = z_0 + re^{i\theta}$$

$$dz = ire^{i\theta}\,d\theta \qquad dz\,\frac{f(z)}{z-z_0} = d\theta\,\frac{f(z_0+re^{i\theta})\,ire^{i\theta}}{re^{i\theta}}$$

Let $r \to 0$ $\quad \int_{C} dz\,\frac{f(z)}{z-z_0} = i f(z_0)\int_0^{2\pi} d\theta$

$$= 2\pi i\, f(z_0)$$

$\Rightarrow$ If you know the value of a function on a boundary C, then you know the function everywhere interior to the boundary.

General Derivatives

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{z - z_0}$$

$$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i} \cdot \frac{1}{\Delta z} \oint_C dz \, f(z) \left[\frac{1}{z - z_0 - \Delta z} - \frac{1}{z - z_0} \right]$$

$$= \frac{1}{2\pi i} \frac{1}{\Delta z} \oint_C \frac{dz \, f(z)}{(z - z_0 - \Delta z)(z - z_0)} \left[z - z_0 - (z - z_0 - \Delta z) \right]$$

$$= \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)}$$

$$\Longrightarrow \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{(z - z_0)^2} \quad \text{as } \Delta z \to 0$$

$$f'(z_0) = \frac{1}{2\pi i} \oint dz \, \frac{f(z)}{(z - z_0)^2}$$

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint dz \, \frac{f(z)}{(z - z_0)^{n+1}}}$$

If $f(z)$ is analytic then all derivatives are analytic.

Analyticity of Path Integrals

$f(z)$ is a continuous function in R and

$$\oint_C dz\, f(z) = 0$$

for every closed contour in R, then $F(z)$ is analytic in R

$\Rightarrow$ note that have not assumed $f(z)$ is analytic

$\Rightarrow$ want to show that can prove analyticity of f if closed contours of f are zero

$\Rightarrow$ inverse of C G Theorem

$$F(z_1) \equiv \int_{z_0}^{z_1} dz\, f(z)$$

$$F(z_1+\Delta z) - F(z_1) = \int_{z_1}^{z_1+\Delta z} dz\, f(z)$$

$$\frac{F(z_1+\Delta z) - F(z_1)}{\Delta z} - f(z_1) = \frac{1}{\Delta z}\left[\int_{z_1}^{z_1+\Delta z} dz\, f(z) - f(z_1)\Delta z\right]$$

$$= \frac{1}{\Delta z}\int_{z_1}^{z_1+\Delta z} dz\, \left(f(z) - f(z_1)\right)$$

$\Rightarrow$ requires path independence so can choose straight line from z_1 to $z_1+\Delta z$.

$$\left|\frac{F(z_1+\Delta z)-F(z_1)}{\Delta z}-f(z_1)\right| \leq \frac{1}{\Delta z}\int_{z_1}^{z_1+\Delta z}|dz|\,|f(z)-f(z_1)|$$

$$\leq \epsilon\frac{1}{|\Delta z|}\int_{z_1}^{z_1+\Delta z}|dz| = \epsilon$$

$$\lim_{\Delta z\to 0}\frac{F(z_1+\Delta z)-F(z_1)}{\Delta z} \equiv F'(z_1) = f(z_1)$$

$\Rightarrow$ also means can carry out integrals without explicitly doing the integration if know indefinite integral of $f(z)$

$\Rightarrow$ e.g. as with $\int dz\,\frac{1}{z} \Rightarrow \ln(z)$

$\Rightarrow$ be careful if need a cut.

$\Rightarrow$ integrals of analytic functions can be carried out like with real functions.

$\Rightarrow$ the integral is independent of path so don't need to specify path

$\Rightarrow$ just endpoints

Morera's Theorem

If a function $f(z)$ is continuous in a simply connected domain R and

$$\oint_C dz\, f(z) = 0 \quad \text{for every closed}$$

contour C in R, then f(z) is analytic

$$F(z_1) = \int_{z_0}^{z_1} dz\, f(z)$$

Have shown

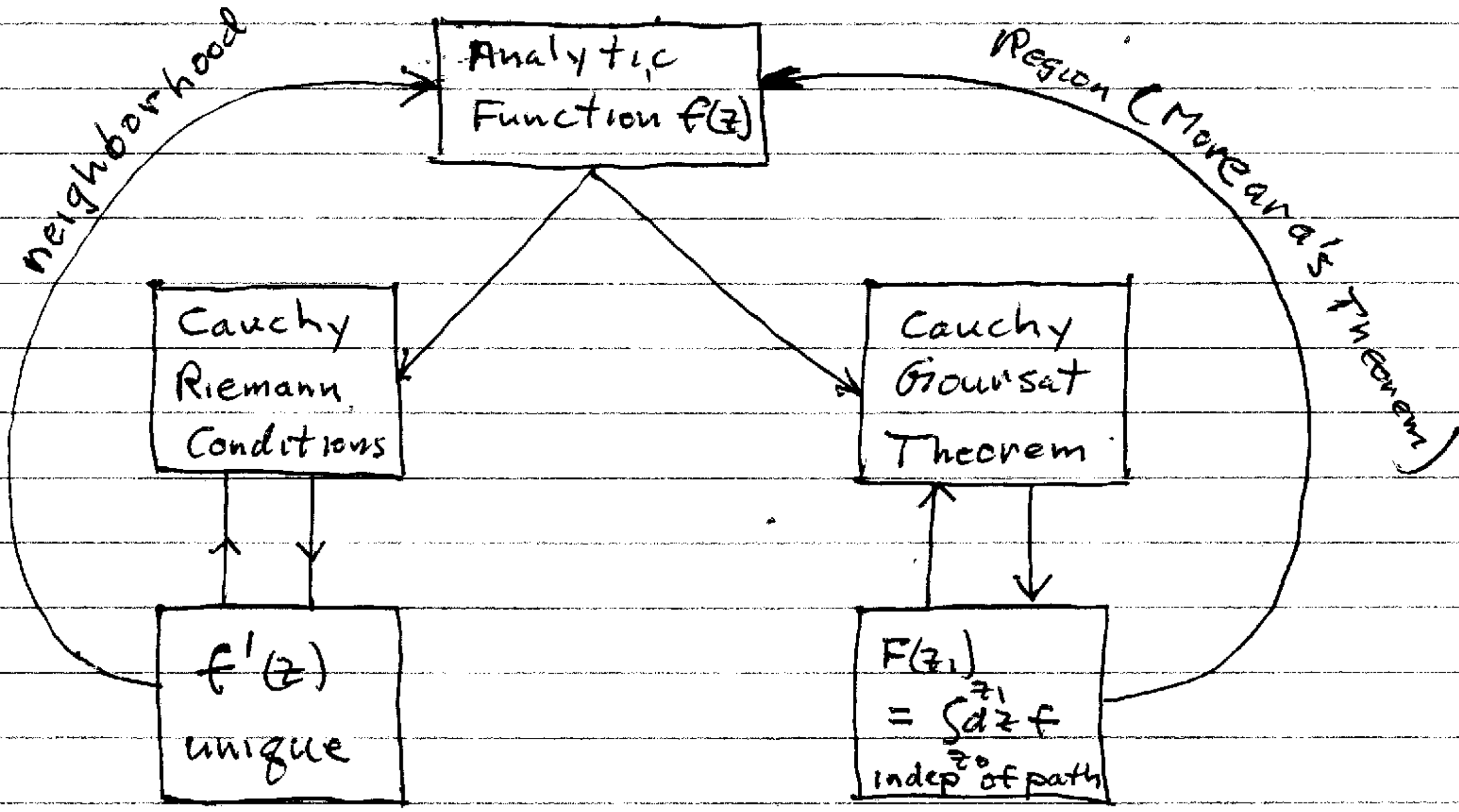
$$F'(z_1) = f(z_1)$$

True for any $z_1 \Longrightarrow F(z_1)$ is analytic

If $F(z_1)$ is analytic, $F'(z_1)$ is analytic

$$\Longrightarrow f(z_1) \text{ is analytic}$$

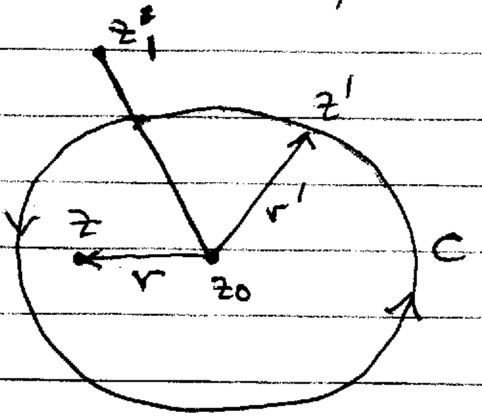
Schematic



Analytic Function f(z)
Cauchy Riemann Conditions
Cauchy Goursat Theorem
f'(z) unique
F(z1) = ∫dz f indep of path
neighborhood
Region (Morera's Theorem)

Taylor Series

Want to expand $f(z)$ around z_0 where z_1 is the nearest point where $f(z)$ is not analytic.



C is a circle of radius r' from z_0.

$$f(z) = \frac{1}{2\pi i}\oint_C dz' \frac{f(z')}{z'-z} = \frac{1}{2\pi i}\oint_C dz' \frac{f(z')}{z'-z_0+z_0-z}$$

$$= \frac{1}{2\pi i}\oint_C dz' \frac{f(z')}{(z'-z_0)\left(1+\frac{z_0-z}{z'-z_0}\right)}$$

Let $t = \frac{z-z_0}{z'-z_0}$, $\quad |t| = \frac{r}{r'} < 1$

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots t^{N-1} + \frac{t^N}{1-t}$$

$\Rightarrow$ note that this is exact

$$f(z) = \frac{1}{2\pi i}\oint dz' \frac{f(z')}{z'-z_0}\left[\sum_{n=0}^{N-1}\left(\frac{z-z_0}{z'-z_0}\right)^n + \frac{(\;)^N}{1-(\;)}\right]$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{N-1}(z-z_0)^n \oint_C dz' \frac{f(z')}{(z'-z_0)^{n+1}} + R_N$$

$$f(z) = \sum_{n=0}^{N-1} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0) + R_N$$

$$R_N = \frac{1}{2\pi i} \oint_C dz' \frac{f(z')}{(z'-z_0)^{N+1}} \frac{(z-z_0)^N}{z'-z} (z'-z_0)$$

$$|R_N| \lesssim \frac{1}{2\pi} \oint_C |dz'| |f(z')| \frac{r^N}{r'^N (r'-r)} \lesssim |f(z)|_{max}^{C} \left(\frac{r}{r'}\right)^N \frac{r'}{r'-r}$$

$|f(z)|_{max}^{C}$ = maximum value of f on C

Since $\frac{r}{r'} < 1$, $\lim_{N \to \infty} R_N = 0$

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0)}$$ Taylor series

$\Rightarrow$ converges only where

$$|z - z_0| < |z_1 - z_0|$$

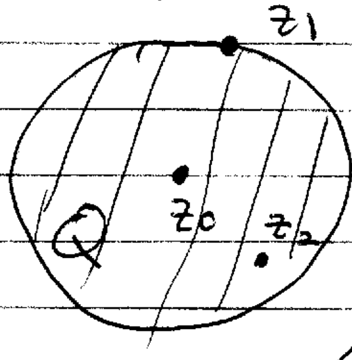
since $|f(z)|_{max}^{C} \longrightarrow \infty$ if C includes z_1

example: Expand $\frac{1}{z-3}$ around $z_0 = 2.5$

$\Rightarrow$ radius of convergence $|z - z_0| < 0.5$

Analytic Continuation

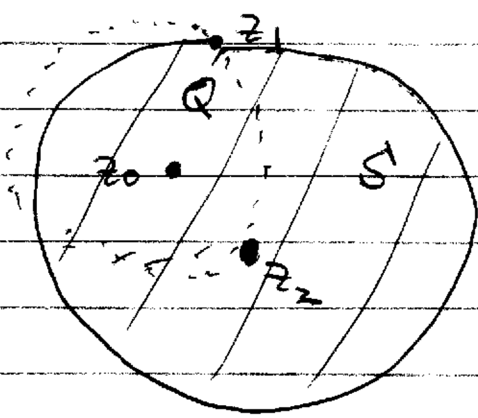
Consider an analytic function $f(z)$ with a singularity at z_1. Expand $f(z)$ in a Taylor series around z_0.



The expansion is valid in domain Q.

Consider a point z_2 within Q. Calculate $f(z_2)$ and all of its derivatives at z_2.

$\Rightarrow$ Can expand f in a Taylor series around z_2.



Region of convergence of Taylor series around z_2 is now S.

$\Rightarrow$ can now calculate f over a larger domain

$\Rightarrow$ continue until can evaluate f everywhere except where non analytic

$\Rightarrow$ the process of extending a region where f can be evaluated is called analytic continuation.

$\Rightarrow$ ~~note that the series around z_2 is not the same as around z_0.~~