Fourier Series and Fourier ~~and Laplace~~ Transforms

Boas Ch. 7

Periodic functions

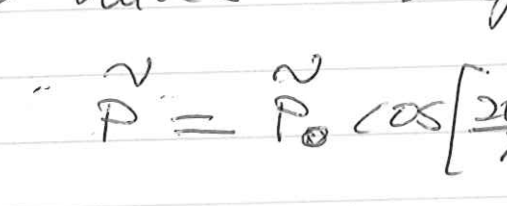
Many phenomena in physics are periodic in time and/or space.

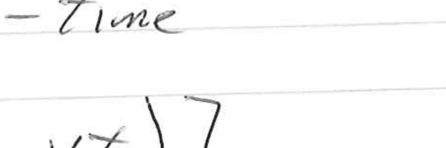
⇒ spring-mass system ⇒ time
⇒ R-C circuit ⇒ time
⇒ waves ⇒ space-time

$$\tilde{P} = \tilde{P}_0 \cos\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

⇒ at a given time periodic in space
⇒ at a given location periodic in time

⇒ A periodic function consist of a complex pattern that repeats





⇒ need to develop techniques to describe complicated but periodic functions

⇒ consider a generic system that is periodic in space. Choose a spatial coordinate such that the periodicity length is 2π

$$\cos\left(\frac{2\pi x}{\lambda}\right) \Rightarrow \text{let } \frac{2\pi x}{\lambda} \to x$$

$$\Rightarrow \cos x, \sin x$$

⇒ periodic over 2π

⇒ same for time

$$\cos \frac{2\pi}{\tau} t \Rightarrow \frac{2\pi}{\tau} t \to t$$

$$\Rightarrow \cos t, \sin t$$

⇒ want to represent an arbitrary function that is periodic over 2π.

⇒ $f(x)$

<u>Sin and Cos Series</u>

⇒ express $f(x)$ in terms of sines and cosines

$$\boxed{f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)}$$

To calculate the a_n's and b_n's we use the orthogonality of all of these functions over the interval $(0, 2\pi)$.

$$I_{sc} = \int_0^{2\pi} dx \, \sin(nx)\cos(mx) = 0 \quad \text{all } m, n$$

$$I_{ss} = \int_0^{2\pi} dx \, \sin(nx)\sin(mx) = 0 \quad m \neq n$$

$$I_{cc} = \int_0^{2\pi} dx \, \cos(nx)\cos(mx) = 0 \quad m \neq n$$

Consider $I_{cc} = \int_0^{2\pi} dx \left(\frac{(e^{inx} + e^{-inx})(e^{imx} + e^{-imx})}{4} \right)$

$\Rightarrow$ all of the form

$$\int_0^{2\pi} dx \, e^{iqx} = \left. \frac{e^{iqx}}{iq} \right|_0^{2\pi} = 0$$

for $m \neq n$, where q is an integer.

$\Rightarrow I_{cc} = 0$

$\Rightarrow$ same for I_{ss}, I_{sc}

$\Rightarrow$ multiply $f(x)$ by $\sin(n'x)$ and integrate over 2π

$$\int_0^{2\pi} dx \, f(x)\sin(n'x) = \sum_{n=1}^{\infty} b_n \int_0^{2\pi} dx \, \sin(nx)\sin(n'x)$$

$$= \sum_{n=1}^{\infty} b_n \frac{2\pi}{2} \delta_{nn'} = b_{n'} \pi$$

since $\int_0^{2\pi} dx \sin^2(n'x) = \frac{1}{2}(2\pi)$.

$$\Rightarrow \boxed{b_n = \frac{1}{\pi} \int_0^{2\pi} dx\, f(x) \sin(nx) \quad \text{for } n = 1, 2, \cdots}$$

Similar for a_n

$$\boxed{a_n = \frac{1}{\pi} \int_0^{2\pi} dx\, f(x) \cos(nx) \quad n = 1, 2, \cdots}$$

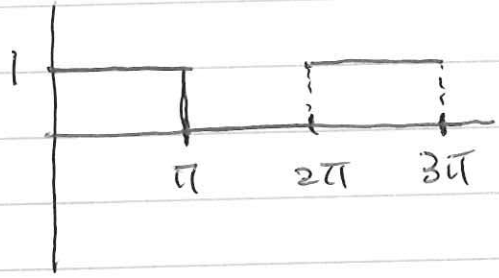
For a_0 integrate f over $(0, 2\pi)$

$$\int_0^{2\pi} dx\, f(x) = \frac{1}{2} a_0\, 2\pi = a_0 \pi$$

$$\boxed{a_0 = \frac{1}{\pi} \int_0^{2\pi} dx\, f(x)}$$

Same as for other a_n's.

Example Consider the "squarewave" function



$$a_0 = \frac{1}{\pi}\int_0^{2\pi} dx\, f(x)$$

$$= \frac{1}{\pi}\int_0^{\pi} dx\, 1 = 1$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} dx \cos(nx) = \frac{1}{\pi}\, \frac{\sin nx}{n}\Big|_0^{\pi} = 0$$

for $n \neq 0$

$$b_n = \frac{1}{\pi}\int_0^{\pi} dx \sin(nx) = -\frac{1}{\pi}\, \frac{\cos nx}{n}\Big|_0^{\pi}$$

$$= -\frac{1}{\pi}\, \frac{\cos n\pi - 1}{n}$$

$$= -\frac{1}{\pi}\, \frac{(-1)^n - 1}{n} = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi} & n \text{ odd} \end{cases}$$

$$\boxed{f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n\ \mathrm{odd}} \frac{1}{n}\sin(n\pi)}$$

Under what conditions does the Fourier Sin and Cos series converge to f?
$\Rightarrow$ are these functions a "complete set"

Dirichlet's Theorem: If f has a finite # of max and min values and discontinuities then the series converges where continuous.

Convergence in the mean

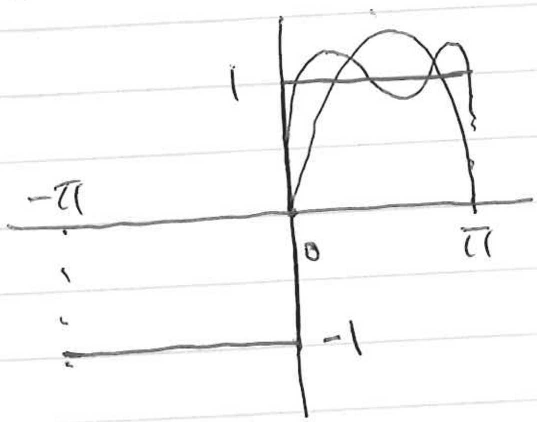
Introduce a weak convergence criterion

$$\lim_{N\to\infty} \frac{1}{2\pi}\int_0^{2\pi} dx \left(f(x) - f_N(x)\right)^2 = 0$$

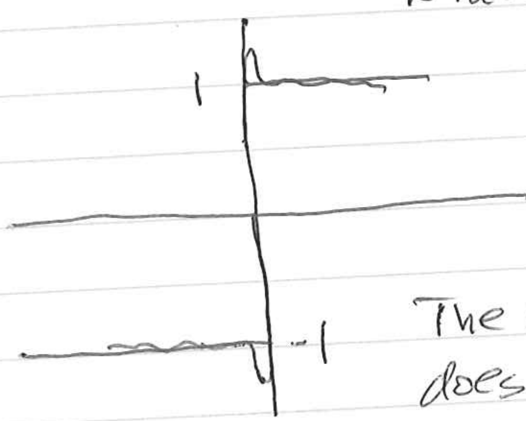
$$f_N = \frac{1}{2}a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

$f(x)$ can deviate from f_N at a finite # of points

Gibbs Phenomenon



N large



The overshoot does not go away for large N. Width ~~gets~~ gets smaller.

Note: can often exploit the symmetry of the problem to simplify the evaluation of a_n, b_n

$\Longrightarrow$ in the above problem all of the a_n are zero.

Complex exponential Series

We know that the sines and cosines can be written as exponentials

$$\cos(nx) = \frac{1}{2}\left(e^{inx} + e^{-inx}\right)$$

$$\sin(nx) = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)$$

It therefore seems reasonable to represent periodic functions in terms of complex exponentials.

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}}$$

To find the c_n we first note that the average of

$$e^{iqx}$$

over 2π is zero — we showed this ~~...~~ earlier. Thus, if we multiply the series by $e^{-in'x}$

and integrate over 2π, we find

$$\int_0^{2\pi} dx\, f(x)\, e^{-in'x} = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} dx\, e^{i(n-n')x}$$

$$= \sum_{n=-\infty}^{\infty} 2\pi c_n \delta_{nn'} = 2\pi c$$

Thus,

$$\boxed{c_n = \frac{1}{2\pi}\int_0^{2\pi} dx\, f(x)\, e^{-inx}}$$

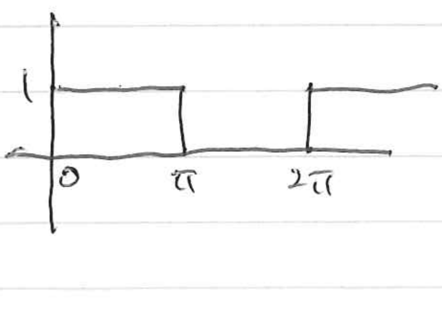
For $f(x)$ real

$$f^* = f(x) = \sum_n c_n^* e^{-inx} = \sum_n c_n e^{inx}$$

comparing the coefficients of e^{-inx}

$$\Longrightarrow \boxed{c_n^* = c_{-n}} \quad \text{for } f \text{ real}$$

<u>example</u> - square wave



$$c_n = \frac{1}{2\pi}\int_0^{\pi} dx\, e^{-inx}$$

$$= \frac{1}{2\pi} \left. \frac{e^{-inx}}{-in} \right|_0^{\pi}$$

$$= \frac{1}{2\pi} \frac{e^{-in\pi} - 1}{-in} = \begin{cases} 0 & n \text{ even and } n \neq 0 \\ \frac{1}{\pi i n} & \end{cases}$$

$$c_0 = \frac{1}{2}$$

$$f(x) = \frac{1}{2} + \frac{1}{\pi i} \sum_{n=1,3,\dots} \frac{1}{n} e^{inx}$$

$$+ \frac{1}{\pi i} \sum_{n=-1,-3,\dots} \frac{1}{n} e^{inx}$$

$$= \frac{1}{2} + \frac{2}{\pi \cancel{i}} \sum_{n=1,3,\dots} \frac{1}{n} \frac{e^{inx} - e^{-inx}}{2i}$$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1,3,\dots} \frac{1}{n} \sin(nx) \quad \text{as before}$$

<u>Other intervals</u>

Suppose $f(x)$ is periodic over the length 2ℓ. Choose as basis functions

$$\sin\left(\frac{\pi x n}{\ell}\right), \quad \cos\left(\frac{\pi x n}{\ell}\right)$$

Again, these are orthogonal over the interval $(0, 2\ell)$ or $(-\ell, \ell)$.

Replace $\frac{1}{2\pi}\int_0^{2\pi} dx$ by $\frac{1}{2\ell}\int_0^{2\ell} dx$ or $\frac{1}{2\ell}\int_{-\ell}^{\ell} dx$

$$a_n = \frac{1}{\ell} \int_0^{2\ell} dx\, f(x) \cos\left(\frac{n\pi x}{\ell}\right)$$

etc.

Parsevall's Theorem

Calculate the average of f^2 over the interval $(0, 2\pi)$

$$\frac{1}{2\pi}\int_0^{2\pi} dx\, f^2(x) = \frac{1}{2\pi}\int dx \left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)\right]$$

Since each of the functions is orthogonal to the others unless the index n is the same

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} dx \left[\left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}\left(a_n^2 \cos^2 nx + b_n^2 \sin^2 nx\right)\right]$$

$$= \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\boxed{\frac{1}{2\pi}\int_0^{2\pi} dx\, f^2 = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)}$$

This is also called the completeness relation.

$\Rightarrow$ not really useful for calculating $\langle f^2 \rangle$

$\Rightarrow$ tells you if the sines and cosines form a complete set.

If some terms in the series are discarded $\Rightarrow$ e.g. $n=1$ then will find

$$\frac{1}{2\pi}\int_0^{2\pi} dx\, f^2 > \left(\frac{a_0}{2}\right)^2 + \sum_{n=2}^{\infty}\left(a_n^2+b_n^2\right)$$

The equality must be satisfied if the basis functions form a complete set.

For the exponential series

$$\frac{1}{2\pi}\int_0^{2\pi} dx\, |f|^2 = \sum_n |c_n|^2$$

Fourier Transform

We have explored how to represent periodic functions but many systems are not periodic $\Rightarrow$ how do we treat such systems

$\Rightarrow$ allow the periodicity length l to go to infinity

Start with a system of length $2l$,

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{i\frac{n\pi x}{l}}$$

where calculate c_n by multiplying by

$$e^{-i\frac{n'\pi x}{l}}$$

and integrating over $(-l, l)$

$$\int_{-l}^{l} dx\, e^{-\frac{in'\pi x}{l}} f(x) = 2l\, c_{n'}$$

Change n' to n and substitute above.

$$f(x) = \sum_{-\infty}^{\infty} e^{i\frac{n\pi x}{l}} \frac{1}{2l} \int_{-l}^{l} dx'\, e^{-i\frac{n\pi x'}{l}} f(x')$$

where use x' in the integral to not confuse it with $e^{in\pi x/l}$

Define $k_n = \frac{n\pi}{\ell}$ and note the increment of k_n is $k_{n+1} - k_n = \frac{\pi}{\ell}(n+1-n) = \frac{\pi}{\ell} \equiv \Delta k_n$

$$f(x) = \sum_{k_n=-\infty}^{\infty} e^{ik_n x} \Delta k_n \frac{1}{2\pi} \int_{-\ell}^{\ell} dx' \, e^{-ik_n x'} f(x')$$

Let

$$F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \, e^{-ikx'} f(x')$$

~~what~~

Over what scale does $F(k)$ vary?

<u>example</u> $f(x') = \dfrac{1}{x'^2 + a^2}$

$\cancel{e^{ikx'}}$

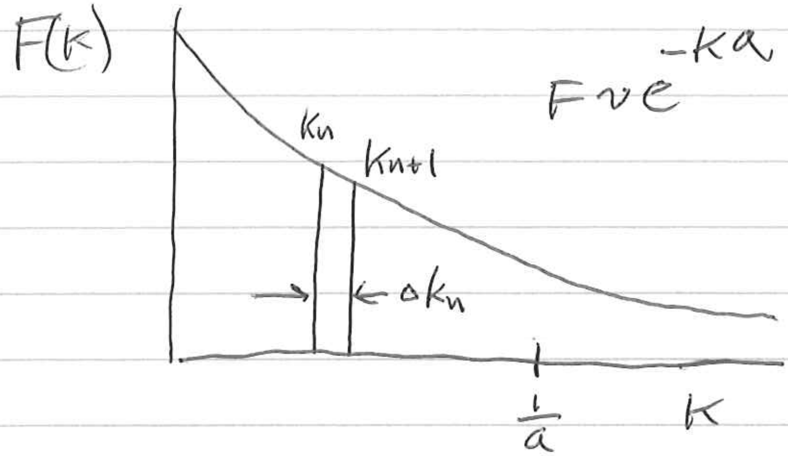
$$F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \, \frac{e^{-ikx'}}{x'^2 + a^2}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \, \frac{e^{-ikz}}{z^2 + a^2}$$

e^{-ikz} bounded in LHP. Close contour and residue at $z = -ia$

$$= -\frac{2\pi i}{2\pi} \left. \frac{e^{-ikz}}{z - ia} \right|_{z=-ia}$$

$$= \frac{-i e^{-ka}}{-2ia} = \frac{1}{2a} e^{-ka}$$



F(k) falls off with a characteristic scale $\sim \frac{1}{a}$

The point is that $\Delta k_n \sim \frac{\pi}{\ell} \ll \frac{1}{a}$

$\Rightarrow$ this means that

$$\sum_{k_n=-\infty}^{\infty} \Delta k_n \Rightarrow \int_{-\infty}^{\infty} dk_n$$

as long as $\ell \gg a$ where "a" is the characteristic scale of $f(x)$. Thus

$$\boxed{\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk\, e^{ikx}\, F(k) \\ F(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx'\, e^{-ikx'}\, f(x') \end{aligned}}$$

$F(k)$ is the Fourier transform of $f(x)$ and the integral over k is the inverse transform, which yields back $f(x)$.

$\Rightarrow$ the Fourier representation is very useful for solving differential equations.

<u>example</u> Represent $f(x) = \begin{cases} 1, & -1<x<1 \\ 0, & |x|>1 \end{cases}$

as a Fourier integral.

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx'\, e^{-ikx'} f(x')$$

$$= \frac{1}{2\pi}\int_{-1}^{1} dx'\, e^{-ikx'} = \frac{1}{2\pi}\left.\frac{e^{-ikx'}}{-ik}\right|_{-1}^{1}$$

$$= \frac{1}{2\pi}\frac{e^{-ik} - e^{ik}}{-ik} = \frac{\sin k}{\pi k}$$

So

$$f(x) = \int_{-\infty}^{\infty} dk\, e^{ikx}\frac{\sin k}{\pi k}$$

or

$$f(x) = \int_{-\infty}^{\infty} dk\, \frac{\sin k}{\pi k}(\cos kx + i\sin kx)$$

$$= \int_{-\infty}^{\infty} dk\, \frac{\sin k \cos kx}{\pi k}$$

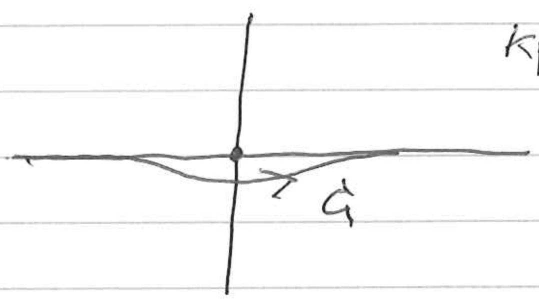
since the $\sin(kx)$ term is an odd function while $\sin(k)/k$ is even

Now complete the inverse transform to see that we obtain $f(x)$

$$f(x) = \int_{-\infty}^{\infty} dk\, e^{ikx} \left(\frac{e^{ik} - e^{-ik}}{2i\pi k} \right)$$

note: no singularity at $k=0$
$\Rightarrow$ move contour integral away from $k=0$

$$f(x) = \int_{C} dk \left(\frac{e^{ik(x+1)} - e^{ik(x-1)}}{2i\pi k} \right)$$



For $x > 1$, close in UHP for both exponentials
$\Rightarrow$ $x+1$ and $x-1$ positive
$\Rightarrow$ residue at $k=0$ is zero since two exponentials cancel.

For $x < -1$, close in LHP since $x+1$ and $x-1$ both negative
$\Rightarrow$ no enclosed pole
$\Rightarrow$ zero

For $-1 < x < 1$, $x+1 > 0$ and $x-1 < 0$. Close $e^{ik(x+1)}$ piece in UHP and $e^{ik(x-1)}$ in LHP. Only $e^{ik(x+1)}$ has a nonzero residue

$$f(x) = \frac{2\pi i}{2\pi i} e^{ik(x+1)} \Big|_{k=0} = 1$$

<u>example</u> Dirac delta function $\delta(x)$

$\delta(x)$ has the property that its amplitude goes to ∞ but its width goes to zero such that its area is unity

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1$$

What is the Fourier representation of $\delta(x)$?

$$F(k) = \int_{-\infty}^{\infty} dx'\, e^{-ikx'}\, \delta(x')\, \frac{1}{2\pi}$$

$$= \frac{1}{2\pi}$$

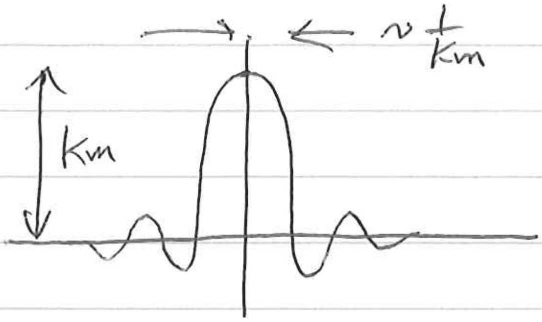
$$\Rightarrow \quad \boxed{\delta(x) = \int_{-\infty}^{\infty} dk\, e^{ikx}\, \frac{1}{2\pi}}$$

Can think of this as

$$\delta(x) = \lim_{k_m \to \infty} \frac{1}{2\pi} \int_{-k_m}^{k_m} dk\, e^{ikx}$$

$$= \lim_{k_m \to \infty} \frac{1}{2\pi}\, \frac{e^{ik_m x} - e^{-ik_m x}}{ix}$$

$$= \lim_{k_m \to \infty} \frac{1}{\pi}\, \frac{\sin k_m x}{x}$$



For $k_m x < \pi$,

$\delta \sim k_m$

Parseval's Theorem for Fourier Integrals

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} dx \underbrace{\int_{-\infty}^{\infty} dk\, e^{ikx} F(k)}_{f(x)} \underbrace{\int_{-\infty}^{\infty} dk'\, e^{-ik'x} F^*(k')}_{f^*(x)}$$

$$= \int_{-\infty}^{\infty} dk\, F(k) \int_{-\infty}^{\infty} dk'\, F^*(k') \underbrace{\int_{-\infty}^{\infty} dx\, e^{ix(k-k')}}_{2\pi\,\delta(k-k')}$$

$$= 2\pi \int_{-\infty}^{\infty} dk\, F(k)\, F(k)^*$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} dk\, |F(k)|^2$$