Solving equations with Variable Coefficients,

For equations in which the coefficients a_j are not constant, the exponential solutions are no longer valid, e.g.

$$\mathcal{L}(y) = y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots a_0(x)\, y^{(0)} = 0$$

If the a_j's are analytic in the neighborhood of a point x_0, they can be expanded in a Taylor series around x_0. We might expect that $e_1, \ldots e_n$ can also be expanded in a power series.

$\Rightarrow$ All the n linearly independent solutions are analytic in the neighborhood of an <u>ordinary</u> point where the a_j's are analytic.

$\Rightarrow$ The expansions are valid to the nearest singularity of the a_j's.

<u>example</u>
$$y' + \frac{2x}{1+x^2}\, y = 0$$

This has a Taylor series solution around $x=0$ valid for $|x| < 1$ since this is the distance to the singularity at $x = \pm i$.

<u>Example</u> Airy's Eqn

Consider Airy's equation

$$y'' - xy = 0$$

This equation is important since it has the generic form of a turning point. To see this, ignore the fact that x is a variable and look for exponential solutions

$$y \sim e^{ikx}$$

We find

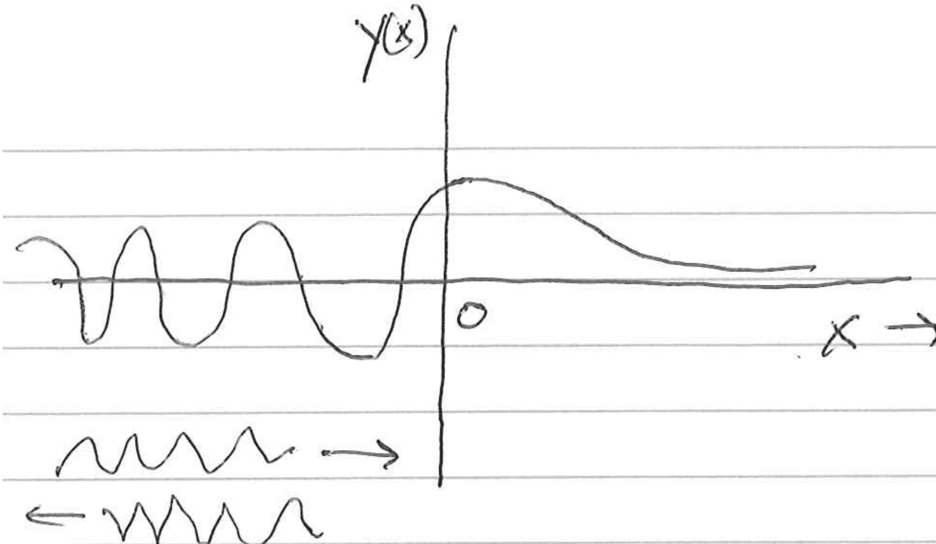
$$k^2 = -x \quad \text{or} \quad k = \pm(-x)^{\frac{1}{2}}$$

For $x<0$, have oscillatory solutions since k is real

For $x>0$, have $k = \pm i x^{1/2}$ so

$$y \sim e^{\pm|k|x}$$

The solutions are typically decaying in space



At $x = 0$, $K \to$

⇒ called a turning point

propagating waves : turning point : decaying wave

The Airy equation describes the reflection of a wave from a region where it can't propagate

⇒ bounded solutions in quantum

⇒ wave propagation in inhomogeneous media

Series solutions

$$y = \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + \cdots$$

$$y' = \sum_{k=0}^{\infty} k c_k x^{k-1}$$

$$y'' = \sum_{k=0}^{\infty} k(k-1) c_k x^{k-2}$$

Substituting into the Airy Eqn,

$$\underbrace{\sum_{k=0}^{\infty} k(k-1) c_k x^{k-2}}_{2c_2 x^0 + 3(2) c_3 x + \dots} - \underbrace{\sum_{k=0}^{\infty} c_k x^{k+1}}_{c_0 x + c_1 x^2 + \dots} = 0$$

Since the equation must be valid for all values of x, the coefficient of each power of x must vanish.

$$\Rightarrow c_2 = 0$$

$\Rightarrow$ For series on the left let
$k \to k+3$

$$\sum_{k=0}^{\infty} \underbrace{\left[(k+3)(k+2) c_{k+3} - c_k \right]}_{=0} x^{k+1} = 0$$

$$c_{k+3} = \frac{c_k}{(k+2)(k+3)}$$

Have two solutions produced by
c_0 and c_1

Co solution

$$c_3 = \frac{c_0}{2(3)}$$

$$c_6 = \frac{c_3}{5(6)} = \frac{c_0}{2\cdot3\cdot5\cdot6}$$

$$c_{3m} = \frac{c_0}{2\cdot3\cdot5\cdot6\cdots(3m-1)(3m)}$$

C1 solution

$$c_4 = \frac{c_1}{3\cdot4}$$

$$c_7 = \frac{c_4}{6\cdot7} = \frac{c_1}{3\cdot4\cdot6\cdot7}$$

$$c_{3m+1} = \frac{c_1}{3\cdot4\cdot6\cdot7\cdots(3m)(3m+1)}$$

Let $$\varphi_0(x) = 1 + \sum_{m=1}^{\infty} \frac{x^{3m}}{2\cdot3\cdot5\cdot6\cdots(3m-1)(3m)}$$

$$\varphi_1(x) = x + \sum_{m=1}^{\infty} \frac{x^{3m+1}}{3\cdot4\cdot6\cdot7\cdots 3m(3m+1)}$$

General solution

$$y = c_0\,\varphi_0(x) + c_1\,\varphi_1(x)$$

what about convergence? Use ratio test

$$\varphi_0 = 1 + \sum_{m=1}^{\infty} R_m$$

$$\lim_{m\to\infty} \frac{R_{m+1}}{R_m} = \lim_{m\to\infty} \frac{x^{3m+3}}{2\cdot 3\cdots(3m+2)(3m+3)} \frac{2\cdot 3\cdots(3m-1)(3m}{x^{3m}}$$

$$= \lim_{m\to\infty} \frac{x^3}{(3m+2)(3m+3)} = \lim_{m\to\infty} \frac{x^3}{9m^2}$$

$$\to 0$$

Converges for all $|x| < \infty$.

$\Rightarrow$ reasonable since $a_0(x)$ is analytic for all x.

Are a_0, a_1 linearly indep?

$\Rightarrow$ Wronskian still applies for variable coeff.

$\Rightarrow$ check at $x=0$

$a_0(0) = 1 \qquad a_0'(0) = 0$

$a_1(0) = 0 \qquad a_1'(0) = 1$

$$W(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\Rightarrow$ linearly indep.

Example : Legendre's Eqn

Consider

$$(1-x^2)y'' - 2xy + \ell(\ell+1)y = 0$$

or

$$y'' - \frac{2x}{1-x^2}y + \frac{\ell(\ell+1)}{1-x^2}y = 0$$

with ℓ a constant. Arises from systems in spherical coordinates.

Are the coefficients a_0, a_1 analytic? Not at $x = \pm 1$ but they are for $|x| < 1$ $\Rightarrow$ try series solutions ~~for~~ around $x = 0$.

$$y = \sum_{k=0}^{\infty} c_k x^k$$

$$y' = \sum_{k=0}^{\infty} k c_k x^{k-1}$$

$$\sum_{k=0}^{\infty} \Big[\underbrace{k(k-1)c_k x^{k-2}}_{(k+2)(k+1)c_{k+2}x^k} - k(k-1)c_k x^k - 2k c_k x^k + \ell(\ell+1)c_k x^k \Big] =$$

$\Rightarrow$ demand coefficient of x^k be zero.

$$c_{k+2} = \frac{k(k+1) - \ell(\ell+1)}{(k+1)(k+2)} c_k$$

$$= - \frac{(\cancel{k} + \ell)(k + \ell + 1)}{(k+1)(k+2)} c_k$$

c_0 series:

$$c_2 = - \frac{\ell(\ell+1)}{2} c_0$$

$$c_4 = \cdots$$

c_1 series:

$$c_3 = \frac{(1-\ell)(2+\ell)}{2 \cdot 3} c_1$$

$$c_5 = \cdots$$

$$\mathcal{Q}_0 = 1 - \frac{\ell(\ell+1)}{2!} x^2 + \frac{\ell(\ell+1)(\ell-2)(\ell+3)}{4!} x^4 - \cdots$$

$$\mathcal{Q}_1 = x - \frac{(\ell-1)(\ell+2)}{3!} x^3 + \frac{(\ell-1)(\ell+2)(\ell-3)(\ell+4)}{5!} x^5$$

$$y = c_0 \mathcal{Q}_0(x) + c_1 \mathcal{Q}_1(x)$$

Convergence? $\mathcal{Q}_0 = \sum_m R_m$

$$\lim_{m \to \infty} \frac{R_{m+1}}{R_m} = x^2 \frac{m(m+1) - \ell(\ell+1)}{(m+1)(m+2)} = x^2$$

$$\Rightarrow |x^2| < 1$$

Truncation of the series

For integer values of l, one of the series truncates

Even l:

$$C_{k+2} = -\frac{(l-k)(l+k+1)}{(k+1)(k+2)} C_k$$

$\Rightarrow C_{k+2} \cancel{\neq} \propto l-k = 0$

for $k = l$

$\Rightarrow$ series for $\mathcal{Q}_0(x)$ truncates

$\Rightarrow$ Highest $C_k \neq 0$ surviving power is

$\sim x^k \sim x^l$

$\Rightarrow$ for even l the $\mathcal{Q}_0(x)$ series truncates to a polynomial with order l.

$\Rightarrow$ The $\mathcal{Q}_1(x)$ series does not truncate

Odd l: The $\mathcal{Q}_1(x)$ series truncates to a polynomial of order l

$\Rightarrow$ The $\mathcal{Q}_1(x)$ series does not truncate

$\Rightarrow$ called Legendre polynomials

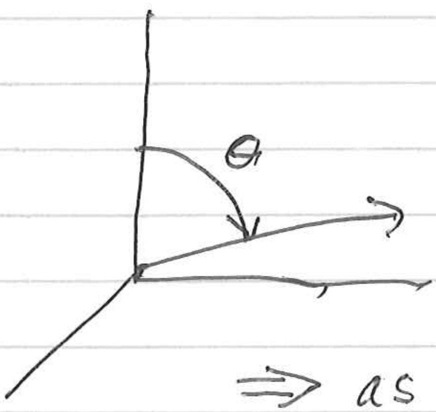
$\Rightarrow$ Legendre polynomials are bounded solutions over the entire interval

$$-1 \leq x \leq 1$$

$\Rightarrow$ in spherical coordinates

$$x = \cos\theta \quad \text{so}$$

$$0 \leq \theta \leq \pi$$



$\Rightarrow$ as will be discussed later these polynomials become a convenient set of basis functions in systems ~~tn~~ with ~~sph~~ spherical geometry

Classification of singular points of differential equations

We have shown that we can obtain series solutions of equations in regions where the coefficients $a_j(x)$ are analytic. What about in regions where the a_j's are singular? Are series solutions possible? Yes, under some conditions.

There are three basic classes of equations:

① Ordinary point – An ordinary point x_0 of a differential equation is a point where all of the a_j's are analytic.

② Regular singular point – A point x_0 is a regular singular point if the equation can be written in the form

$$(x-x_0)^n y^n(x) + (x-x_0)^{n-1} a_{n-1}(x) y^{(n-1)}(x) + \ldots\ldots (x-x_0)^0 a_0(x) y^{(0)}(x) = 0$$

where all of the a_j's are analytic at x_0.
$\Rightarrow$ series solutions possible

③ Irregular singular point – If the equation is ~~neg~~ neither an ordinary nor a regular singular point at x_0 $\Rightarrow$ no general technique.

example $x^2 y'' + x y' - y = 0$

Has a regular singular point at $x=0$.

Euler Equation

Consider an equation of the following form

$$f(y) = x^n y^{(n)} + x^{n-1} a_{n-1} y^{(n-1)} + \cdots x^0 a_0 y^{(0)} = 0$$

where the a_i's are constants. This is an Euler equation. It has a regular singular point (RSP) at $x=0$.

$\Rightarrow$ The Euler equation has powerlaw solutions

$$y = x^r$$

$$\frac{dy}{dx} = r x^{r-1}$$

$$y^{(n)} = r(r-1) \cdots\cdots (r-n+1)\, x^{r-n}$$

$$x^n y^{(n)} = r(r-1) \cdots (r-n+1)\, x^r$$

$$f(x^r) = g(r)\, x^r$$

$$g(r) = r(r-1) \cdots (r-n+1) + r(r-1) \cdots (r-n+2)\, a_{n-1} + \cdots a_0 = 0,$$

$\Rightarrow$ nth order polynomial for r

$\Rightarrow$ n powerlaw solutions to the Euler equation.

$\Rightarrow$ If have degenerate solutions for $r = r_0$ then have solutions

$$x^{r_0}, \quad x^{r_0}\ln(x), \quad x^{r_0}(\ln(x))^2 \ldots$$

$\Rightarrow$ similar to the case of degeneracy of the exponential solutions for constant coeff. equations

Solutions of eqns. with RSP : local behavior

The existence of the power law solutions of Euler's eqn is the reason the equations with RSP can be solved. Consider an equation with a RSP at x_0.

$$(x-x_0)^n y^{(n)} + (x-x_0)^{n-1} a_{n-1}(x) y^{(n-1)} + \cdots = 0$$

where the a_j's are analytic at x_0.

$\Rightarrow$ expand the a_j's in a Taylor series around x_0

$$a_j(x) = a_j(x_0) + a_j'(x-x_0) + \cdots$$

What is the form of the equation very close to x_0?

$$(x-x_0)^n y^{(n)} + (x-x_0)^{n-1} a_{n-1}(x_0)\, y^{(n-1)} + \cdots = 0$$

$\Rightarrow$ this is Euler's eqn.

$\Rightarrow$ very close to the singularity have powerlaw solutions

<u>example</u>

$$x^2 y'' + \alpha x y' + x y = 0$$

$\Rightarrow$ RSP at $x = 0$

$a_1 = \alpha \quad \Rightarrow a_1(0) = \alpha$

$a_0(x) = x \quad \Rightarrow a_0(0) = 0$

$\Rightarrow$ Close to the singularity

$$x^2 y'' + \alpha x y' = 0$$

$$y \sim x^r$$

$$r(r-1)x^r + \alpha r x^r = 0$$

$$[r(r-1) + \alpha r]\, x^r = 0$$

$$\Rightarrow r = 0,\ 1-\alpha$$

$$y = c_1 + c_2 x^{1-\alpha}$$

$\Rightarrow$ If $\alpha = 1$

$$y = c_1 + c_2 \ln(x)$$

example Legendre's Eqn.

We previously obtained the series solution for Legendre's equation around $x=0$ and found that the solutions fail at $x = \pm 1$. What is the behavior of the solutions near $x = \pm 1$?

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

Let $t = x-1 \Rightarrow \frac{d}{dx} = \frac{d}{dt}$

$$\underbrace{[1-(t+1)^2]}_{1-t^2-1-2t \,=\, -t(2+t)} \frac{d^2y}{dt^2} - 2(t+1)\frac{dy}{dt} + \alpha(\alpha+1)y = 0$$

Near $t=0$,

$$-2t\, y_{tt} - 2y_t + \alpha(\alpha+1)y = 0$$

Multiply by t

$$t^2 y_{tt} + t\, y_t - \frac{\alpha(\alpha+1)t}{2} y = 0$$

$a_1(t) = 1$ $\qquad a_1(0) = 1$

$a_0(t) = -\frac{\alpha(\alpha+1)}{2}t$ $\qquad a_0(0) = 0$

Near $t=0$,

$$t^2 y_{tt} + t\, y_t = 0$$

powerlaw solutions $\Rightarrow y \sim t^r$

$$\underbrace{\left[r(r-1)+r\right]}_{r^2} t^r = 0$$

$\Rightarrow r^2 = 0$

The roots are degenerate so the behavior near $x = 1$ is

$$y = c_1 + c_2 \ln(x-1)$$

Series solutions for equations with RSP

A differential equation with a RSP has at least one solution of the form

$$\varphi(x) = x^r \sum_{k=0}^{\infty} c_k x^k \Rightarrow \text{Frobenius solution}$$

$\Rightarrow$ seems plausible since very close to the singularity the equation reduces to Euler's eqn which has power law solutions

example Bessel's Equation

Bessels equation arises in systems with cylindrical geometry and we will show later that the solutions of the equation can be used to construct basis functions in cylindrical geometry.

$$x^2 y'' + x y' + (x^2 - p^2)\, y = 0$$

where the constant p is the "order" of the equation.

Let's first find the solution near $x=0$

$\Rightarrow$ the power law solution

$$y \sim x^r$$

Near $x=0$

$$x^2 y'' + x y' - p^2 y = 0$$

$$(r(r-1) + r - p^2)\, x^r = 0$$

$$(r^2 - p^2)\, x^r = 0$$

$$g(r)\, x^r = 0$$

$$g(r) = r^2 - p^2 = 0 \Rightarrow \text{indicial equation}$$

$$r = \pm p$$

Near $x=0$, $\quad y \sim x^{\pm p}$

For $p=0$, $y \sim 1, \ln(x)$

To obtain the full series, re-write the equation as

$$x(xy')' + (x^2 - p^2)\, y = 0$$

$$y = \sum_{k=0}^{\infty} c_k x^{k+r}$$

$$y' = \sum_{k=0}^{\infty} (k+r) c_k x^{k+r-1}$$

$$xy' = \sum (k+r) c_k x^{k+r}$$

$$x(xy')' = \sum (k+r)^2 c_k x^{k+r}$$

$$\sum_{k=0}^{\infty} [\underbrace{(k+r)^2 - p^2}_{g(k+r)}] c_k x^{k+r} + \sum_{k=0}^{\infty} c_k x^{k+r+2} = 0$$

$$g(r) c_0 x^r + g(1+r) c_1 x^{r+1} \qquad c_0 x^{r+2} + c_1 x^{r+3} \cdots$$

$$+ g(2+r) c_2 x^{r+2} + \cdots$$

First two terms on left yield

$g(r) = 0 \Rightarrow r = \pm p$ as before from Euler

$c_1 = 0$

To equate coefficients of each power of x, let $k \to k-2$ in the sum on right

$$\sum \left(g(k+r) c_k + c_{k-2} \right) x^{k+r} = 0$$

$$c_k = - \frac{c_{k-2}}{g(k+r)}$$

First solution : $r = p$

$$c_2 = - \frac{c_0}{g(2+p)}$$

$$c_4 = - \frac{c_2}{g(4+p)} = \frac{c_0}{g(2+p)g(4+p)}$$

$c_6 = \ldots$

$\Rightarrow$ Bessel function

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+1+p)} \left(\frac{x}{2}\right)^{2k+p}$$

$$= \frac{1}{\Gamma(1)\Gamma(1+p)}\left(\frac{x}{2}\right)^p - \frac{1}{\Gamma(2)\Gamma(2+p)}\left(\frac{x}{2}\right)^{2+p}$$

$$+ \ldots$$

where $\Gamma(n+1) = n\Gamma(n)$ is the Gamma function

Second Solution: $r = -p$

$\Rightarrow$ same procedure

$$J_{-p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+1-p)} \left(\frac{x}{2}\right)^{2k-p}$$

Because for integer values of p $J_{-p} = (-1)^n J_p$ and therefore not linearly independent of J_p, a combination of J_{-p} and J_p are used for the second solution

$$N_p(x) = \frac{\cos(\pi p)\, J_p(x) - J_{-p}(x)}{\sin \pi p}$$

$=$ Neumann function.

Note that $N_p(x)$ is singular at $x=0$ (Euler solution)