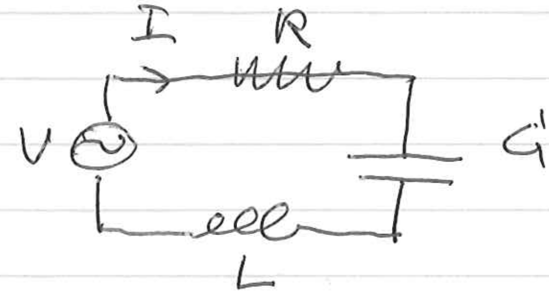
Ordinary Differential Equations

Boas Ch. 8

An equation having differential operators is a differential equation. If the operators are only functions of a single variable, it is an ordinary differential equation.

<u>example</u> Consider a circuit containing a resistor R, inductor L, ~~and~~ capacitor C, driven by a voltage V.



Sum of potential drops around loop is zero.

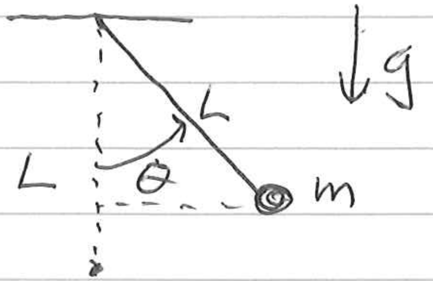
$$V - IR - \frac{Q}{C} - L\frac{dI}{dt} = 0$$

where $I = dQ/dt$ so ~~that the derivative~~

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = V(t)$$

This is a second order (highest derivative), linear (Q appears only as a linear term) inhomogeneous (the driven $V(t)$ is nonzero) equation.

<u>example</u> As an example of a nonlinear equation, consider a pendulum with a mass m attached to a massless rod of length. L.



From energy conservation

$$\frac{1}{2} m L^2 \dot{\theta}^2 + mgL(1-\cos\theta) = \text{const.}$$

Take a derivative

$$\cancel{\frac{m}{2}} L^{\cancel{2}} \cancel{2} \ddot{\theta} \cancel{\dot{\theta}} + \cancel{m} g \cancel{L} \sin\theta \cancel{\dot{\theta}} = 0$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

$\Rightarrow$ second order homogeneous nonlinear equation

$\Rightarrow$ expand $\sin\theta$ around $\theta = 0$

$$\ddot{\theta} + \frac{g}{L}\left(\theta - \frac{1}{6}\theta^3 + \cdots\right) = 0$$

$\Rightarrow$ powers of θ not equal to 1 tells us it is a nonlinear equation.

$\Rightarrow$ For small θ

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \qquad \theta(t) \sim \sin\omega t, \cos\omega t$$
$$\omega^2 = g/L$$

⇒ what about large θ oscillations?

$\theta_{max} \lesssim \pi$?

⇒ period can become very long.

⇒ nonlinear system.

⇒ There are few generic approaches to solving nonlinear equations.

⇒ focus on linear equations

⇒ of course the nonlinear pendulum can be solved exactly from the energy conservation equation.

<u>Linear first order equations</u>

Consider the equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

⇒ has exact solution

⇒ solve through several examples

example $P = \text{const}$ and $Q = 0$

$$\frac{dy}{dx} + P y = 0$$

$\Rightarrow$ combine y together and x together (Separation of variables)

$$\frac{dy}{y} = -P\,dx$$

$\Rightarrow$ integrate $(0, x)$

$$\ln(y)\Big|_0^x = -Px$$

$$y(x) = y(0)\, e^{-Px}$$

$\Rightarrow$ one free parameter $y(0)$

$\Rightarrow$ exponential solution

$\Rightarrow$ this is generic for equations with constant coefficients.

example Inhomogeneous equation with constant coeff.

$$\frac{dy}{dx} + Py = Q$$

$\Rightarrow$ by inspection can see that have an additional ~~sol~~ solution

$$y_P = \frac{Q}{P}$$

This is known as the particular solution. The exponential solution is the homogeneous solution, which ~~remains a solution so~~ can be added to y_p

$$y = y_h + y_p = y(0)e^{-Px} + \frac{Q}{P}$$

$\Rightarrow$ one free parameter.

<u>example</u> Homogeneous eqn with $P(x)$

$$\frac{dy}{dx} + P(x)y = 0$$

$$\frac{dy}{y} = -P(x)\,dx$$

$\Rightarrow$ integrate $(0, x)$

$$\ln\left(\frac{y}{y(0)}\right) = -\int_0^x dx'\,P(x')$$

$$y = y(0)\,e^{-\int_0^x dx'\,P(x')}$$

<u>example</u> Inhomogeneous eqn with $P(x), Q(x)$

$$\frac{d}{dx}y + P(x)y = Q(x)$$

Multiply eqn by $e^{\int_0^x dx' P(x')}$ and note that

$$\frac{d}{dx} e^{\int_0^x dx' P(x')} = P(x)\, e^{\int_0^x dx' P(x')}$$

$$\underbrace{e^{\int_0^x dx' P(x')} \frac{dy}{dx} + y \frac{d}{dx} e^{\int_0^x dx' P(x')}}_{\frac{d}{dx}\left(y e^{\int_0^x dx' P(x')}\right)} = Q e^{\int_0^x dx' P(x')}$$

Integrate $(0, x)$

$$y e^{\int_0^x dx' P(x')} - y(0) = \int_0^x dx'' Q(x'')\, e^{\int_0^{x''} dx' P(x')}$$

$$y(x) = y(0)\, e^{-\int_0^x dx' P(x')} + e^{-\int_0^x dx' P(x')} \int_0^x dx'' Q(x'')\, e^{\int_0^{x''} dx' P(x')}$$

$\Rightarrow$ again have one free variable $y(0)$

example Let

$$\frac{dy}{dx} + \underbrace{\frac{6x}{1+x^2}}_{P} y = \underbrace{\frac{2x}{1+x^2}}_{Q}$$

$$\int_0^x dx' \frac{6x'}{1+x'^2} = 3 \int_0^x dx' \frac{d}{dx'} \ln(1+x'^2) = 3 \ln(1+x^2)$$

$$e^{\int_0^x dx' P(x')} = e^{3\ln(1+x^2)} = (1+x^2)^3$$

$$y(x) = y(0)\frac{1}{(1+x^2)^3} + \frac{1}{(1+x^2)^3}\int_0^x dx'' \underbrace{\frac{2x''}{\cancel{1+x''^2}}\left(1+x''^2\right)^{\cancel{3}\,2}}_{\frac{1}{3}\frac{d}{dx}(1+x''^2)^3}$$

$$y(x) = y(0)\frac{1}{(1+x^2)^3} + \frac{1}{(1+x^2)^3}\frac{1}{3}\left[(1+x^2)^3 - 1\right]$$

Linear homogeneous eqns with const. coeff.

Consider an nth order eqn

$$\mathcal{L}(y) \equiv y^{(n)} + a_{n-1}y^{(n-1)} + \ldots\ldots a_0 y^{(0)} = 0$$

with all a_j independent of x. Consider some m solutions $\varphi_j(x)$. Any combination

$$y = \sum_j c_j \varphi_j(x)$$

is also a solution since

$$\mathcal{L}y = \sum_j c_j \mathcal{L}\varphi_j = 0$$

Exponential solutions

Look for solutions of the form $y = e^{kx}$

$$\frac{d^j}{dx^j} y = k^j e^{kx} = k^j y$$

$$\mathcal{L} y = 0 \Rightarrow P(k) = \sum_{j=0}^{n} a_j k^j = 0$$

$P(k)$ is an nth order polynomial

$\Rightarrow$ has n solutions $k_1, k_2 \cdots k_n$

$\Rightarrow$ y has n solutions

$$e^{k_1 x}, e^{k_2 x} \cdots e^{k_n x}$$

What if have a repeated root?
Do we only have $n-1$ solutions? No.
Normally $P(k)$ takes the form

$$P(k) = (k-k_1)(k-k_2)\cdots(k-k_n)$$

With a repeated root k_0 of 2nd order

$$P(k) = g(k)(k-k_0)^2$$

with $g(k_0) \neq 0$

so

$$\mathcal{L}(e^{kx}) = g(k)(k-k_0)^2 e^{kx}$$

Take derivative with respect to k

$$\frac{d}{dk}\mathcal{L}(e^{kx}) = \mathcal{L}(xe^{kx}) = g'(k)(k-k_0)^2 e^{kx}$$
$$+2g(k-k_0)e^{kx}$$
$$+g(k-k_0)^2 xe^{kx}$$

Set $k=k_0 \Rightarrow$ RHS is zero

$$\mathcal{L}(xe^{k_0x}) = 0$$

$\Rightarrow$ xe^{k_0x} is a solution.

$\Rightarrow$ still have n solutions

$\Rightarrow$ for an mth order repeated root have

$$e^{k_0x}, xe^{k_0x}, \dots\dots, x^{m-1}e^{k_0x}$$

as the solutions.

<u>example</u> RLC circuit with no voltage source.

$$\ddot{Q} + \frac{R}{L}\dot{Q} + \frac{Q}{LC} = 0$$

Let $\omega_0^2 = \frac{1}{LC}$, $\gamma = \frac{1}{2}\frac{R}{L}$

$$\ddot{Q} + 2\gamma\dot{Q} + \omega_0^2 Q = 0$$

For $\gamma = 0$ (no dissipation from R)

$$\ddot{Q} + \omega_0^2 Q = 0 \quad \Rightarrow \text{take } Q \sim e^{-i\omega t}$$

$$-\omega^2 + \omega_0^2 = 0 \qquad \omega = \pm\omega_0$$

$$Q \sim e^{\pm i\omega_0 t} \text{ or } \sin(\omega_0 t), \cos(\omega_0 t)$$

For $\omega_0 = 0$ (no capacitor)

$$\ddot{Q} + 2\gamma\dot{Q} = 0 \Rightarrow \text{take } Q \sim e^{-\Gamma t}$$

$$\Gamma^2 - 2\gamma\Gamma = 0$$

$$\Rightarrow \Gamma = 0, \ \Gamma = 2\gamma$$

$$Q \sim 1, \ e^{-2\gamma t}$$

$I = 0 \Rightarrow$ trivial solution

General case $\Rightarrow$ take $Q \sim e^{kt}$

$$k^2 + 2\gamma k + \omega_0^2 = 0$$

$$k = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2}$$

$$= -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

Three possible solutions:

overdamped for $\gamma > \omega_0$
critically damped for $\gamma = \omega_0$
underdamped for $\gamma < \omega_0$

Overdamped solution: k real and negative

$$Q \sim \underbrace{e^{-(\gamma + \sqrt{\gamma^2 - \omega_0^2})t}}_{Q_1(t)}, \quad \underbrace{e^{-(\gamma - \sqrt{\gamma^2 - \omega_0^2})t}}_{Q_2(t)}$$

$$Q = c_1 Q_1(t) + c_2 Q_2(t)$$

Critically damped: $\gamma = \omega_0 \Rightarrow k = -\gamma$
$\Rightarrow$ degenerate

$$Q = (c_1 + c_2 t)e^{-\gamma t}$$

Under damped: $\gamma < \omega_0$, $k = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}$

$$Q = e^{-\gamma t}\left(c_1 e^{i\sqrt{\omega_0^2-\gamma^2}\,t} + c_2 e^{-i\sqrt{\omega_0^2-\gamma^2}\,t}\right)$$

Or

$$Q = e^{-\gamma t}\left(c_1 \cos\sqrt{\omega_0^2-\gamma^2}\,t + c_2 \sin\sqrt{\omega_0^2-\gamma^2}\,t\right)$$

$\Rightarrow$ damped, oscillatory motion

$\Rightarrow$ oscillation due to the exchange of energy between L and C and dissipation due to R.

Linear Independence

The n exponential solutions of $\mathcal{L}(y)=0$ are linearly independent. Recall that a set of functions φ_j are linearly dependent if

$$\sum_j c_j \varphi_j(x) = 0$$

with $c_j \neq 0$ ~~for~~ at least some values of j. If the only solution is $c_j = 0$ for all j, the solutions are linearly independent.

Take exponential solutions (no repeated roots)

Proof by contradiction:

Take
$$\sum_{j=1}^{n} c_j e^{k_j x} = 0$$

Suppose at least one $c_q \neq 0$

Divide by $e^{k_1 x}$

$$\sum_{j=1}^{n} c_j e^{(k_j - k_1)x} = 0$$

Take derivative with respect to x.

$j=1$ term gone

$$\sum_{j=2}^{n} c_j (k_j - k_1) e^{(k_j - k_1)x}$$

Repeat for all $j \neq q$, are left with

$$c_q(k_q - k_1)(k_q - k_2)(k_q - k_{q-1})(k_q - k_{q+1}) \times \cdots\cdots (k_q - k_n) e^{k_q x} = 0$$

Requires $c_q = 0 \Rightarrow$ contradiction

$\Rightarrow$ only solution is $c_j = 0$ for all j

$\Rightarrow$ exponential solutions are linearly independent.

<u>The Wronskian</u>

The Wronskian $W(\varphi_1, \dots, \varphi_n)$ of n functions having $n-1$ derivatives on some interval is

$$W = \begin{vmatrix} \varphi_1 & \cdots & \varphi_n \\ \varphi_1' & \cdots & \varphi_n' \\ \vdots & & \\ \varphi_1^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}$$

Theorem: If $\varphi_1 \dots \varphi_n$ are n solutions of $\mathcal{L}y = 0$ on an interval, they are linearly independent if and only if $W \neq 0$ for all x on the interval.

proof: Suppose $W(\phi_1 \dots \phi_n) \neq 0$ for all x on the interval, ~~bet~~

$\Rightarrow$ assume linearly dependent and find contradiction

Let $c_1 \dots c_n$ be constants such that

$$c_1 \phi_1(x) + \cdots + c_n \phi_n(x) = 0$$

then can take derivatives

$$c_1 \phi_1'(x) + \cdots c_n \phi_n'(x) = 0$$

$$\vdots$$

$$c_1 \phi_1^{(n-1)}(x) + \cdots c_n \phi_n^{(n-1)}(x) = 0$$

$\Rightarrow$ set of n equations for $c_1 \cdots c_n$. The condition that they have a nonzero solution is that the determinant of the coefficients is zero. But this is the Wronskian which is non-zero so the only solution is $c_j = 0$ for all j.

$\Rightarrow$ If $W \neq 0$ then the $\phi_j(x)$ are linearly independent

$\Rightarrow$ good test of linear independence.

<u>example</u>

$$\frac{d^2}{dx^2}y + \alpha^2 y = 0$$

$$y \sim e^{kx} \Rightarrow \text{exponential solution}$$

$$k^2 + \alpha^2 = 0 \quad \Rightarrow k = \pm i\alpha$$

$$u_1 = e^{i\alpha x}, \quad u_2 = e^{-i\alpha x}$$

$$W = \begin{vmatrix} e^{i\alpha x} & e^{-i\alpha x} \\ i\alpha e^{i\alpha x} & -i\alpha e^{-i\alpha x} \end{vmatrix} = -2i\alpha$$

$W \neq 0$ for $\alpha \neq 0 \Rightarrow$ linearly indep.

Suppose $\alpha = 0. \Rightarrow u_1 = u_2 = 1$

$$\Rightarrow W = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$\Rightarrow$ not linearly indep.

Other solution is x

$$u_1 = 1, \quad u_2 = x$$

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1 \quad \Rightarrow \text{linearly indep.}$$

For $\alpha \neq 0$ could also choose

$$u_1 = \sin \alpha x \qquad W = \begin{vmatrix} \sin\alpha x & \cos\alpha x \\ \alpha\cos\alpha x & -\alpha\sin\alpha x \end{vmatrix}$$
$$u_2 = \cos\alpha x$$

$$= -\alpha\left(\sin^2\alpha x + \cos^2\alpha x\right)$$

$$= -\alpha \quad \neq 0$$

$\Rightarrow$ linearly indep.

<u>Equation for W</u> (proof not shown in class)

$$W = \begin{vmatrix} u_1 & \cdots & u_n \\ \vdots & & \\ u_1^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix}$$

Take the derivative of W. Generally ~~mult~~ must take the derivative of each row

0 $\Rightarrow$ two rows the same

$$\frac{dW}{dx} = \begin{vmatrix} u_1' & \cdots & u_n' \\ u_1' & \cdots & u_n' \\ \vdots & & \\ u_1^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix} + \cdots \begin{vmatrix} u_1 & \cdots & u_n \\ \vdots & & \\ u_1^{(n-2)} & \cdots & u_n^{(n-2)} \\ u_1^{(n)} & \cdots & u_n^{(n)} \end{vmatrix}$$

$$\frac{dw}{dx} = \begin{vmatrix} u_1 & \cdots & u_n \\ \vdots & & \\ u_1^{(n-2)} & \cdots & u_n^{(n-2)} \\ u_1^{(n)} & \cdots & u_n^{(n)} \end{vmatrix}$$

$$u_1^{(n)} = -a_{n-1} u_1^{(n-1)} - \cdots a_0 u_1^{(0)}$$

$\Rightarrow$ use other rows to subtract the terms $u_1^{(0)} \cdots u_1^{(n-2)}$ leaving only $u_1^{(n-1)}$ term in bottom row

$$\frac{dw}{dx} = \begin{vmatrix} u_1 & \cdots \cancel{u} \cdots & u_n \\ \vdots & & \\ -a_{n-1} u_1^{(n-1)} & \cdots & -a_{n-1} u_n^{(n-1)} \end{vmatrix}$$

$$= -a_{n-1} \bar{w}$$

$$\Rightarrow \frac{d\bar{w}}{dx} + a_{n-1} \bar{w} = 0$$

$$\Rightarrow w \sim e^{kx}$$

$$k + a_{n-1} = 0 \quad \Rightarrow k = -a_{n-1}$$

Suppose know $\bar{w}(x_0)$

$$w(x) = w(x_0)\, e^{-a_{n-1}(x - x_0)}$$

If $W(x_0) \neq 0$, then $W(x) \neq 0$ every where

$\Rightarrow$ can evaluate $W(x)$ anywhere (any x) to show is nonzero to demonstrate $u_1 \cdots u_n$ are linearly independent.

example Back to

$$\frac{d^2}{dx^2} y + \alpha^2 y = 0$$

$u_1 = \sin \alpha x$ $\qquad W = \begin{vmatrix} \sin \alpha x & \cos \alpha x \\ \alpha \cos \alpha x & -\alpha \sin \alpha x \end{vmatrix}$

$u_2 = \cos \alpha x$

$$W(0) = \begin{vmatrix} 0 & 1 \\ \alpha & 0 \end{vmatrix} = -\alpha$$

$\Rightarrow u_1, u_2$ linearly indep.

Initial Value Problem

In an initial value problem for an nth order equation $\mathcal{L}y = 0$, specify

$$y(x_0) = \alpha_0$$

$$y'(x_0) = \alpha_1$$

$$\vdots$$

$$y^{(n-1)}(x_0) = \alpha_{n-1}$$

For x_0 any real number and α_j any real constants, there exists a solution of $\mathcal{L}y = 0$ satisfying these conditions on y at x_0 valid for $-\infty < x < \infty$.

Let $y(x) = c_1 \varphi_1 + \cdots c_n \varphi_n$

with $\varphi_1, \ldots, \varphi_n$ linearly independent solutions of $\mathcal{L}$. Want to solve for the c_j's with the specified boundary conditions at x_0.

$$c_1 \varphi_1(x_0) + \cdots c_n \varphi_n(x_0) = \alpha_0$$

$$\vdots$$

$$c_1 \varphi_1^{(n-1)}(x_0) + \cdots c_n \varphi_n^{(n-1)}(x_0) = \alpha_{n-1}$$

The determinant of the matrix eqn for the c_j's is W, which is nonzero for $\varphi_1 \ldots \varphi_n$ linearly independent

$\Longrightarrow$ therefore the c_j's have a unique solution satisfying the boundary conditions

<u>example</u> Consider the LC circuit

$$\ddot{Q} + \frac{Q}{LC} = 0 \qquad \text{Let } \omega_0^2 = \frac{1}{LC}$$

$$Q = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

$\Rightarrow$ 2nd order equation

$\Rightarrow$ specify $Q = Q_0$ at $t=0$ and
$I = \frac{dQ}{dt}$ at $t=0$

At $t=0$:

$$Q = Q_0 = c_1$$

$$I = \frac{dI}{dt} = 0 = -\omega_0 c_1 \sin \omega_0 t + \omega_0 c_2 \cos \omega_0 t$$

$$= \omega_0 c_2$$

$$\Rightarrow c_2 = 0$$

so $\boxed{Q = Q_0 \cos \omega_0 t}$

Linear inhomogeneous equs with constant coefficients

Consider an nth order equation with a driving term on the RHS.

$$\mathcal{L}(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots a_0 y^{(0)} = F(x)$$

~~the~~ The general solution can be written as a particular solution for which

$$\mathcal{L}(y_p) = F(x)$$

plus any ~~poa~~ contribution from the homogeneous solution

$$\mathcal{L}(y_h) = 0$$

as

$$y = y_h + y_p$$

where the freedom to choose y_h allows to ~~to~~ match intial conditions as for initial value problems

example For the ~~R~~LC circuit with $V(t)$

$$\ddot{Q} + \frac{Q}{LC} = \frac{V_0}{L} \sin \omega t$$

For $\omega_0^2 = \frac{1}{LC}$

$$\ddot{Q} + \omega_0^2 Q = \frac{V_0}{L} \sin\omega t$$

We take $Q_p(t)$ to be some constant times $\sin(\omega t)$

$$Q_p = C_p \sin\omega t$$

$$-\omega^2 C_p \cancel{\sin\omega t} + \omega_0^2 C_p \cancel{\sin\omega t} = \frac{V_0}{L} \cancel{\sin\omega t}$$

$$C_p = \frac{V_0}{L} \frac{1}{\omega_0^2 - \omega^2}$$

$$Q_p = \frac{V_0}{L} \frac{1}{\omega_0^2 - \omega^2} \sin\omega t$$

$\Rightarrow$ this is the method of <u>undetermined coefficents</u>

$\Rightarrow$ C_p was the undetermined coefficient.

<u>example</u> RLC circuit with $V(t)$

$$\ddot{Q} + \frac{R}{L}\dot{Q} + \frac{Q}{LC} = \frac{V_0}{L} \sin(\omega t)$$

A solution with $Q \sim \sin(\omega t)$
since $\dot{Q}$ term changes $\sin(\omega t)$ to $\cos(\omega t)$

Using earlier notation

$$\ddot{Q} + 2\gamma \dot{Q} + \omega_0^2 Q = \frac{V_0}{L} \sin\omega t$$

Try $Q = C_s \sin\omega t + C_c \cos\omega t$

$$-\omega^2 (C_s \sin\omega t + C_c \cos\omega t)$$
$$+ 2\gamma\omega (C_s \cos\omega t - C_c \sin\omega t)$$
$$+ \omega_0^2 (C_s \sin\omega t + C_c \cos\omega t) = \frac{V_0}{L} \sin\omega t$$

$\Rightarrow$ since must be satisfied for all time, equate coefficients of sines and cosines separately.

sines:

$$-\omega^2 C_s - 2\gamma\omega C_c + \omega_0^2 C_s = \frac{V_0}{L}$$

cosines:

$$-\omega^2 C_c + 2\gamma\omega C_s + \omega_0^2 C_c = 0$$

$$C_c = \frac{2\gamma\omega C_s}{\omega^2 - \omega_0^2} \quad \Rightarrow \text{prop. to } \gamma$$

$$\Rightarrow \left(\omega_0^2 - \omega^2 + \frac{2\gamma\omega (2\gamma\omega)}{-\omega^2 + \omega_0^2} \right) C_s = \frac{V_0}{L}$$

$$\boxed{C_s = \frac{V_0}{L} \frac{(-\omega^2 + \omega_0^2)}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

$\Rightarrow$ For $\gamma = 0$, same as before

$$\boxed{C_c = -\frac{2\gamma\omega}{\omega_0^2 - \omega^2} C_s}$$

Exponential Sources

For general n-th order eqn with exponential source

$$\mathcal{L}(y) = A e^{kx}$$

Assume $y_p = B e^{kx}$

$$\mathcal{L}(Be^{kx}) = B \sum_{j=0}^{n} a_j k^j e^{kx} = B\, P(k) e^{kx}$$
$$= A e^{kx}$$

$\Rightarrow B = \frac{A}{P(k)}$ with $P_k = \sum_{j=0}^{n} a_j k^j$

$\Rightarrow$ ok as long as no repeated roots and k is not one of the homogeneous roots (e.g., $P(k) = 0$

If e^{kx} is a solution of the homogeneous eqn $\Rightarrow f(e^{kx}) = 0$ have a solution of the form

$$y_p = Bxe^{kx}$$

$\Rightarrow$ if k is not a repeated root

If e^{kx} is a solution of homogeneous equation and is also a 2nd order repeated root,

$$y_p = Bx^2e^{kx}$$

<u>example</u>

$$y'' + y' - 2y = e^x$$

Homogeneous solution

$$(k^2 + k - 2)y_h = 0$$

$$k = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$$

$$= 1, -2$$

$\Rightarrow$ one solution is e^{x}

$\Rightarrow$ trying $y_p = Be^x$ gives

$0 = e^x \Rightarrow$ doesn't work

$\Rightarrow$ try $\quad y_p = B x e^x$

$$B(xe^x)'' + B(xe^x)' - 2Be^x x = e^x$$

$$B(x\cancel{e^x} + 2\cancel{e^x}) + B(x+1)\cancel{e^x} - 2B\cancel{e^x}x = \cancel{e^x}\,1$$

$$B[x+2+x+1-2x] = 1$$

$$3B = 1 \Rightarrow B = \frac{1}{3}$$

$$\Rightarrow y_p = \frac{1}{3} x e^x$$

Principle of Superposition

$\Rightarrow$ If have more than one driven on the RHS of the equation

$$\mathcal{L}(y) = Q_1(x) + Q_2(x)$$

$\Rightarrow$ treat each driven separately

$$\mathcal{L}(y_1) = Q_1 \qquad \Rightarrow \quad y = y_1 + y_2$$
$$\mathcal{L}(y_2) = Q_2$$

$$\mathcal{L}(y_1 + y_2) = \mathcal{L}y_1 + \mathcal{L}y_2 = Q_1 + Q_2$$

$\Rightarrow$ ok if $\mathcal{L}$ is a linear operator