Laplace Transforms

The Fourier transforms and inverse transforms that we defined earlier ~~have~~ are not useful in some systems.

For example, in the time domain

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' f(t') e^{-i\omega t'}$$

$$f(t) = \int_{-\infty}^{\infty} d\omega F(\omega) e^{i\omega t}$$

Suppose we are interested in an initial value problem in which derivatives of $f(t)$ are specified at $t=0$. It is not clear how to implement such boundary conditions in the usual Fourier representation. We might also have a problem in which $f(t)$ is growing in time

$$f(t) \sim e^{\Gamma t}$$

In this case the time integral in the definition of $F(\omega)$ diverges so we can't use the usual Fourier integrals.

The Laplace transform allows us to solve both of these types of problems.

The best way to obtain the Laplace transform and its inverse is to use what we know about the FT and modify it appropriately.

Suppose $f(t) = 0$ for $t < 0$. Suppose also that $f(t)$ becomes large as $t \to \infty$. Let's define a function

$$G(t) = f(t) e^{-\gamma t}$$

with $\gamma > 0$ and with γ ~~sup~~ sufficiently large so that $G \to 0$ as $t \to \infty$. We also have $G = 0$ for $t < 0$. Since G is bounded for both $t = \pm\infty$, we can use the FT acting on G.

$$G(t) = \int_{-\infty}^{\infty} d\omega \, e^{i\omega t} \frac{1}{2\pi} \underbrace{\int_{0}^{\infty} dt' \, G(t') e^{-i\omega t'}}$$

we used ~~note~~ $G = 0$ for $t' < 0$

Thus, since $G = f e^{-\gamma t}$

$$f(t) = \int_{-\infty}^{\infty} d\omega \, e^{(\gamma + i\omega)t} \, \frac{1}{2\pi} \int_{0}^{\infty} dt' \, f(t') \, e^{-(\gamma+i\omega)t'}$$

$$\left.\begin{aligned} \text{Let } p &= \gamma + i\omega \\ dp &= i\, d\omega \end{aligned}\right\} \text{change variables from } \omega \text{ to } p$$

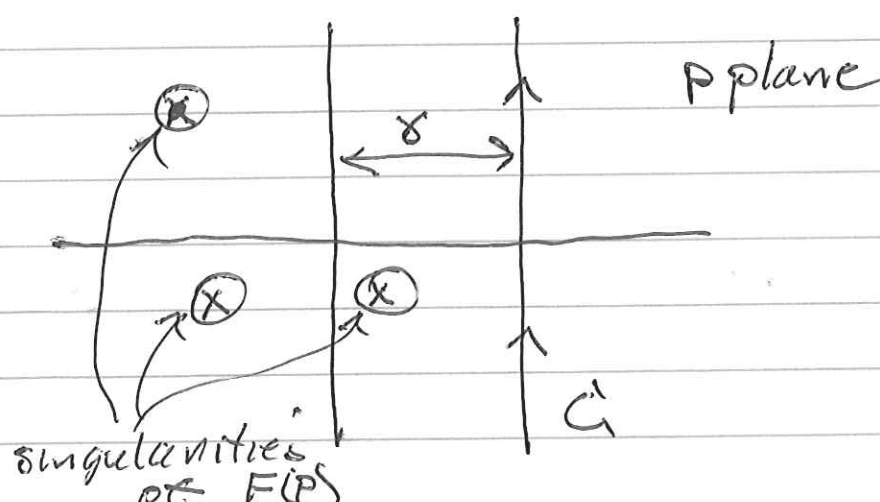
$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dp \, e^{pt} \underbrace{\int_{0}^{\infty} dt' \, f(t') \, e^{-pt'}}_{F(p)}$$

Laplace transform:

$$F(p) = \int_{0}^{\infty} dt' \, f(t') \, e^{-pt'}$$

Inverse transform:

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dp \, e^{pt} \, F(p)$$



The integral is a contour integral in the complex p plane.

F(p) will generally have singularities

$\Rightarrow$ singularities must lie to the left of G

$\Rightarrow$ for $t < 0$ can close contour in the RH plane where

$$e^{pt} = e^{-p|t|} \Rightarrow 0$$

$\Rightarrow$ since we want $f = 0$ for $t < 0$, can't have any singularities enclosed by the contour when we close it in the RH plane

$\Rightarrow$ just make sure γ is large enough

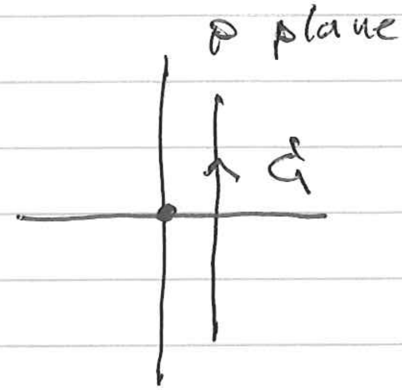
Example: Let $f(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

$$F(p) = \int_0^\infty dt' \, f(t') e^{-pt'} = \int_0^\infty dt' \, e^{-pt'}$$

$$= \frac{1}{p}$$ with $p > 0$ so integral converges at ∞.

Inverse transform

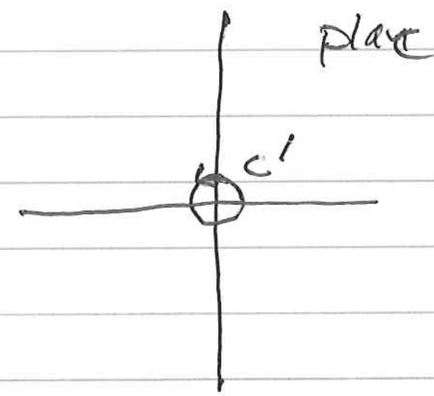
$$f(t) = \frac{1}{2\pi i}\int_C dp\, e^{pt} F(p)$$



For $t<0$, close integral in RH plane $\Rightarrow f = 0$

For $t>0$, close in LH plane

$$f(t) = \frac{1}{2\pi i}\oint_{C'} dp\, e^{pt}\,\frac{1}{p}$$



$$= 1$$

<u>example</u> $\quad f(t) = \begin{cases} \cos(at) & t>0 \\ 0 & t<0 \end{cases}$

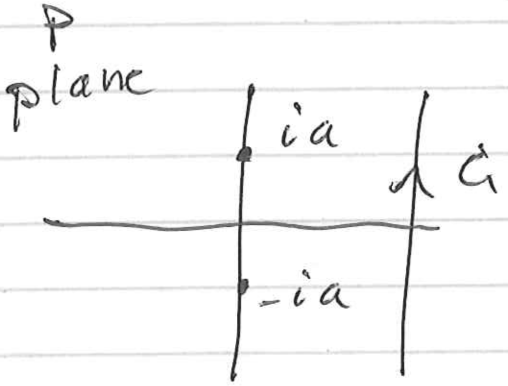
$$F(p) = \int_0^\infty dt'\, e^{-pt'}\cos(at')$$

$$= \frac{1}{2}\int_0^\infty dt'\left(e^{-(p-ia)t'} + e^{-(p+ia)t'}\right)$$

$$= \frac{1}{2}\left(\frac{-1}{-(p-ia)} + \frac{-1}{-(p+ia)}\right)$$

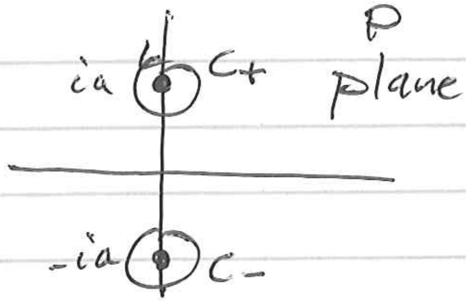
$$= \frac{1}{2}\,\frac{p+ia+p-ia}{p^2+a^2} = \frac{p}{p^2+a^2}$$

$$f(t) = \frac{1}{2\pi i}\int_C dp\, e^{pt}\frac{p}{(p+ia)(p-ia)}$$



$$\Rightarrow 0 \quad t<0$$

$$= \frac{2\pi i}{2\pi i}\left[\frac{e^{pt}p}{p+ia}\bigg|_{p=ia} + \frac{e^{pt}p}{p-ia}\bigg|_{-ia}\right]$$



$$= \frac{ia\,e^{iat}}{2ia} + \frac{e^{-iat}(-ia)}{-2ia}$$

$$= \cos at$$

Solving differential equations with Laplace transforms

Often have to solve differential equations with specified values of the function ~~as initia~~ and/or derivatives at an initial time. Can use the Laplace transform to solve such problems.

$\Rightarrow$ first need to know how to carry out the transform of a differential equation

$\Rightarrow$ transforms of derivatives.

Take the transform of

$$\dot{y} = \frac{dy}{dt}$$

Let the operator L represent the Laplace transform

$$L(\dot{y}) = \int_0^\infty dt' \, \frac{dy}{dt'} \, e^{-pt'}$$

$\Rightarrow$ do an integration by parts

$$L(\dot{y}) = y(t')\, e^{-pt'} \Big|_0^\infty + p \int_0^\infty dt' \, y(t')\, e^{-pt'}$$

$$= -y(0) + p\, \underline{Y}(p)$$

with $\underline{Y}(p)$ the transform of $y(t)$.

Higher derivatives follow in a similar way

$$L(\ddot{y}) = -\dot{y}(0) + p\, L(\dot{y})$$

$$= -\dot{y}(0) + p\left(-y(0) + p\, \underline{Y}(p)\right)$$

$$= -\dot{y}(0) - p\, y(0) + p^2\, \underline{Y}(p)$$

<u>example</u> $\ddot{y} + 4y = 0$

with $y(0) = 1$ and $\dot{y}(0) = 0$

Take the transform of the equation

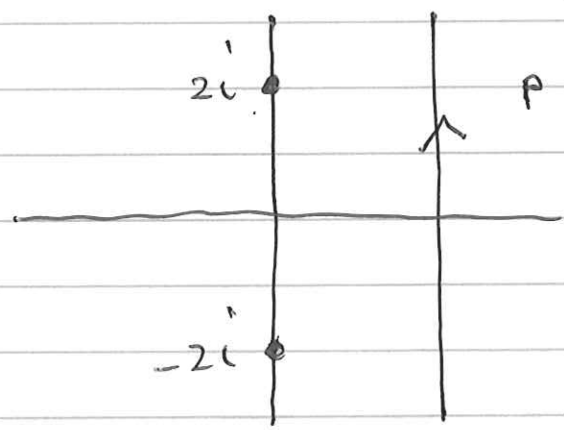
$$L(\ddot{y}) + 4\,Y(P) = 0$$

$$-\overset{0}{\cancel{\dot{y}(0)}} - P\,y(0) + P^2\,Y(P) + 4\,Y(P) = 0$$

$$Y(P) = y(0)\,\frac{P}{P^2+4} = \frac{P}{P^2+4}$$

$\Rightarrow$ inverse transform

$$y(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dp\; e^{pt}\, \frac{p}{p^2+4}$$



for $t < 0$, close contour in RHP

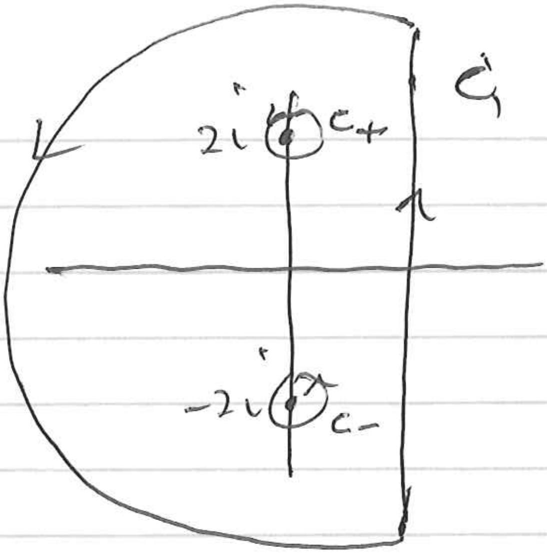
$\Rightarrow$ no singularities

$\Rightarrow y = 0$

for $t > 0$ close in LHP

$\Rightarrow$ by Jordan's Lemma contour at large R is zero since

$$\frac{P}{P^2+4} \to \frac{1}{P} \to 0$$



For $t > 0$

$$y(t) = \frac{1}{2\pi i} \oint_C dp \, \frac{e^{pt} \, p}{(p+2i)(p-2i)}$$

$$= \frac{1}{2\pi i} \oint_{C_+} dp \, \frac{e^{pt} \, p}{(p+2i)(p-2i)}$$

$$+ \frac{1}{2\pi i} \oint_{C_-} dp \, \frac{e^{pt} \, p}{(p+2i)(p-2i)}$$

$$= \cancel{\frac{2\pi i}{2\pi i}} \frac{e^{2it} \, 2i}{4i} + \cancel{\frac{2\pi i}{2\pi i}} \frac{e^{-2it} \, (-2i)}{-4i}$$

$$\boxed{y(t) = \cos(2t)}$$

note that $y(0) = 1$ and $\dot{y}(0) = 0$

<u>example</u>:

$$\ddot{y} + 4y = \sin t$$

with $y(0) = 0$ and $\dot{y}(0) = 0$

Take Laplace transform of eqn.

$$L(\ddot{y}) + 4\,\underline{Y}(p) = L(\sin t)$$

$$(p^2+4)\,\underline{Y}(p) = L(\sin t)$$

$$L(\sin t) = \int_0^\infty dt' \sin(t')\, e^{-pt'}$$

$$= \frac{1}{2i} \int_0^\infty dt' \left(\underline{e^{(-p+i)t'} - e^{(-p-i)t'}} \right)$$

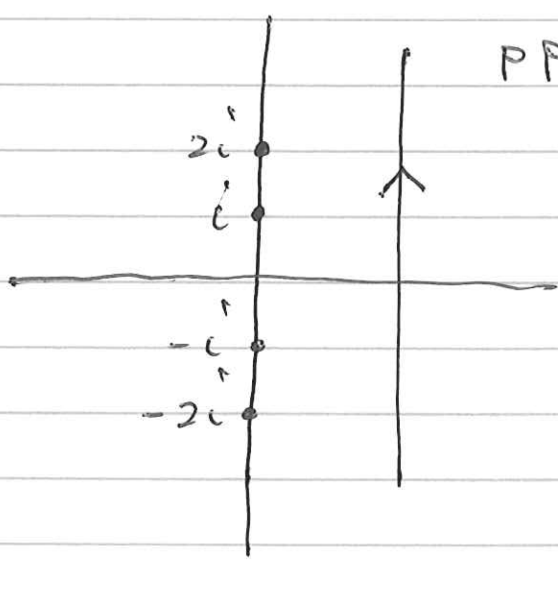
$$= \frac{1}{2i} \left(\frac{e^{(-p+i)t'}}{-p+i} \Big|_0^\infty - \frac{e^{(-p-i)t'}}{-p-i} \Big|_0^\infty \right)$$

$$= \frac{1}{2i} \left(- \frac{1}{-p+i} + \frac{1}{-p-i} \right)$$

$$= -\frac{1}{2i} \left(\frac{1}{-p+i} + \frac{1}{p+i} \right) = -\frac{1}{2i} \, \frac{2i}{p^2+1}$$

$$= - \frac{1}{p^2+1}$$

$$\cancel{(p^2+4)} \quad Y(p) = - \frac{1}{(p^2+1)(p^2+4)}$$



$$y(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dp \; e^{pt} \frac{1}{(p^2+1)(p^2+4)}$$

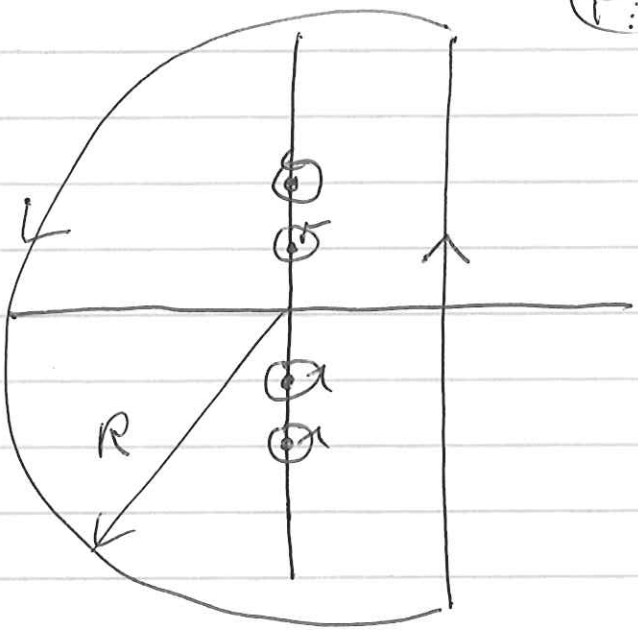
$= 0$ for $t < 0$

$\Rightarrow$ close in LHP for $t > 0$

$\Rightarrow$ pick up pole contributions

$\Rightarrow$ contribution from circle at large R is zero

$\Rightarrow$ Jordan's Lemma



$$\frac{1}{(p^2+1)(p^2+4)} = \frac{1}{(p+i)(p-i)(p+2i)(p-2i)}$$

$\Rightarrow$ first order poles

$$y(t) = \cancel{\frac{2\pi i}{2\pi i}}\left[\frac{e^{2it}}{4i(-4+1)} + \frac{e^{it}}{2i(-1+4)} + \frac{e^{-it}}{-2i(-1+4)} + \frac{e^{-2it}}{(-4+1)(-4i)}\right]$$

$$y(t) = \frac{1}{3}\left[\frac{1}{4i}\left(-e^{2it} + e^{-2it}\right) + \frac{1}{2i}\left(e^{it} - e^{-it}\right)\right]$$

$$= \frac{1}{3}\left(-\frac{1}{2}\sin(2t) + \sin t\right)$$

Check boundary conditions:

$$y(0) = 0$$

$$\dot{y}(0) = \frac{1}{3}\left(-\cos(2t) + \cos t\right)\Big|_{t=0} = 0$$

Note: $\sin(t)$ term is particular solution and $\sin(2t)$ term is homogenious solution.