Basis functions from Bessel's eqn

Bessels eqn is given by

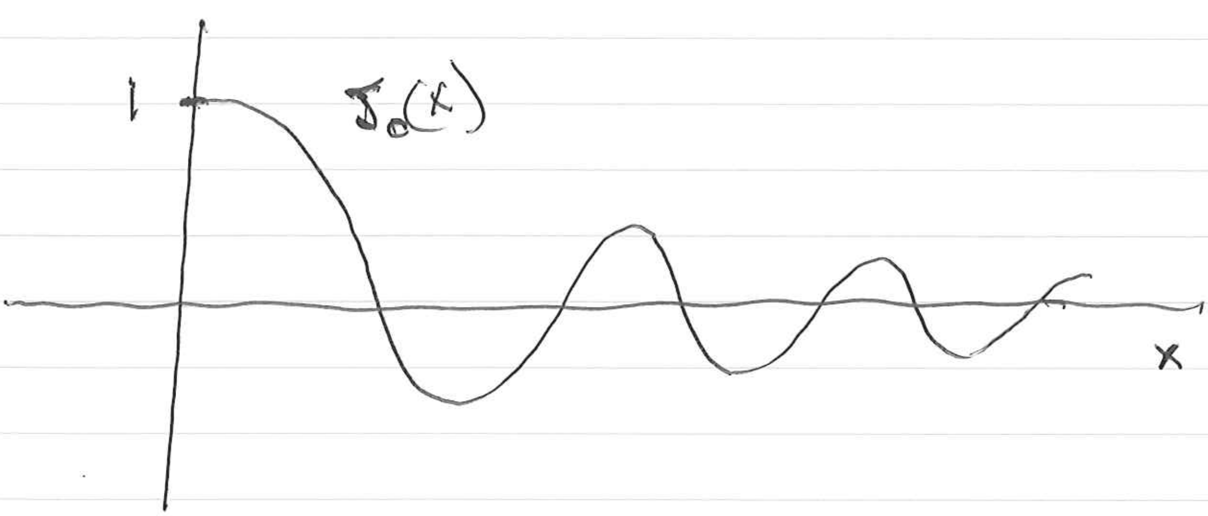
$$x^2 y'' + x y' + (x^2 - p^2) y = 0$$

This equation arises in physics problems in cylindrical and spherical geometry. The form of the solutions $J_p(x)$ for $x \gg 1$ is oscillatory,

$$J_p(x) \simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right)$$

$\Rightarrow$ behavior similar to the sines and cosines in rectangular geometry

$\Rightarrow$ motivates the construction of basis functions



The form that typically appears in physics ~~B~~ actually includes an additional parameter k which can be obtained by changing variables. Let $x \equiv kr$ and note that

$$\cancel{\frac{1}{x}\frac{d}{dx}} \quad x\frac{d}{dx} = kr\frac{d}{d(kr)} = r\frac{d}{dr}$$

so Bessels equation can be written as

$$r^2\frac{d^2}{dr^2}y + r\frac{d}{dr}y + (k^2r^2 - p^2)y = 0$$

with the solutions $J_p(kr)$, $N_p(kr)$.

J_p and N_p, however, are not valid basis functions since the Bessel Eqn is not of the Stürm-Liouville form. As discussed earlier, any second order eqn can be converted to S-L form. For Bessel's eqn it is particularly easy. Let

$$g = \cancel{\frac{d}{kr}} \; y\,\frac{1}{r^{1/2}}$$

~~Substitute $y =$ into the Bessel eqn~~

Substitute $y = r^{1/2}g$ into Bessel's eqn

$$r^2\left[r^{1/2} g_{rr} + r^{-\frac{1}{2}} g_r + \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) g\, r^{-\frac{3}{2}} \right]$$

$$+ r\left[r^{1/2} g_r + \tfrac{1}{2} r^{-\frac{1}{2}} g \right] + (k^2 r^2 - p^2) r^{1/2} g = 0$$

Divide by $r^{1/2}$ and sort terms

$$r^2 g_{rr} + r g_r (1+1) + \left(-\tfrac{1}{4} + \tfrac{1}{2} + k^2 r^2 - p^2\right) g = 0$$

$$\frac{d}{dr} r^2 \frac{dg}{dr} + \left(k^2 r^2 + \tfrac{1}{4} - p^2\right) g = 0$$

$\Rightarrow$ Stürm-Liouville form with

$$p(r) = r^2$$
$$\lambda = k^2$$
$$w(r) = r^2$$
$$q = \tfrac{1}{4} - p^2$$

$\Rightarrow$ Basis functions are

$$g_p = \frac{J_p(k_n r)}{r^{1/2}}, \quad \frac{N_p(k_n r)}{r^{1/2}}$$

where $g_p(k_n r)$ satisfies the B.C.s

$$r^2 g_p(k_n r)\, g_p'(k_m r) \Big|_a^b = 0$$

$\Rightarrow$ need to check the behavior near $r=0$ to be sure that the BC are satisfied.

$\Rightarrow$ near $r=0$

$$\frac{d}{dr} r^2 \frac{d}{dr} g_p(k_n r) + \left(\frac{1}{4} - p^2\right) g_p(k_n r) = 0$$

$\Rightarrow$ Euler form $g_p \sim r^s$

$$s(s+1) + \left(\frac{1}{4} - p^2\right) = 0$$

$$s = \frac{-1 \pm \sqrt{1 - 4\left(\frac{1}{4} - p^2\right)}}{2} = -\frac{1}{2} \pm p$$

$s_+ = -\frac{1}{2} + p \Rightarrow \frac{J_p}{r^{1/2}}$ ok

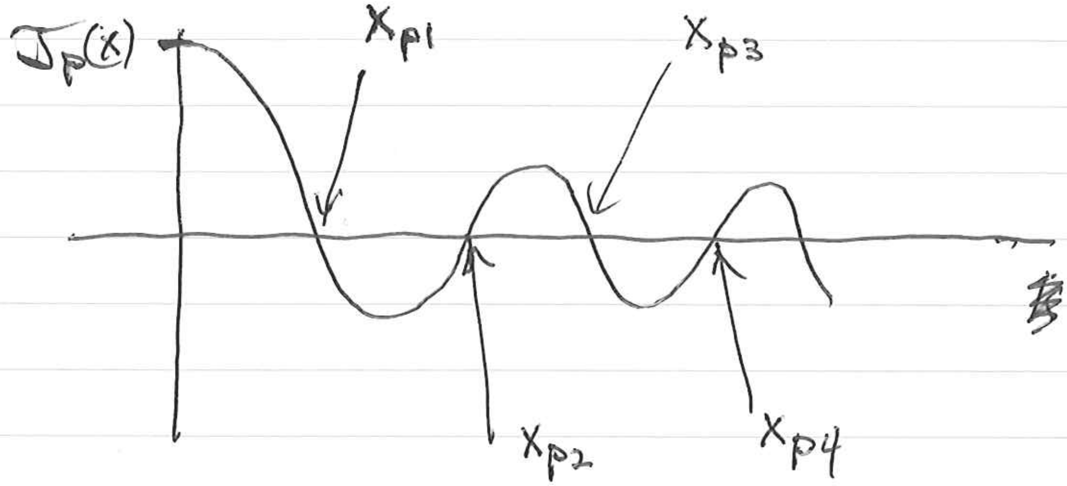
$s_- = -\frac{1}{2} - p \Rightarrow \frac{N_p}{r^{1/2}}$ no good. Goes to ∞ at $r=0$.

$\Rightarrow$ BC at $r = a$ $\Rightarrow$ a = radius of the system

$\Rightarrow$ Want $g_p(k_n a) = 0$

$\Rightarrow$ Could also choose $g_p'(k_n a) = 0$

$\Rightarrow$ a different set of functions



$\Rightarrow$ the zeros of $J_p(x)$ are tabulated.

$\Rightarrow$ x_{pn} is the nth zero of $J_p(x)$

$\Rightarrow$ This determines k_n

$$k_n = \frac{x_{pn}}{a}$$

Basis functions for $r \in (0, a)$ are

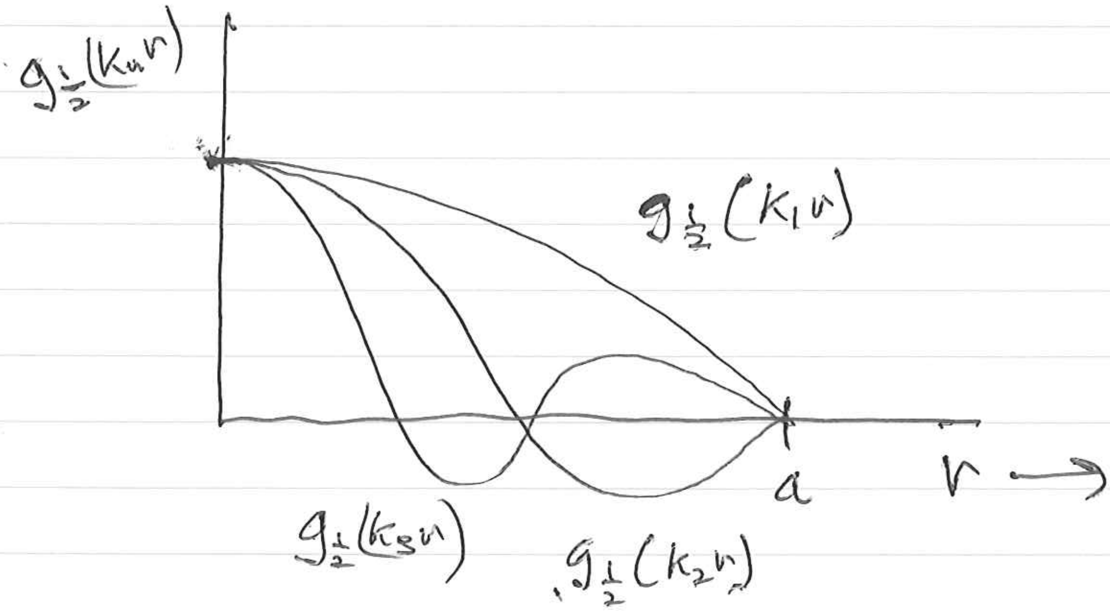
$$\frac{J_p(k_n r)}{r^{1/2}}$$

with $k_n = \frac{x_{pn}}{a}$ and $n = 1, 2, 3 \ldots\ldots$

~~Orthogonality~~

For spherical geometry find $p = \frac{1}{2}$ (to be discussed later).

$$g_{\frac{1}{2}}(k_n r) \sim r^0$$



Normalization and Orthogonality : $w(r) = r^2$

$$\cancel{\int_0^a dr \left[J_p(k_n r) J_p(k_m r) r \right]}$$

$$\int_0^a dr\, g_p(k_n r)\, g_p(k_m r)\, r^2 = \delta_{mn} N_n^2$$

$$\int_0^a dr\, \frac{J_p(k_n r)}{r^{1/2}} \frac{J_p(k_m r)}{r^{1/2}}\, r^2 = \delta_{mn} N_n^2$$

$$\boxed{\int_0^a dr\, r\, J_p(k_n r)\, J_p(k_m r) = \delta_{mn} N_m^2}$$

Can show (but it is not illuminating) that

$$N_m^2 = \frac{a^2}{2} J_{p+1}^2(k_n a)$$

$\Rightarrow$ Note that since $k_n a = x_{pn}$ that $J_{p+1}(x_{pn}) \neq 0$

Recursion Formulae: (Boas p. 592)

$$J_p'(x) = J_{p-1}(x) - \frac{p}{x} J_p(x)$$

$$2 J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$$

$\vdots$

Once we have established the orthogonality relation above we can express functions in terms of $J_p(k_n r)$, which for a complete set over $(0, a]$.

$\Rightarrow$ when using orthogonality multiply by r before carrying out the integral.

<u>example</u> Represent $\delta(r-r_0)$ in a spherical coordinate system

In a system with spherical symmetry

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}$$

and often leads to equations of the form

$$\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} y + k^2 r^2 y = 0$$

which is a particular form of the spherical Bessel equation. This is the same form that we found previously from Bessel's equation with $p = \frac{1}{2}$. The basis functions are

$$J_{\frac{1}{2}}(k_n r) \qquad \text{with} \quad k_n = \frac{x_{\frac{1}{2}n}}{a}$$

We can represent $\delta(r-r_0)$ in terms of these functions.

$$\delta(r-r_0) = \sum_{n=0}^{\infty} c_n J_{\frac{1}{2}}(k_n r)$$

Multiply by $r J_{\frac{1}{2}}(k_m r)$ and integrate from 0 to a.

$$\int_0^a dr\, r\, \delta(r-r_0)\, J_{\frac{1}{2}}(k_m r) = C_m \frac{a^2}{2} J_{\frac{3}{2}}^{2}(X_{\frac{1}{2}m})$$

$\Rightarrow$ orthogonality eliminated all other terms in the sum

$$C_n = \frac{2 r_0}{a^2} J_{\frac{1}{2}}\left(\frac{X_{\frac{1}{2}n} r_0}{a}\right) \frac{1}{J^2_{\frac{3}{2}}(X_{\frac{1}{2}n})}$$

$$\delta(r-r_0) = \frac{2 r_0}{a^2} \sum_{n=0}^{\infty} \frac{J_{\frac{1}{2}}(X_{\frac{1}{2}n} r_0/a)\, J_{\frac{1}{2}}(X_{\frac{1}{2}n} r/a)}{J^2_{\frac{3}{2}}(X_{\frac{1}{2}n})}$$