Physics 604 Homework #10 Fall ‘20

Dr. Drake

This assignment due on Tuesday December 1.

1. The inhomogeneous Airy equation is given by

$$\frac{d^2y}{dx^2} - xy = 1 \tag{1}$$

This equation arises in mode conversion problems in plasma where an electromagnetic wave drives an electrostatic wave at its cutoff. The "1" on the right hand side represents the electromagnetic driver and the left side the electrostatic response. The time dependence $e^{-i\omega t}$ has been factored out of the equation.

   (a) For $|x| >> 1$ write down approximate expressions for the particular and homogeneous solutions to the equation. The complete solution must correspond to a linear combination of these solutions for $|x|$ large. Which of these large $|x|$ solutions are physical (think about causality with a wave source around $x = 0$)?

   (b) To obtain the full solution, write

$$y(x) = \int_C dk Y(k) e^{ikx}, \tag{2}$$

   where the contour C can be chosen so that one of the endpoints of the integral yields the "1" on the right hand side of Eq. (2). The other endpoint of the contour must be chosen to yield the physical boundary conditions as $x \to +\infty$ and $x \to -\infty$.
   Hint: Proceed by inserting the integral expression in (2) into the differential equation in (1).

2. Find the solution of

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - \frac{\eta(\eta+1)}{x^2}y = \delta(x-a)$$

with $a > 0$ on the interval $0 < x < \infty$ with $y(0) = y(\infty) = 0$ and $\eta > 0$.

3. A stretched string of length L satisfies the equation

$$y_{tt} - c^2 y_{xx} = \frac{F(x)}{\rho}\cos(\omega_0 t)$$

with $F(x)$ the force and ρ the density per unit length of the string. Assume that there is sufficient dissipation in the string so that all transients have died away, leaving the steady state solution. Write the solution in terms of a Green's function and solve for the Green's function.
Hint: Eliminate the time dependence first.

4. Consider the Green's function of the operator L,

$$LG(x, x') = \delta(x - x')$$

where

$$L = \frac{\partial}{\partial x} p(x) \frac{\partial}{\partial x} + q(x)$$

and $p(x)$ and $q(x)$ are real. Given two solutions of the homogeneous equation $\phi_1(x)$ and $\phi_2(x)$, construct a solution for G over the interval (a, b) with $G(a, x'), G(b, x') = 0$ and show that

$$G(x, x') = G(x', x).$$

This is the reciprocity relation for self-adjoint operators. The response at x to a disturbance at x' is the same as the response of x' due to a disturbance at x.