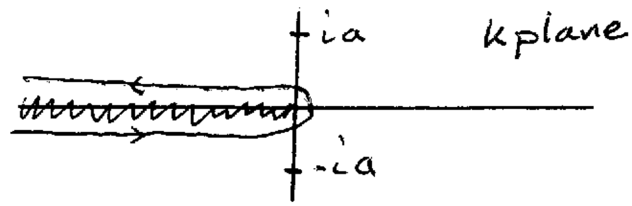
Physics 604 Final Exam Fall '20

Dr. Drake

1. (30 pts) Evaluate the following integral

$$I = \int_c dk \frac{\ln(k)}{(k^2 + a^2)}$$

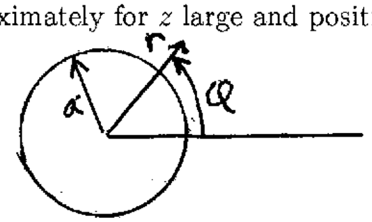
with C given by



2. (40 points) An integral representation of 2nd Hankel function, $H_\nu^{(2)}(z)$, is

$$H_\nu^{(2)}(z) = \frac{1}{\pi i} \int_{\infty e^{-i\pi}}^{0} e^{i(z/2)(t-1/t)} \frac{dt}{t^{1+\nu}},$$

where a cut extends from zero to negative infinity in the t plane and the integral lies below the cut, starting at negative infinity below the cut and ending at zero. Evaluate the integral approximately for z large and positive.



3. (60 pts) Consider a two-dimensional cylinder of radius a with radius r and angle ϕ as given in the figure. At $r = a$ the electric potential V is maintained at V_0 for $0 < \phi < \pi$ and at $-V_0$ for $\pi < \phi < 2\pi$. In the region $r > a$ the potential $V(r, \phi)$ satisfies Laplace's equation

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} V + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} V = 0.$$

The goal of this problem is to solve for the potential in the region $r > a$.

(a) Write the solution for V in terms of a separable set of eigenfunctions $\Phi_m(\phi)$ and $R_m(r)$ as follows:

$$V(r, \phi) = \sum_m c_m R_m(r) \Phi_m(\phi)$$

Write equations for $\Phi_m(\phi)$ and $R_m(r)$ and solve these equations using boundary conditions appropriate for the solution V. Normalize the basis functions Φ_m so they have unit norm.

Hint: Use the symmetry in ϕ to simplify the solution for Φ_m.

(b) Sketch the lowest three eigenfunctions Φ_m over the interval $(0, 2\pi)$ and plot $V(\phi)$ over the same interval. Which of these solutions match the symmetry of $V(\phi)$?

(c) Solve for c_m by matching the potential at $r = a$.

(d) In the limit $r \gg a$ what is the lowest order non-trivial behavior of the solution? How does the solution fall off with radius r?

4. (70 points) A liquid is placed in a hollow sphere of radius b. The equation satisfied by the liquid is

$$\frac{\partial T}{\partial t} - \kappa \nabla^2 T = 0 \tag{1}$$

where

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}$$

and T remains zero at the boundary for all time. At $t = 0$ the temperature of the liquid is given by $r_0 T_0 \delta(r - r_0)$. The goal of this problem is to determine how the temperature of the liquid decreases with time. Write the temperature in terms of basis functions,

$$T(r, t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(r).$$

(a) Estimate the time scale over which the temperature of the fluid decays in time based on a simple dimensional argument.

(b) Write a differential equation for the basis functions $\phi_n(r)$ suitable for the spherical geometry of the cavity. Show that by making the substitution $\phi_n(r) = g_n(r)/r^{1/2}$ that the equation satified by g_n is a Bessel equation. What is ν? What is the behavior of the two solutions of the Bessel equation $J_\nu(kr)$ and $Y_\nu(kr)$ and the corresponding possible behavior of ϕ_n near $r = 0$. Do either or both of the solutions satisfy the Stürm-Liouville boundary conditions at $r = 0$? Give expressions for the eigenvalues of your basis functions (in terms of the known properties of $J_\nu(kr)$ and $Y_\nu(kr)$) and define the normalization so that the $\phi_n(r)$ have unity norm. State why the basis functions are orthogonal (you don't have to prove orthogonality). Sketch the lowest three eigenfunctions.
Hint: Bessel's equation is $r^2 y'' + ry' + (k^2r^2 - \nu^2)y = 0$ and $\int_0^1 dr r[J_\nu(x_{\nu n}r)]^2 = [J_{\nu+1}(x_{\nu n})]^2/2$ with $x_{\nu n}$ the nth zero of the Bessel function of order ν.

(c) Derive an equation for $c_n(t)$. What is the characteristic damping rate of the eigenfunctions? Solve for $c_n(t)$ and then write a solution for the complete space/time dependence of T.

(d) What is the lowest order non-trivial form of the solution at late time? What is the decay time of the temperature based on this behavior?

Formula Sheet - Complex Variables

Cauchy Riemann Conditions:

$$f(z) = u(x,y) + i\,v(x,y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Integral:

$$\int_{-\infty}^{\infty} dz\, e^{-\frac{z^2}{a^2}} = a\sqrt{\pi}$$

Schwarz Inequality: $\left|\int_C dz\, f(z)\right| \leq \int_C |dz|\,|f(z)|$

Cauchy's Integral Formula:

$$f(z_0) = \frac{1}{2\pi i}\oint_C dz\, \frac{f(z)}{(z-z_0)}$$

Derivative Formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint dz\, \frac{f(z)}{(z-z_0)^{n+1}}$$

Taylor Series: $f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_0)\,\frac{(z-z_0)^n}{n!}$

Laurent Series: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

$$a_n = \frac{1}{2\pi i}\oint_C dz'\, \frac{f(z')}{(z'-z_0)^{n+1}}$$

Residue Theorem: $\oint_C dz\, f(z) = 2\pi i \sum_j a_{-1}(z_j)$

Residues: <u>mth order pole at z_j</u>

$$a_{-1}(z_j) = \frac{1}{(m-1)!}\left[(z-z_j)^m f(z)\right]^{(m-1)}\Big|_{z=z_j}$$

<u>1st order pole</u> $f = \frac{P(z)}{Q(z)}$, $a_{-1}(z_j) = \frac{P(z_j)}{Q'(z_j)}$

Formula Sheet Exam 2

WKB Theory

$$\frac{d^2}{dx^2}y + k^2(x)\,y = 0$$

$$y \sim \frac{e^{\pm i \int^x dx'\, k(x')}}{k(x)^{1/2}}$$

Fourier Transform

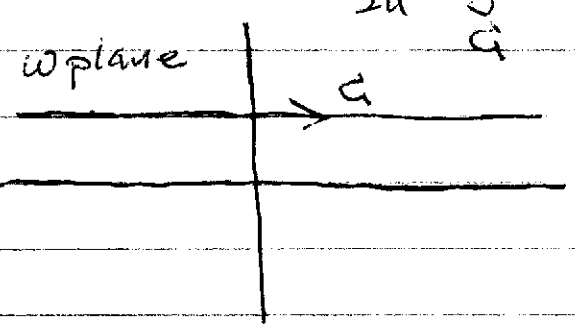
$$F(k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, F(k)\, e^{ikx}$$

Laplace Transform

$$F(\omega) = \int_0^{\infty} dt\, e^{i\omega t} f(t)$$

$$f(t) = \frac{1}{2\pi}\int_{C} d\omega\, e^{-i\omega t} F(\omega)$$



C above poles of $F(\omega)$

Dirac delta function

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}$$

Cauchy Derivative Formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint dz\, \frac{f(z)}{(z-z_0)^{n+1}}$$

Taylor Series

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_0)\,\frac{(z-z_0)^n}{n!}$$

Residue Theorem

$$\oint_C dz\, f(z) = 2\pi i \sum_j a_{-1}(z_j)$$

Residues: mth order pole

$$a_{-1}(z_j) = \frac{1}{(m-1)!}\left[(z-z_j)^m f(z)\right]^{(m-1)}\Big|_{z=z_j}$$