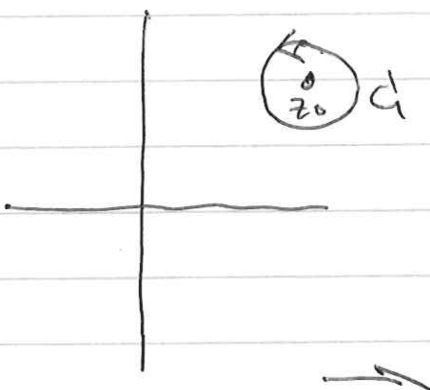
Residues

Consider a function $f(z)$ with an isolated singular point at z_0. The Laurent series is

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-z_0)^j$$



$$\oint_C (z-z_0)^j \, dz = \begin{cases} 0 & j \neq -1 \\ 2\pi i & j = -1 \end{cases}$$

$$\Rightarrow \oint_C dz \, f(z) = 2\pi i \, a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C dz \, f(z)$$

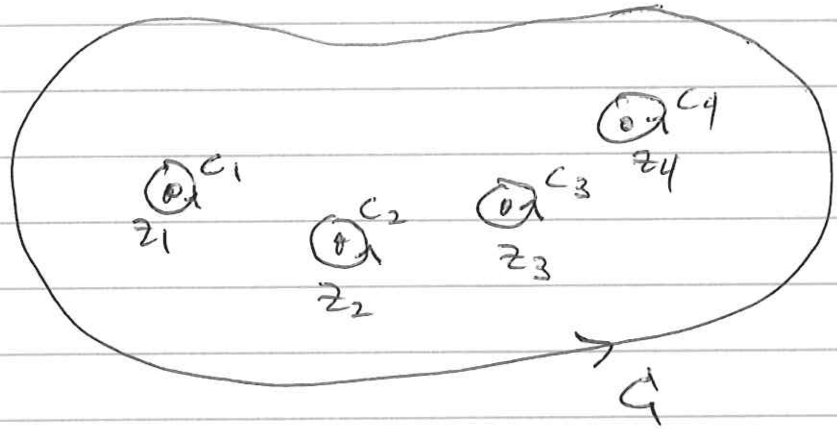
$R \equiv a_{-1}$ is the residue of $f(z)$ at $z = z_0$

Residue Theorem

Consider a function f which has a finite # of isolated singularities z_j within a closed contour C. Then

$$\oint_C f(z)\,dz = 2\pi i \sum_j a_{-1}(z_j)$$

$$= 2\pi i \,(\text{sum of residues})$$



Shrink the contour around the singularities

$$\oint_C f(z)\,dz = \sum_j \oint_{C_j} dz\, f(z)$$

Expand $f(z)$ in a Laurent series around each singularity.

$$\oint_{C_j} dz\, f(z) = 2\pi i\, a_{-1}(z_j)$$

$$\Longrightarrow \quad \oint_C dz\, f(z) = 2\pi i \sum_j a_{-1}(z_j)$$

$\Rightarrow$ The integral is simply $2\pi i$ times the sum of the residues.

$\Rightarrow$ How to find $a_{-1}(z_j)$?

How to calculate $a_{-1}(z_j) \equiv R(z_j)$

① Suppose f has a first order pole of the form at $z = z_j$.

$$f(z) = \frac{P(z)}{q(z)}$$

where P, q are analytic but $q(z_j) = 0$

Expand q in a Taylor series around z_j

$$q(z) \simeq q'(z_j)(z - z_j) + \cdots\cdots$$

Near z_j

$$f(z) = \frac{P(z_j)}{q'(z_j)\,(z - z_j)}$$

$$R(z_j) \equiv a_{-1}(z_j) = \frac{P(z_j)}{q'(z_j)}$$

example

$$f(z) = \frac{\cos z}{e^z - 1}$$ near $z = 0$. Note also poles at $z = 2\pi n i$

$P(z) = \cos z$

$q(z) = e^z - 1 \qquad q'(z) = e^z$

$$q(z) \simeq e^z\Big|_{z=0} z = z$$

$$R(0) = \frac{\cos(0)}{1} = 1$$

(2) $f(z)$ has an mth order pole at z_j.

$$g(z) = (z - z_j)^m f(z) \text{ is}$$

analytic near z_j

$$I_j \equiv \oint_{C_j} dz\, f(z) = \oint_{C_j} \frac{g(z)}{(z - z_j)^m}$$

$\Longrightarrow$ use Cauchy's derivative formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C dz \frac{f(z)}{(z - z_0)^{n+1}}$$

$$\Longrightarrow \quad f(z) \Longrightarrow g(z)$$

$$n+1 = m$$

$$I_j = \frac{2\pi i}{(m-1)!} g^{(m-1)}(z_j)$$

$$= \frac{2\pi i}{(m-1)!} \left[(z - z_j)^m f(z) \right]^{(m-1)} \Big|_{z = z_j}$$

$$R(z_j) = \frac{1}{(m-1)!} \left[(z - z_j)^m f(z) \right]^{(m-1)} \Big|_{z = z_j}$$

Example

$$f(z) = \frac{1+z}{z^2} \quad \text{at} \quad z=0$$

$\Rightarrow$ second order pole $\Rightarrow m=2$

$$g = z^2 f = 1+z$$

$$R(0) = \frac{1}{1!} (1+z)^{(1)} \Big|_{z=0}$$

$$= 1$$

3 Essential singularity $\Rightarrow$ calculate the Laurent series

$$f(z) = e^{\frac{1}{z}} \quad \text{at } z=0.$$

$$= 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots$$

$$R = 1$$

Evaluation of Integrals

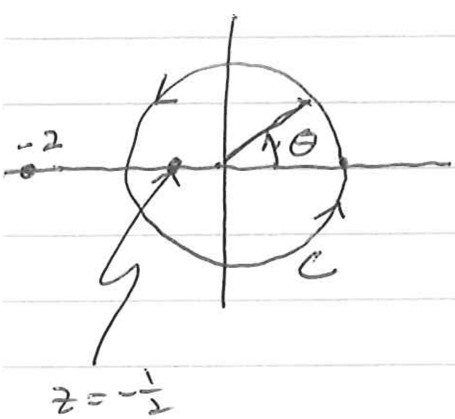
(1) Trig functions (finite integrals)

e.g. $$I = \int_0^{2\pi} d\theta \, f(\sin\theta, \cos\theta)$$

Let
$$I = \int_0^{2\pi} d\theta \, \frac{\cancel{d\theta}\ 1}{5+4\cos\theta}$$

$\Rightarrow$ consider this as an integral over the unit circle in the complex plane

z plane



$$z = e^{i\theta}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$$

$$dz = i\,d\theta\, e^{i\theta} = i z\, d\theta$$

$$d\theta = \frac{dz}{iz}$$

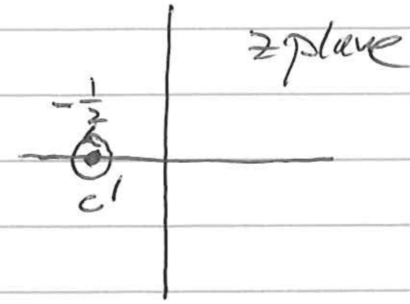
$$I = \cancel{\tfrac{4}{2}} \int_C \frac{dz}{iz} \frac{1}{5 + \frac{4}{2}\left(z + \frac{1}{z}\right)} = \frac{1}{i} \oint_C dz \frac{1}{2z^2 + 5z + 2}$$

$$= \frac{1}{i} \oint_C dz \frac{1}{(2z+1)(z+2)}$$

$\Rightarrow$ simple pole at $z = -\frac{1}{2}$ inside C
pole at $z = -2$ outside C

$\Rightarrow$ shrink contour around $z = -\frac{1}{2}$

$$I = \frac{1}{2i} \oint_{c'} dz \, \frac{1}{(z+\frac{1}{2})(z+2)}$$



$$= \frac{1}{2i} \frac{1}{(-\frac{1}{2}+2)} \oint_{c'} \frac{dz}{z+\frac{1}{2}}$$

$$= \frac{1}{2i} \, \frac{2}{3} \, 2\pi i = \frac{2\pi}{3}$$

(2) Algebraic functions $f(x) = \frac{P(x)}{Q(x)}$

$$I = \int_{-\infty}^{\infty} dx \, f(x)$$

$\Rightarrow$ no poles on real axis

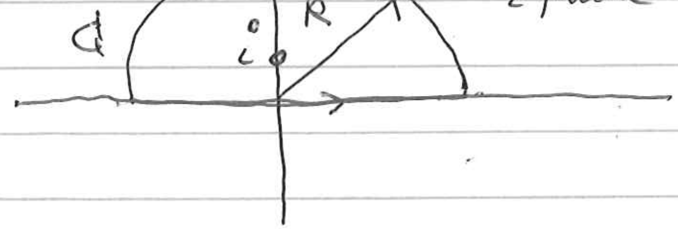
$\Rightarrow$ P, Q polynomials
degree Q − degree $P \geq 2$
$\Rightarrow$ why?

Let $I = \int_{-\infty}^{\infty} dx \, \frac{1}{1+x^2}$

$$= \int_{-\infty}^{\infty} dz \, \frac{1}{1+z^2}$$

$\Rightarrow$ note poles at $z = \pm i$

Consider closed contour integral over C

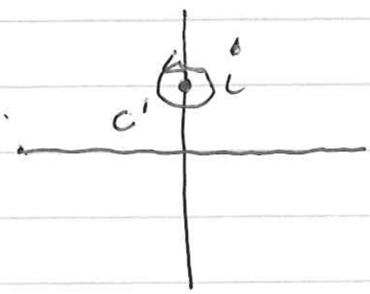


$\Rightarrow R \to \infty$

~~Breaks~~

$$I_c = \oint_c dz \frac{1}{(z+i)(z-i)}$$

$\Rightarrow$ shrink around $z = i$



$$I_c = \oint_{c'} dz \frac{1}{(z+i)(z-i)}$$

$$= \frac{1}{2i} \oint_{c'} \frac{dz}{z-i} = \frac{2\pi i}{2i} = \pi$$

$\Rightarrow$ How is this related to I ?

$\Rightarrow$ Break C into integral along the real axis C_{ra} and semi-circle C_R

$$I_c = \int_{\substack{C_{ra} \\ R\to\infty}} \frac{dz}{1+z^2} + \int_{\substack{C_R \\ R\to\infty}} \frac{dz}{1+z^2}$$

$z = Re^{i\theta}$

$dz = Ri\,d\theta\, e^{i\theta}$

$$= I + \underbrace{\cancel{\int_0^\pi d\theta \frac{R i\, d\theta\, e^{i\theta}}{R^2 e^{2i\theta}}}}_{R\to\infty} \to 0$$

$$= I$$

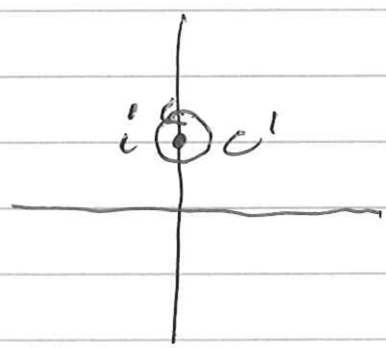
$$\Rightarrow \boxed{I = \pi}$$

$\Rightarrow$ For degree q − degree $p \geq 2$
integral over C_R always zero

$\Rightarrow$ Doesn't matter if C_R in
UHP or LHP

Let $$I = \int_{-\infty}^{\infty} dx \, \frac{1}{(1+x^2)^2} = \int_{-\infty}^{\infty} dz \, \frac{1}{(1+z^2)^2}$$

$$= \oint_{c'} \frac{dz}{(z+i)^2 (z-i)^2}$$



$\Rightarrow$ second order pole at $z = i$

Recall
$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_c \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$
for c around z_0
and f analytic
there

$\Rightarrow$ $f = \frac{1}{(z+i)^2}$, $n = 1$

$$I = \frac{2\pi i}{1!} \left(\frac{1}{(z+i)^2} \right)' \Bigg|_{z=i} = 2\pi i \, \frac{-2}{(z+i)^3} \Bigg|_{i}$$

$$= 2\pi i \, \frac{-2}{8 i^3} = \frac{\pi}{2}$$

(3) Infinite integrals over trig function or exponentials e^{ikx}

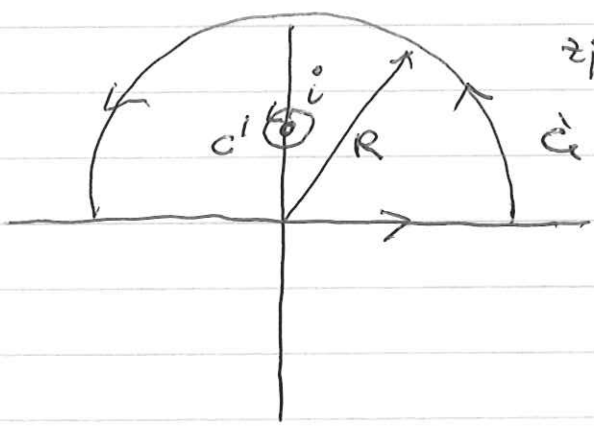
$$I = \int_0^\infty dx \, \frac{\cos x}{1+x^2}$$ ⟹ integral an even function

so $$I = \frac{1}{2} \int_{-\infty}^{\infty} dx \, \frac{\cos x}{1+x^2}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dx \, \frac{\cos x + i \sin x}{1+x^2}$$ ⟹ $\sin x$ odd ⟹ 0

$$I = \frac{1}{2} \int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{1+x^2}$$

Consider $$I_c = \frac{1}{2} \oint_{C} dz \, \frac{e^{iz}}{1+z^2}$$



z plane

⟹ shrink to C'

$$I_c = \frac{1}{2} \oint_{C'} dz \, \frac{e^{iz}}{(z-i)(z+i)}$$

$$= \frac{1}{2} e^{-1} \frac{1}{2i} \oint_{C'} dz \, \frac{1}{z-i}$$

$$= \frac{1}{4ie} (2\pi i) = \frac{\pi}{2e}$$

Again write I_c interms of C_{ra} and C_R

$$I_C = I + \underbrace{\int_{C_R} dz \, \frac{e^{iz}}{1+z^2}}_{\substack{R\to\infty \\ I_R}} \qquad z = Re^{i\theta}$$

$$dz = Ri\,d\theta\, e^{i\theta}$$

$$I_R = \int_0^{\pi} \frac{Ri\,d\theta\, e^{i\theta}\, e^{iR(\cos\theta + i\sin\theta)}}{R^2 e^{2i\theta}}$$

$$|I_R| \le \frac{1}{R}\int_0^{\pi} d\theta \, e^{-R\sin\theta}$$

~~since R~~

$\Rightarrow$ $\sin\theta$ even around $\pi/2$

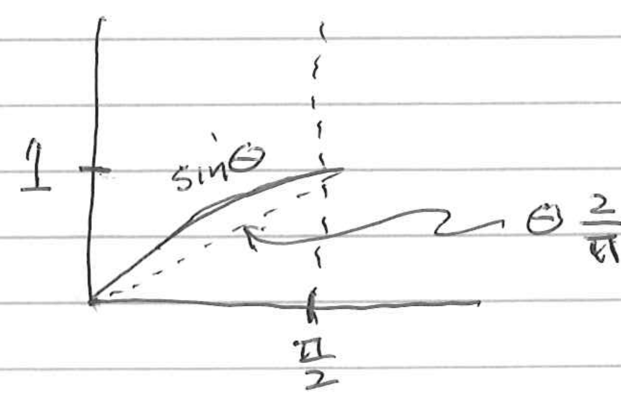
$$= \frac{2}{R}\int_0^{\frac{\pi}{2}} d\theta \, e^{-R\sin\theta}$$

$\Rightarrow$ ~~since $R \to \infty$ dominate contribution around $\theta = 0$~~

$$\le \lim_{R\to\infty} \frac{2}{R}\int_0^{\frac{\pi}{2}} d\theta \, e^{-R\theta \frac{2}{\pi}}$$

$$= \lim_{R\to\infty} \frac{2}{R}\,\frac{1}{R\frac{2}{\pi}}$$

$$= 0$$



$\sin\theta > \frac{2}{\pi}$

$$\Rightarrow \boxed{I = I_c = \frac{\pi}{2e}}$$

~~Q~~

More generally let

$$I = \int dx \, \frac{P(x)}{Q(x)} e^{ikx}$$

with P, Q polynomials with $Q \neq 0$ on real axis and k real.

$$I = 2\pi i \left(\text{sum of residues of } \frac{P(z)}{Q(z)} e^{ikz} \text{ in UHP} \right)$$

Require $k > 0$ and degree Q − degree $P \geq 1$

$\Rightarrow$ then I_R is zero

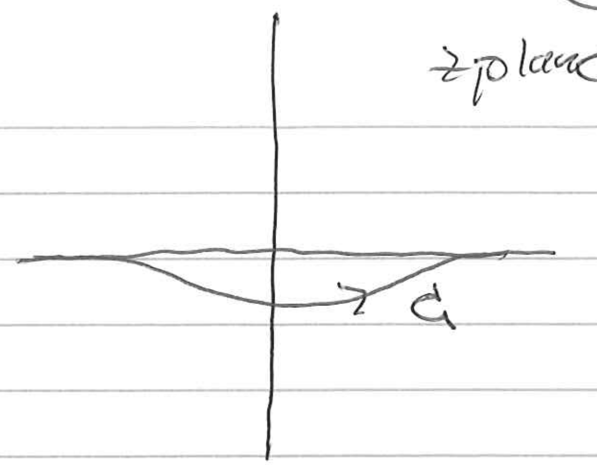
$\Rightarrow$ Jordan's Lemma

Let

$$I = \int_{-\infty}^{\infty} dx \, \frac{\sin x}{x}$$

$\Rightarrow$ no singularity at $x = 0$

$\Rightarrow$ deform ~~cont~~ integral below real axis



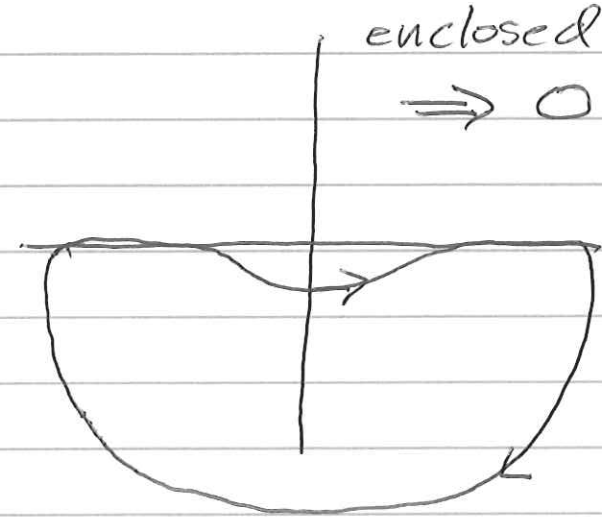
$$I = \int_C dz \, \frac{\sin z}{z}$$

$$= \underbrace{\int_{C_1} dz \, \frac{e^{iz}}{2iz}}_{\text{close in UHP as before. Find residue at } z=0} - \underbrace{\int_{C_1} dz \, \frac{e^{-iz}}{2iz}}_{\text{Close in LHP}}$$

close in UHP as before. Find residue at $z=0$

$$\underbrace{\qquad}\quad \frac{1}{2i} \, 2\pi i \, e^{i0} = \pi$$

Close in LHP where e^{-iz} goes to zero at ∞. NO enclosed poles $\Rightarrow 0$



$$\Rightarrow \boxed{I = \pi}$$

$\Rightarrow$ Be careful. Do not separate the integral over $\sin x$ into integrals over e^{ix} and e^{-ix} until you move the contour.

$\int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{x}$ is not a proper integral

$\Rightarrow$ don't follow Boas on this topic!!

④ Fractional powers

Let $I = \int_0^\infty dx \, \frac{x^{p-1}}{1+x} \quad 0 < p < 1$

$\Rightarrow$ integral is ~~bound~~ bounded

since for large x, integrand goes like $\frac{x^p}{x^2}$ with $p < 1$.

At $p = 1$ diverges since $\sim \frac{1}{x}$

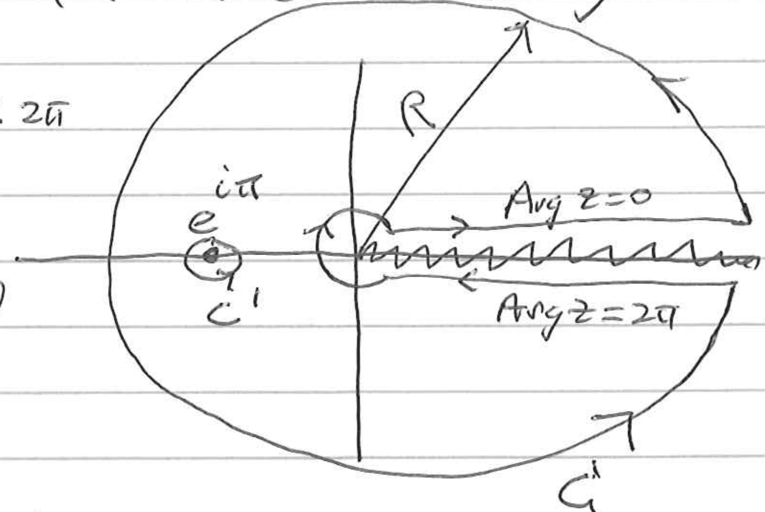
and $\int^\infty dx \frac{1}{x}$ diverges $\sim \ln(x) \Big|^\infty$

Need a branch cut to make x^p single valued. Take BC along real axis.

Choose $0 \leq \mathrm{Arg}(z) < 2\pi$

Pole at $z = e^{i\pi}$.

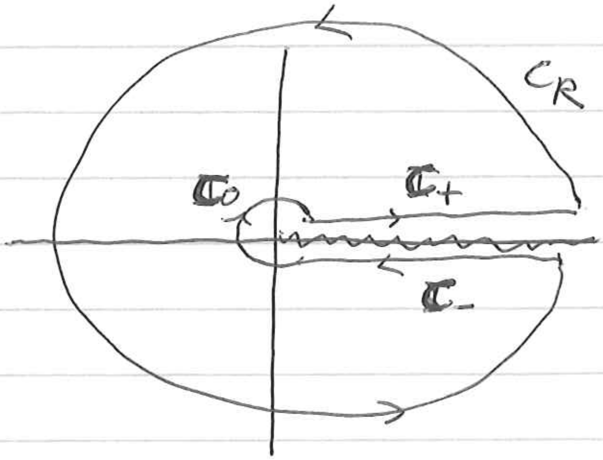
Shrink C around the pole



$$I_c = \oint_C dz \, \frac{z^{p-1}}{1+z} = \oint_{C'} dz \, \frac{z^{p-1}}{(z - e^{i\pi})}$$

$$= \left(e^{i\pi}\right)^{p-1} 2\pi i = -2\pi i \, e^{ip\pi}$$

Divide C into C_R, C_0, C_+, C_- as shown



$$I_R = \lim_{R \to \infty} \int_{C_R} dz \, \frac{z^{p-1}}{z}$$

$$z = Re^{i\theta}$$

$$dz = Ri\,d\theta\, e^{i\theta}$$

$$I_R = \int_0^{2\pi} Ri\,d\theta\, e^{i\theta} R^{p-2} e^{i\theta(p-2)}$$

$$\sim \frac{1}{R^{1-p}} \longrightarrow 0$$

$$I_0 = \int_{C_0} dz \, \frac{z^{p-1}}{1+z} \qquad z = re^{i\theta}$$

$$dz = ri\,d\theta\, e^{i\theta}$$

$$= \lim_{r \to 0} \int_{2\pi}^{0} ri\,d\theta\, e^{i\theta} r^{p-1} e^{i\theta(p-1)}$$

$$\sim \lim_{r \to 0} r^p \longrightarrow 0$$

$$I_+ = \int_{C_+} dz \, \frac{z^{p-1}}{1+z} \qquad z = re^{i\theta} = r$$

since $\theta = 0$ on top of cut

$dz = dr$

$$= \int_0^{\infty} dr \, \frac{r^{p-1}}{1+r} = I$$

~~$I_- =$~~

$$I_- = \int_{C_-} dz \, \frac{z^{p-1}}{1+z} \qquad z = re^{i\theta} = re^{2\pi i}$$

$$dz = dr\, e^{2\pi i}$$

$$= \int_\infty^0 \frac{dr\, e^{2\pi i}\, r^{p-1}\, e^{2\pi i(p-1)}}{1 + re^{2\pi i}}$$

$$= -e^{2\pi i p} \int_0^\infty dr \frac{r^{p-1}}{1+r} = -e^{2\pi i p} I$$

$$I_c = I - e^{2\pi i p} I = -2\pi i\, e^{ip\pi}$$

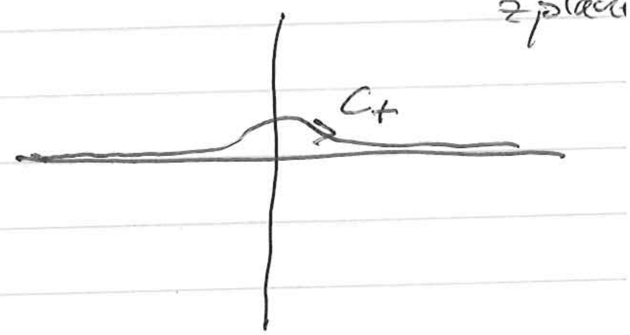
$$I = \frac{-2\pi i\, e^{ip\pi}}{1 - e^{2\pi i p}} = \frac{-2\pi i}{e^{-ip\pi} - e^{i\pi p}}$$

$$\boxed{I = \frac{\pi}{\sin p\pi}}$$

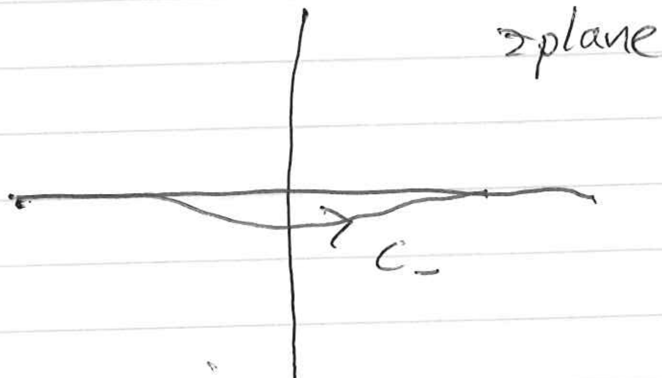
Principal Value Integrals

It is essential to properly define an integral to specify the contour with respect to all singularities in the system. Let

$$I_+ = \int_{C_+} dz \, \frac{\cos(z)}{z}$$



$$I_- = \int_{C_-} dz \, \frac{\cos z}{z}$$



$$I_+ = \underbrace{\frac{1}{2}\int_{C_+} dz \, \frac{e^{iz}}{z}}_{\substack{\text{close in UHP} \\ \Rightarrow \text{ no enclosed singularities}}} + \underbrace{\frac{1}{2}\int_{C_+} dz \, \frac{e^{-iz}}{z}}_{\substack{\text{close in LHP} \\ \Rightarrow \text{ enclosed sing. at } z=0}}$$

$$= 0 - \frac{2\pi i}{2} e^{-i(0)} = -\pi i$$

$$I_- = \pi i$$

$\Rightarrow$ different values depending on how go around singularity

Can also define an integral that goes through the singularity as the "Principal Value" of the integral

$$I \equiv P\int_{-\infty}^{\infty} dx \, \frac{\cos x}{x} \equiv \lim_{\epsilon \to 0} \int_{-\infty}^{-\epsilon} dx \, \frac{\cos x}{x}$$

$$+ \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} dx \, \frac{\cos x}{x}$$

$\Rightarrow$ note that this is not a proper contour integral.

$\Rightarrow$ note that because x is odd and $\cos(x)$ is even the integral yields zero

$$I = 0$$

$\Rightarrow$ in physics it is occasionally useful to define the PV integral.