

This assignment is due on Tuesday Dec. 8.

Note that the Arfken problems correspond to the 7th Edition with the corresponding 6th Edition problems in the parentheses.

1. Arfken Chapter 8.2.2 (10.1.2)
2. Show that the first derivatives of the Legendre polynomials satisfy a self-adjoint differential equation with eigenvalue $\lambda = n(n+1) - 2$. Show that these functions satisfy an orthogonality relation

$$\int_{-1}^1 dx (1-x^2) P'_m(x) P'_n(x) = 0,$$

for $n \neq m$.

3. Expand the square-wave function $f(x)$ [where $f = -1$ for $x \in (-1, 0)$ and $f = 1$ for $x \in (0, 1)$] in a series of Legendre polynomials over the interval $(-1, 1)$.
4. The temperature in a one-dimensional medium satisfies a diffusion equation

$$\frac{\partial T}{\partial t} - \kappa \frac{\partial^2}{\partial x^2} T = 0$$

with diffusion coefficient κ over the interval (a, b) , where $T(a, t) = T(b, t) = T_0$. At $t = 0$, $T(x, 0) = 0$ for $x \in (a, b)$. Solve for $T(x, t)$ by expanding in a series of $\sin()$ and/or $\cos()$ functions. At late time what is the approximate (non-trivial) time dependence?

Hint: Choose your basis functions to match your B.C.'s.

5. Consider a solid sphere of radius " a " in a world with *four spatial dimensions*. At $t = 0$ the sphere, with initial temperature of T_0 is immersed in a heat bath with $T = 0$. The temperature inside the sphere satisfies a diffusion equation.

$$\frac{\partial T}{\partial t} - \kappa \nabla^2 T = 0$$

where

$$\nabla^2 = \frac{1}{r^3} \frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r}.$$

- (a) Estimate how long it will take for the temperature of the center of the sphere to change significantly.
- (b) Construct a set of basis functions $\Phi_n(r)$ such that $\Phi_n(a) = 0$ which are appropriate for solving for $T(r, t)$. Write the basis functions as a linear combination of the two solutions of Bessel's equation $Y_\nu(kr)$ and $J_\nu(kr)$. Write down expressions for the eigenvalues of your basis functions and normalize $\Phi_n(r)$ so that

$$\int_0^a dr \, w \, \Phi_n^2(r) = 1.$$

You do not have to prove that the eigenfunctions are orthogonal, but state why you are confident that they are. Sketch the lowest three eigenfunctions.

- (c) Write the space/time dependence of T as

$$T(r, t) = \sum_{n=1}^{\infty} c_n(t) \Phi_n(r)$$

and solve for $c_n(t)$. At late time find an approximate expression for $T(r, t)$. How does T decay at late time?