Physics 373 Sample Final Exam Fall '18

Dr. Drake

1. (a) Carry out the integral
$$\int_{-\infty}^{\infty} dz \frac{sin(z)}{z}$$
   (b) Evaluate $(-i)^{1/3}$ in the cut z plane with a branch cut along the negative real axis and $Arg(z) = 0$ along the positive real axis. Evaluate $(-1)^{1/3}$ just below the cut.
   (c) Calculate the Laurent series of the function
$$f(z) = \frac{1}{z(z-i)^2}$$
   around $z = i$.
   (d) Legendre's differential equation is given by
$$\frac{d}{dx}(1-x^2)\frac{dy}{dx} + l(l+1)y = 0.$$
   Write the equation in the terms of the variable $t = x - 1$ and write the form of the equation very close to the singularity at $t = 0$. What is the lowerst order behavior of the solutions near $t = 0$. Show that the two solutions are linearly independent.

2. Consider the differential equation
$$\frac{d^2y}{dt^2} + y = sin(2t),$$
   where the source on the right side of the equation is turned on at $t = 0$ and the initial conditions $y(0) = 0$ and $dy(0)/dt = 0$.
   (a) Take the Laplace transform of the equation to obtain an equation for $L(y) = Y(p)$.
   (b) Write down the inverse transform for $y(t)$ and evaluate the integral in the complex p plane to obtain $y(t)$.

3. The Legendre Polynomials $P_l(x)$ form a complete set over the interval $x \in (-1, 1)$.

(a) Use fact that $P_0(1) = 1$ and $P_1(1) = 1$ and the recursion formula

$$lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x)$$

to evaluate $P_l(1)$ for all l. What is $P_l(-1)$ for l even? For l odd?

(b) The function $f(x)$ is -1 for $x \in (-1, 0)$ and 1 for $x \in (0, 1)$. Express $f(x)$ as an infinite sum of Legendre polynomials.

Hint: Use symmetry arguments to simplify the problem and the recursion formula

$$(2l+1)P_l(x) = P'_{l+1} - P'_{l-1}$$

to carry out some of the integration. You can leave your answer in terms of $P_l(0)$, which is a known function.

4. Consider the heat conduction in an infinitely long cylindrical solid object of radius R in which the boundary temperature is maintained at zero. The equation for the temperture $T(r,t)$ is given by

$$\frac{\partial T}{\partial t} - D\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}T = 0,$$

where at $t = 0$ the temperature is initially uniform $T(r, t = 0) = T_0$. Express the space-time dependence of T in terms of a set of basis functions as

$$T(r,t) = \sum_n C_n R_n(r) T_n(t).$$

(a) Calculate the equation satisifed by $R_n(r)$, define the boundary conditions satisfied by R_n and check that they are of the correct Sturm-Liouville form. Write the eigenvalues and the orthogonality condition.

(b) Write down the equation satisfied by $T_n(t)$ and its solution.

(c) Using the initial condition for T at $t = 0$, calculate the coefficients C_n and write the full solution for $T(r,t)$.

Hint: the recursion formula

$$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$$

will be helpful.

(d) At late time what is the approximate solution for $T(r,t)$? What is the characteristic decay time of the temperature in the cylinder at this late time?