Steady state diffusion

\[ \nabla^2 T = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T = 0 \]

\[ T = \sum \frac{C_n}{\alpha_n} \mathbf{X}_n(x) \mathbf{Y}_n(y) \]

=> insert into \( \nabla^2 T = 0 \)

=> divide by \( \mathbf{X}_n \mathbf{Y}_n \)

\[ \Rightarrow \frac{1}{\mathbf{X}_n} \frac{\partial^2}{\partial x^2} \mathbf{X}_n + \frac{1}{\mathbf{Y}_n} \frac{\partial^2}{\partial y^2} \mathbf{Y}_n = 0 \]

\[ -k_n^2 \mathbf{X}_n + \frac{\partial^2}{\partial x^2} \mathbf{X}_n = 0 \]

\[ \mathbf{X}_n \sim \sin(k_n x), \cos(k_n x) \]

Want \( \mathbf{X}_n = 0 \) at \( x = a \)

\[ \mathbf{X}_n = \frac{\sin(k_n a)}{k_n a} \Rightarrow k_n = \frac{n\pi}{a} \]

\[ T = \sum \frac{C_n}{\alpha_n} \sin(k_n x) e^{-k_n y} \]

Match at \( y = 0 \)

\[ T(x, y = 0) = T_0(x) = \sum \frac{C_n}{\alpha_n} \sin(k_n x) \]

\[ T(x, y = 0) = T_0(x) = \sum \frac{C_n}{\alpha_n} \sin(k_n x) \]
To find $c_m$ multiply by $\sin(kmx)$ and integrate from 0 to $a$.

\[ \int_0^a T(x) \sin(kmx) \, dx = C_m \int_0^a \sin^2(kmx) \, dx = \frac{1}{2} C_m a \]

\[ C_m = \frac{2}{a} \int_0^a x \sin(kmx) \, dx = \frac{2}{a} \left[ \frac{-x \cos(kmx)}{km} \right]_0^a - \int_0^a \left( \frac{\cos(kmx)}{km^2} \right) \, dx \]

\[ = \frac{2}{a} \left[ -a \frac{\cos(kma)}{km} + \frac{\sin(kma)}{km^2} \right] \bigg|_0^a 
\]

\[ = \frac{2}{km} (-1)^{m+1} \]

\[ T(x,y) = \frac{2a}{T} \sum_{n=1}^{m+1} (-1)^{m+1} n \sin \left( \frac{n\pi x}{a} \right) e^{-\frac{n\pi y}{a}} \]

with $a = 10$

\[ 2.14 \quad \text{Insulated container} \quad \Rightarrow \frac{\partial T}{\partial x} \bigg|_{x=a} = 0 \]

As above,

\[ \frac{\partial^2}{\partial x^2} \sum n + \kappa_n \frac{\partial^2}{\partial x^2} \sum n = 0 \]

\[ \sum n \sim \sin(knx) \cos(knx) \]

Require \( \frac{\partial}{\partial x} \sum n = 0 \) at $x=a$
\[ X_n = \cos(k_n x) \]

\[ \frac{d}{dx} X_n = -k_n \sin(k_n x) = 0 \text{ at } x = 0 \]
\[ = -k_n \sin(k_n a) = 0 \text{ at } x = a \]
\[ \Rightarrow k_n = \frac{n \pi}{a} \text{ with } n = 0, 1, \ldots \]

\[ \Rightarrow Y_n \text{ as before } \Rightarrow Y_n = e^{-k_n y} \]
\[ T = \sum_{n=0}^{\infty} C_n \cos(k_n x) e^{-k_n y} \text{ with } k_n = \frac{n \pi}{a} \]

**Case I**

\[ T_0(x) = x - 5 \text{ with } a = 10 \]
\[ = x - \frac{a}{2} \]
\[ = x - \frac{10}{2} \]
\[ = x - 5 \]

\[ T_0(x) = \sum_{n=0}^{\infty} C_n \cos k_n x \]

\[ \Rightarrow \text{ multiply by } \cos(k_m x) \text{ and integrate} \]
\[ \int_0^a (x - \frac{a}{2}) \cos k_m x \, dx = C_m \int_0^a \cos^2 k_m x \, dx \]
\[ = C_0 \text{ for } m = 0 \]
\[ = C_m \frac{1}{2} \text{ for } m \neq 0 \]

\[ C_0 = \frac{1}{a} \int_0^a (x - \frac{a}{2}) \, dx = 0 \]

\[ C_m = \frac{2}{a} \int_0^a (x - \frac{a}{2}) \cos(k_m x) \, dx \]
\[ = \frac{2}{a} \left[ \left( \frac{x - \frac{a}{2}}{k_m} \sin k_m x \right) \bigg|_0^a - \int_0^a \frac{\sin k_m x}{k_m} \, dx \right] \]
\[ C_m = -\frac{2}{a} \cdot \frac{\cos kmx}{km^2} \left| \begin{array}{c} a \\ \frac{a}{km} \end{array} \right| = \frac{2}{a km^2} \left( \cos(ma) - 1 \right) \]

\[ = \frac{2 \cdot 2}{a km^2} \quad \text{n odd} \]

\[ = 0 \quad \text{neven} \]

\[ T(x,y) = -\frac{4a}{\pi^2} \sum \frac{1}{n^2} \cos \left( \frac{n\pi x}{a} \right) e^{-\frac{n\pi y}{a}} \]

with \( a = 10 \)

For large \( y \) \( \Rightarrow \) dominated by \( n = 1 \)

\[ T = \frac{4a}{\pi^2} \cos \left( \frac{\pi x}{a} \right) e^{-\frac{\pi y}{a}} \]

**Case II**

\[ T_0(x) = X \quad \text{As before} \]

\[ \int_0^a x \cos kmx = a \cos \quad \text{for } m = 0 \]

\[ = \frac{a}{2} C_m \quad \text{for } m \neq 0 \]

\[ C_0 = \frac{1}{a} \int_0^a x = \frac{1}{2} \cdot \frac{a^2}{2} = \frac{a}{2} \]

\[ C_m = \text{same as before} \]

\[ T = \frac{a}{2} - \frac{4a}{\pi^2} \sum \frac{1}{n^2} \cos \left( \frac{n\pi x}{a} \right) e^{-\frac{n\pi y}{a}} \]

with \( a = 10 \). Large \( y \)

\[ T = \frac{a}{2} \]
Consider the Schrödinger equation in 2-D rectangular coordinates with \( V = 0 \):

\[
-\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi = i\hbar \frac{\partial}{\partial t} \psi
\]

where \( x \in (0, l) \), \( y \in (0, l) \) where \( \psi = 0 \) on all edges.

Write

\[
\psi = \sum_n \left( X_n(x) Y_n(y) T_{nm}(t) \right)
\]

Insert into the equation, carry out derivatives and divide by \( \frac{\hbar^2}{2m} X_n \) \( \frac{\partial^2}{\partial y^2} Y_n \)

\[
-\frac{\hbar^2}{2m} \left( \frac{1}{X_n} \frac{\partial^2}{\partial x^2} X_n + \frac{1}{Y_n} \frac{\partial^2}{\partial y^2} Y_n \right) = i\hbar \frac{1}{T_n} \frac{\partial}{\partial t} T_{nm}
\]

Each must separate be a constant

\[
\frac{\partial^2}{\partial x^2} X_n + k_{xn}^2 X_n = 0 \quad \text{with} \quad k_{xn} = \frac{n \pi}{L}
\]

\[
X_n = \sin(k_{xn} x)
\]

\[
\int_{0}^{l} x^n x^m dx = \frac{l^{n+m+1}}{n+m+1}
\]
\[ \frac{\partial^2}{\partial y^2} Y_m + k_{ym}^2 Y_m = 0 \]

\[ \Rightarrow Y_m(y) = \sin k_{ym} y \text{ with } k_{ym} = \frac{m\pi}{L} \]

Time dependence

\[ + \frac{\hbar^2}{2M} k_{mn}^2 = i\hbar \frac{\hbar}{T_{nm}} \frac{2}{\hbar} T_{mn} \]

with \( k_{mn}^2 = k_{xn}^2 + k_{ym}^2 = \left( \frac{n^2 + m^2}{L^2} \right) \pi^2 \).

Let \( E_{mn} = \frac{\hbar^2 k_{mn}^2}{2M} \)

\[ \frac{\partial}{\partial t} T_{mn} = -i \frac{E_{mn}}{\hbar} T_{mn} \Rightarrow T_{mn} = e^{-i\omega t} \]

\[ \omega = \frac{E_{mn}}{\hbar} \quad T_{mn} = e^{-i\frac{E_{mn} t}{\hbar}} \]

\[ \mathcal{H} = \sum c_{mn} \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{m\pi y}{L} \right) e^{-i\frac{E_{mn} t}{\hbar}} \]

Find \( c_{mn} \) from the initial condition. The particle has equal probability of being anywhere in the box.

\[ \psi(x, y, t=0) = A \]

\[ P = \psi^* \psi = A^2 \int_0^L \int_0^L A^2 = 1 \Rightarrow A^2 L^2 = 1 \]

\[ A = \frac{1}{L} \]
Initial condition

\[ \frac{1}{x} = \sum c_{mn} \sin \left( \frac{n\pi x}{l} \right) \sin \left( \frac{m\pi y}{l} \right) \]

By symmetry only, odd \( n, m \) will have \( c_{mn} \) non-zero. Multiply by

\[ \sin \left( \frac{n'\pi x}{l} \right) \sin \left( \frac{m'\pi y}{l} \right) \]

and integrate over \( x \) and \( y \).

\[ \frac{1}{l} \int_0^l \int_0^l \sin \left( \frac{n'\pi x}{l} \right) \sin \left( \frac{m'\pi y}{l} \right) \sin \left( \frac{n\pi x}{l} \right) \sin \left( \frac{m\pi y}{l} \right) \, dx \, dy \]

\[ \times \frac{\sin \frac{n\pi x}{l} \sin \frac{n\pi x}{l} \sin \frac{m\pi y}{l} \sin \frac{m\pi y}{l}}{l} \]

\[ = \sum_{m'n'} c_{mn} c_{m'n'} \sin \frac{1}{2} \frac{1}{2} l^2 \]

\[ = \frac{l^2}{4} c_{m'n'} \]

\[ c_{m'n'} = \frac{4}{l^3} \cos \frac{\pi n' x}{l} \left| \begin{array}{cc} \cos \frac{\pi m' y}{l} & \frac{l}{m'} \\ \frac{n' \pi}{l} & \frac{m' \pi}{l} \end{array} \right| \]

\[ = \frac{1}{l^2} \frac{16}{\pi^2} \frac{1}{n'm'} \quad \text{for } n' \text{ and } m' \text{ odd} \]

\[ = 0 \quad \text{for } n' \text{ or } m' \text{ even} \]
\[ \psi = \frac{16 \pi^2 \ell}{\pi^2 l^2} \sum_{m,n \text{ odd}} \frac{1}{m} \sin\left(\frac{\pi x}{\ell}\right) \sin\left(\frac{m \pi y}{\ell}\right) e^{-i \frac{E_{mn} \ell}{\eta}} \]

\[ E_{mn} = \frac{n^2}{2M} K_{mn}^2 \]

Consider the oscillation of a circular membrane of radius \( R \) with \( \frac{\partial^2 \psi}{\partial \theta^2} = 0 \). Take \( u(r=R_1 t) = 0 \) with \( u(r_1 t=0) = U_0 \) with \( \dot{u}(r_1 t=0) = 0 \).

Let

\[ u(r,t) = \sum_{n} C_n R_n(r) T_n(t) \]

wave equation in a cylinder with \( \frac{\partial^2 \psi}{\partial \theta^2} = 0 \)

\[ \frac{\partial^2 \psi}{\partial t^2} - V^2 \frac{1}{\ell^2} \frac{1}{n \Delta r} \frac{1}{n \Delta \theta} \psi = 0 \]

Insert basis functions and then divide by \( R_n T_n \)

\[ \frac{1}{T_n} \sum_{n} R_n - \sum_{n} \frac{\partial^2}{\partial t^2} \left( \frac{1}{R_n} \frac{1}{n \Delta r} \frac{1}{n \Delta \theta} \right) R_n = 0 \]

\[ -k_n^2 \Rightarrow \text{oscillatory solution} \]

\[ \text{can have } R_n = 0 \text{ at } r = R \]
\[ r^2 \frac{\partial^2}{\partial r^2} R_n + r \frac{\partial}{\partial r} R_n + k_n^2 r^2 R_n = 0 \]

\[ \Rightarrow \text{Bessel function with } p = 0 \]

\[ \Rightarrow R_n = J_0(k_n r) \]

with \( k_n r = \frac{x_{on}}{R} \)

**Orthogonality**

\[ \int_0^R r \, J_0(k_n r) J_0(k_m r) \, dr = \frac{R^2}{2} \, J_1^2(x_{on}) \, \delta_{nm} \]

**Time dependence**

\[ T_n + k_n^2 V^2 T_n = 0 \quad \Rightarrow \quad T_n \sim e^{-i \omega_n t} \]

\[ (-\omega_n^2 + k_n^2 V^2) e^{-i \omega_n t} = 0 \]

\[ \omega_n = \pm k_n V \quad T_n \sim \sin(\omega_n t) \text{ or } \cos(\omega_n t) \]

Use sines and cosines to match initial conditions
\[ U(r,t) = \sum_n b_n \left( A_n \sin(\omega_n t) + B_n \cos(\omega_n t) \right) J_0(k_n r) \]

with \( \omega_n = k_n V \)

\[ U(r, t=0) = \sum_n b_n \left( A_n \omega_n \right) J_0(k_n r) = 0 \]

\[ \Rightarrow A_n = 0 \]

\[ U(r,t) = \sum_n b_n \cos(\omega_n t) J_0(k_n r) \]

At \( t=0 \)

\[ U_0 = U(r, t=0) = \sum_n b_n J_0(k_n r) \]

Multiply by \( r J_0(k_n r) \) and integrate \( 0, R \)

\[ U_0 \left( \int_0^R r J_0(k_n r) \, dr \right) = b_n \frac{R^2}{2} J_1(k_n R) \]

\[ \Rightarrow k_n r = x \]

\[ U_0 \left( \int_0^{x_{om}} x J_1(x) \, dx \right) \]

\[ \frac{U_0}{k_m^2} \left[ \frac{d}{dx} (x J_1(x)) \right] \left|_{x_{om}} \right. \]

from Eqn 15.1 in Boas
\[ \frac{R^2}{x_{om}} \frac{U_0}{x_{om}} J_1(x_{om}) = B_m \frac{\hbar^2}{2m} J_1^2(x_{om}) \]

\[ B_m = 2 \frac{U_0}{x_{om} J_1(x_{om})} \]

\[ U(n,t) = \sum_{n=1,2,...} \frac{2U_0}{x_{on}} \cos(\omega_n t) \frac{J_0(k_n r)}{J_1(k_n x_{on})} \]

with \( k_n = \frac{x_{on}}{r} \) and \( \omega_n = k_n V \)

(3)

Consider the Schrödinger eqn with \( \nabla = 0 \)

\[ i\hbar \frac{\psi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \psi = 0 \quad \text{with} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} \]

Consider \( P = \psi \bar{\psi} = \text{local probability of finding a particle} \)

\[ P = \psi \bar{\psi} + \bar{\psi} \psi \]

Use the SE to evaluate \( \psi \) and the complex conjugate of the SE to find \( \bar{\psi} \)

\[ \bar{\psi} = \psi^* \left( - \frac{\hbar^2}{2m} i\hbar \frac{\partial^2}{\partial x^2} \right) + \psi \left( - \frac{\hbar^2}{2m} \left( -i \hbar \frac{\partial}{\partial x} \right) \right) \]
\[ p = \frac{i \hbar}{2m} \left( 2y^2 \frac{\partial^2}{\partial y^2} - 2 \frac{\partial^2}{\partial x^2} \right) \]

Integrate over the box

\[ \frac{1}{L} \int_0^L dx \ y^2 \psi^2 = \frac{i \hbar}{2m} \int_0^L dx \left[ y^2 \frac{\partial^2}{\partial y^2} - y \frac{\partial^2}{\partial x^2} \right] \psi^2 \]

\[ \Rightarrow \text{integrate by parts, since } \psi^2 = 0 \text{ at } x = 0 \]

\[ \Rightarrow \frac{i \hbar}{2m} \left[ \frac{y^4}{4} \bigg|_0^L - \frac{y^2}{2} \psi^2 \bigg|_0^L \right] \psi^2 \]

\[ = \frac{i \hbar}{2m} \left[ \frac{y^4}{4} \bigg|_0^L - \frac{y^2}{2} \psi^2 \bigg|_0^L \right] \psi^2 \]

\[ = 0 \]

\[ \Rightarrow \int_0^L dx \ y^2 \psi^2 = 0 \]

\[ \Rightarrow \text{total probability of finding a particle in a box does not change in time} \]