A set of functions that are orthogonal and normalized is an <u>orthonormal</u> set.

<u>example</u> Legendre Polynomials

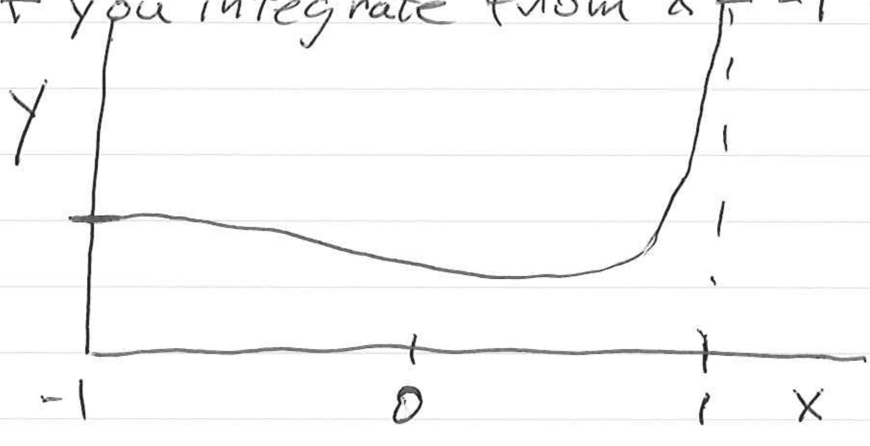
We can illustrate these ideas with Legendre's equation. Recall that Legendre's eqn is

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}y + \alpha(\alpha+1)\, y = 0$$

and we showed the near $x=\pm 1$ one solution is singular. Close to $x=-1$

$$y \sim A + B\ln(1+x)$$

Generally if we demand that $B=0$ at $x=-1$, the solution will diverge at $x=1$ if you integrate from $x=-1$ to $x=1$.



However, if the eigenvalue α is an integer the solution will be bounded at $x=\pm 1$ because the series for y truncates

$\Rightarrow$ integer α yields the basis functions for Legendre's eqn.

$\Rightarrow$ these are the Legendre polynomials

$\Rightarrow$ the important point is that for Legendre's equation the required Stürm - Liouville BC. is

$$(1-x^2)\, y_n\, y_m' \Big|_{-1}^{1} = 0$$

Since the singular solution $\propto \ln(1-x)$

$$y_m' \sim \frac{1}{1-x}.$$

and the BC. is not satisfied. Because the Legendre Polynomials are bounded at $x = \pm 1$

$$(1-x^2)\, P_n\, P_m' \Big|_{-1}^{1} = 0$$

because $1-x^2 = 0$ at ± 1. The Legendre polymomials then are a complete set of basis functions for $-1 \leq x \leq 1$.

A useful representation for $P_n(x)$ is Rodrigues' formula

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} (x^2-1)^n$$

where $\alpha = n$ is now an integer.

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{2}\, 2x = x$$

$$P_2(x) = \frac{1}{4(2)} \frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1)$$

$$= \frac{1}{8}(4\cdot 3x^2 - 4) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$= \frac{1}{2}(3x^2-1)$$

⋮

$\Rightarrow$ note that $P_n(x)$ is even for n even and is odd for n odd.

The Rodrigues formula can be proved from our series representation

For n even

$$P_n(x) = \sum_{\ell=0}^{n/2} \frac{(-1)^\ell (2n-2\ell)!}{2^n \ell! (n-\ell)! (n-2\ell)!} x^{n-2\ell}$$

$$= \sum_{\ell=0}^{n/2} \frac{(-1)^\ell}{(n-\ell)!} \frac{1}{2^n \ell!} \frac{d^n}{dx^n} x^{(2n-2\ell)}$$

note

$$= \sum_{\ell=0}^{n} \frac{(-1)^\ell}{(n-\ell)!} \frac{1}{2^n \ell!} \frac{d^n}{dx^n} x^{2n-2\ell}$$

since $\frac{d^n}{dx^n} x^p = 0$ for $n > p$

$$\Rightarrow P_n(x) = \frac{1}{n!\, 2^n} \frac{d^n}{dx^n} \sum_{\ell=0}^{n} \frac{(-1)^\ell n!}{(n-\ell)!\, \ell!} (x^2)^{n-\ell}$$

$$= \frac{1}{n!\, 2^n} \frac{d^n}{dx^n} (x^2-1)^n$$ from binomial theorem

Use the Rodrigues formula to investigate the orthogonality of the $P_n(x)$ over the interval $(-1, 1)$.

First show that for $m \neq n$

$$I = \int_{-1}^{1} dx\, P_n(x)\, P_m(x) = 0$$

Note that for Legendre's eqn that $w(x) = 1$.
Take $m > n$.

$$I = \frac{1}{2^{n+m}\, n!\, m!} \int_{-1}^{1} dx \left[\frac{d^m}{dx^m}(x^2-1)^m\right]\left[\frac{d^n}{dx^n}(x^2-1)^n\right]$$

Integrate by parts m times and note that have no endpoint contribution

$$I = \frac{1}{2^{n+m}\, n!\, m!} \int_{-1}^{1} dx\, (x^2-1)^m \frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n$$

Since $n+m > 2n$, $\dfrac{d^{n+m}}{dx^{n+m}}(x^2-1)^n = 0$

$\Longrightarrow$ P_n, P_m are orthogonal for $m \neq n$.

For $m = n$,

$$I_n \equiv \int_{-1}^{1} dx\, P_n^2(x) = \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} dx \left[\frac{d^n}{dx^n}(x^2-1)^n\right]^2$$

Integrate by parts n times

$$I_n = \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^{1} dx\, (x^2-1)^n \underbrace{\frac{d^{2n}}{dx^{2n}}(x^2-1)^n}_{(2n)!}$$

$$I_n = \frac{(2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} dx \, (1-x^2)^n$$

Obtain answer by induction

$$(1-x^2)^n = (1-x^2)(1-x^2)^{n-1}$$

$$= (1-x^2)^{n-1} + \frac{x}{2n} \frac{d}{dx}(1-x^2)^n$$

$$I_n = \frac{2n(2n-1)(2n-2)!}{4 \, 4^{n-1} \, n^2[(n-1)!]^2} \int_{-1}^{1} dx \, (1-x^2)^{n-1}$$

$$+ \frac{(2n)!}{4^n (n!)^2} \int_{-1}^{1} dx \, \underbrace{\frac{x}{2n} \frac{d}{dx}(1-x^2)^n}_{\text{integrate by parts}}$$

$$= \frac{2n-1}{2n} I_{n-1} - \frac{1}{2n} I_n$$

$$(2n+1) I_n = (2n-1) I_{n-1} = [2(n-1)+1] I_{n-1}$$

so $\quad I_n = \frac{\text{const.}}{2n+1}$

For $n=0$

$$I_0 = \int_{-1}^{1} dx \, P_0^2 = \int_{-1}^{1} dx = 2$$

$$I_n = \frac{2}{2n+1}$$

$\Rightarrow$ normalized polynomials $\sqrt{\frac{2n+1}{2}} \, P_n(x)$

Legendre Polynomials – Properties

Recursion relations are sometimes useful in carrying out integrals with Legendre Polynomials

$$(2\ell+1)P_\ell(x) = P'_{\ell+1} - P'_{\ell-1}$$

$$\ell P_\ell(x) = (2\ell-1)\,x P_{\ell-1} - (\ell-1)P_{\ell-2}$$

$\vdots$

See Ch 12 Eq. 5.8 in Boas.

$P_\ell(1)$:

$$\ell P_\ell(1) = (2\ell-1)P_{\ell-1}(1) - (\ell-1)P_{\ell-2}$$

$$P_0(1) = 1, \quad P_1(1) = 1$$

If $P_{\ell-1}(1) = 1$ and $P_{\ell-2}(1) = 1$

$$\ell P_\ell(1) = (2\ell-1) - (\ell-1) = \ell$$

$$P_\ell(1) = 1$$

$\Rightarrow P_\ell(1) = 1$ for all ℓ.

What about $P_\ell(-1)$?

$P_\ell(0)$: $P_\ell(0) = 0$ for ℓ odd since P_ℓ is an odd function of x

For even ℓ use recursion formula

$$\ell P_\ell(0) = -(\ell-1) P_{\ell-2}$$

$$P_\ell(0) = -\frac{(\ell-1)}{\ell} P_{\ell-2}$$

$$P_2(0) = -\frac{1}{2} P_0 = -\frac{1}{2}$$

$$P_4(0) = -\frac{3\cdot 1}{4} P_2 = +\frac{3\cdot 1}{4\cdot 2}$$

$$P_6(0) = -\frac{5}{6} P_4(0) = -\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}$$

$$P_\ell(0) = (-1)^{\frac{\ell}{2}} \frac{(\ell-1)!!}{\ell!!}$$

example Let

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$$

Represent $f(x)$ in a series of Legendre polynomials

$$f(x) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(x)$$

Use orthogonality $\Rightarrow$ multiply by $P_{\ell'}$ and integrate $(-1,1)$

$$\int_{-1}^{1} dx\, P_{\ell'}\, f(x) = \sum_{\ell=0}^{\infty} c_\ell \int_{-1}^{1} dx\, P_{\ell'} P_\ell(x)$$

$$= \sum_{\ell=0}^{\infty} c_\ell \frac{2\delta_{\ell\ell'}}{2\ell+1} = \frac{2\, c_{\ell'}}{2\ell'+1}$$

$$c_{\ell'} = \left(\frac{2\ell'+1}{2}\right) \int_0^1 dx\, P_{\ell'}(x)$$

$$= \frac{(2\ell'+1)}{2(2\ell'+1)} \int_0^1 dx \left(P'_{\ell'+1} - P'_{\ell'-1}\right)$$

$$= \frac{1}{2}\Big[P_{\ell'+1}(x) - P_{\ell'-1}(x)\Big]_0^1$$

$= 0$ for ℓ' even except $\ell' = 0$

$$\boxed{c_0 = \frac{1}{2}}$$

For ℓ' odd, $P_{\ell'+1}(1) = P_{\ell'-1}(1) = 1$

$$c_{\ell'} = \frac{-1}{2}\Big[P_{\ell'+1}(0) - P_{\ell'-1}(0)\Big]$$

$$= -\frac{1}{2}\left[(-1)^{\frac{\ell'+1}{2}} \frac{\ell'!!}{(\ell'+1)!!} - (-1)^{\frac{\ell'-1}{2}} \frac{(\ell'-2)!!}{(\ell'-1)!!}\right]$$

$$-(-1)^{\frac{l'-1}{2}} = (-1)^{\frac{l'+1}{2}}$$

$$(l'+1)(l'-1)!! = (l'+1)!!$$

$$l'!! = l'(l'-2)!!$$

$$c_{l'} = -\frac{1}{2}(-1)^{\frac{l'+1}{2}}\left[\frac{l'!! + (l'+1)\, l'!!\, \frac{1}{l'}}{(l'+1)!!}\right]$$

$$c_{l'} = -\frac{1}{2}(-1)^{\frac{l'+1}{2}} \frac{l'!!}{(l'+1)!!}\left(\frac{2l'+1}{l'}\right), \quad l' = 1, 3, 5$$

$$f(x) = \frac{1}{2}P_0(x) + \frac{1}{2}\sum_{\substack{l \\ \text{odd}}} (-1)^{\frac{l+1}{2}} \frac{l!!}{(l+1)!!} \frac{2l+1}{l} P_l(x)$$

$$= \frac{1}{2}P_0(x) + \frac{3}{2\cdot 2}P_1(x) - \frac{3\cdot 1}{2\cdot 4\cdot 2}\frac{7}{3}P_3(x) + \cdots$$

$$= \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \cdots$$

Generating function

$$\frac{1}{(1-2xh+h^2)^{1/2}} = \sum_{l=0}^{\infty} h^l P_l(x) \quad \text{for } |h| < 1$$