Partial Differential Equations

Most problems in physics involve 2-D or 3-D systems and also time dependence

$\Rightarrow$ may typically have 4 variables rather than a single variable

$\Rightarrow$ have partial differential equations rather than ordinary differential equations.

examples

① Laplaces equation

$$\nabla^2 u = 0$$

$\Rightarrow$ electrostatic potential or gravitational potential ~~with~~ in a region with no charge or mass

② Diffusion equation

$$\frac{\partial}{\partial t} u - D \nabla^2 u = 0$$

D = diffusion rate $\sim \frac{m^2}{s}$

$\Rightarrow$ heat flow

$\Rightarrow$ transport of impurities

(3) Wave equation

$$\frac{\partial^2 u}{\partial t^2} - v^2 \nabla^2 u = 0$$

$\Rightarrow$ wave equation with v the phase speed of the wave

(4) Helmholtz eqn

$$\nabla^2 u + k^2 u = 0$$

$\Rightarrow$ arises when solving the diffusion or wave eqns

(5) Schrödinger eqn.

$$i\hbar \frac{\partial}{\partial t} \psi - V(\underset{\sim}{x}) \psi + \frac{\hbar^2}{2m} \nabla^2 \psi = 0$$

with $\hbar = \frac{h}{2\pi}$ with h being Planck's constant

and $V(\underset{\sim}{x})$ the potential with ψ complex

$\psi^* \psi = |\psi|^2 =$ probability of finding a particle

We can solve many of such problems using the basis functions that we have developed.

Laplace's Equation: Rectangular system:

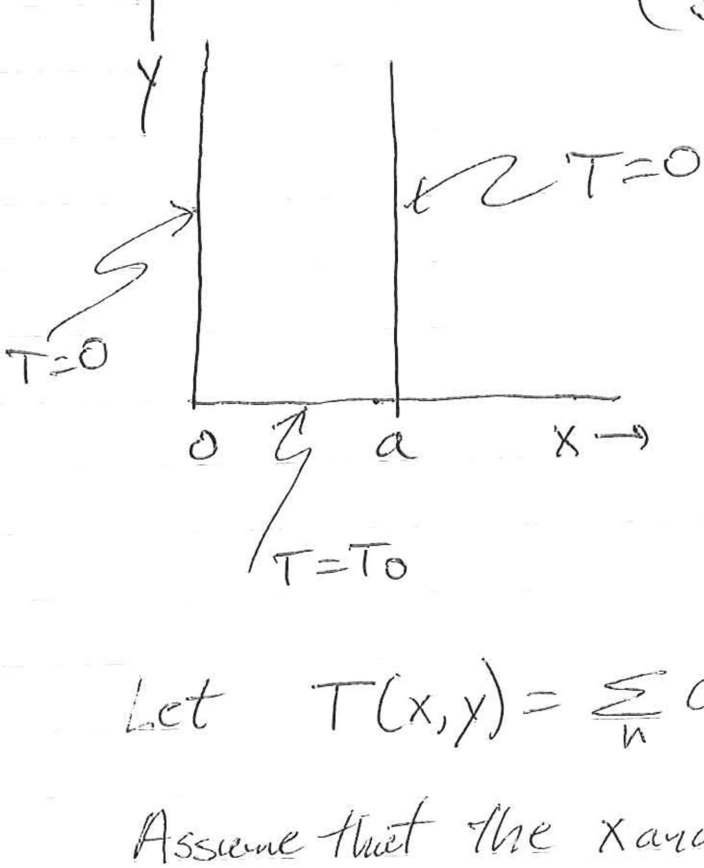
Consider a heat conduction problem in a 2-D rectangular system that has reached a steady state

$$\frac{\partial}{\partial t} T - D \nabla^2 T = 0$$

$$\Rightarrow \nabla^2 T = 0$$

$\Rightarrow$ uniform in z

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T = 0$$



A rectangular system has a temperature T_0 at a plate at $y=0$ and zero temperature at the two sides. The system is open at the top

Let $T(x,y) = \sum_n c_n G_n(x,y)$ with $\nabla^2 G_n = 0$

Assume that the x and y dependence is separable so that

$$G_n(x,y) = X_n(x) Y_n(y)$$

$\Rightarrow$ separation of variables

Thus,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) X_n(x)\, Y_n(y) = 0$$

$$\Rightarrow \quad \frac{\partial^2 X_n(x)}{\partial x^2}\, Y_n + X_n(x)\, \frac{\partial^2 Y_n}{\partial y^2} = 0$$

$\Rightarrow$ divide by $X_n Y_n$

$$\underbrace{\frac{1}{X_n}\frac{\partial^2 X_n}{\partial x^2}}_{\substack{\text{only a} \\ \text{function of } x \\ \Rightarrow -k^2}} + \underbrace{\frac{1}{Y_n}\frac{\partial^2 Y_n}{\partial y^2}}_{\substack{\text{only a function of } y \\ \Rightarrow k^2}} = 0$$

The sum can only be zero for all x and y if each term is a constant. Let

$$\frac{\partial^2 X_n}{\partial x^2} + k^2 X_n = 0\,, \qquad \frac{\partial^2 Y_n}{\partial y^2} - k^2 Y_n = 0$$

$$X_n \sim \sin kx,\ \cos kx \qquad Y_n \sim e^{ky},\ e^{-ky}$$

Why oscillatory in x and exponentially growing or decaying in y?

Two reasons:

① Want $T \to 0$ as $y \to \infty$. Why?

② Have to match $T(x, y=0)$ using the basis functions along x
$\Rightarrow$ need oscillatory functions

How to determine K? Want $T \to 0$ at $x = 0, a$

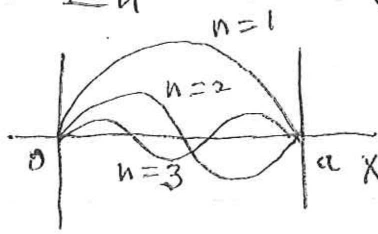
$$\Rightarrow \ X_n \sim \sin(kx) \ \Rightarrow X_n = 0 \text{ at } x = 0$$

Also want $X_n(x=a) = 0 = \sin(ka)$

$$\Rightarrow ka = n\pi$$

$$k_n = \frac{n\pi}{a} \quad \text{with } n = 1, 2, 3 \ldots$$

$$\Rightarrow X_n = \sin(k_n x) \ , \ Y_n = e^{-k_n y}$$



$\Rightarrow$ so $T \to 0$ as $y \to \infty$.

So

$$T = \sum_n c_n \sin k_n x \, e^{-k_n y}$$

To find c_n need to match $T(x, y=0) = T_0$

$$T_0 = \sum_n c_n \sin(k_n x)$$

$\Rightarrow$ $\sin(k_n x)$ form a complete, orthogonal set over $x \in (0, a)$

$\Rightarrow$ multiply by $\sin(k_m x)$ and integrate $(0, a)$

$$T_0 \int_0^a dx \sin k_m x = \sum_n c_n \int_0^a dx \sin(k_n x) \sin(k_m x)$$

$$= \sum_n c_n \delta_{mn} \frac{1}{2} a = c_m \frac{a}{2}$$

$\Rightarrow$ use symmetry $\Rightarrow$ T_0 is even

For m odd

$$C_m = \frac{2T_0}{a}\int_0^a dx\ \sin(k_m x)$$

$$= \frac{2T_0}{a}\left(-\frac{\cos(k_m x)}{k_m}\right)\Big|_0^a$$

$$= \frac{2T_0}{a}\frac{1}{k_m}\left(1-\underbrace{\cos m\pi}_{-1 \text{ for } m \text{ odd}}\right)$$

$$= \frac{4T_0}{m\pi}$$

$$T = \sum_{n=1,3,5\ldots} \frac{4T_0}{n\pi}\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{n\pi y}{a}}$$

For large y only n = 1 is important since

$$\sim e^{-\frac{n\pi y}{a}} \to 0$$

faster with larger n

$$T(x,y) \simeq \frac{4T_0}{\pi}\sin\left(\frac{\pi x}{a}\right)e^{-\frac{\pi y}{a}}$$

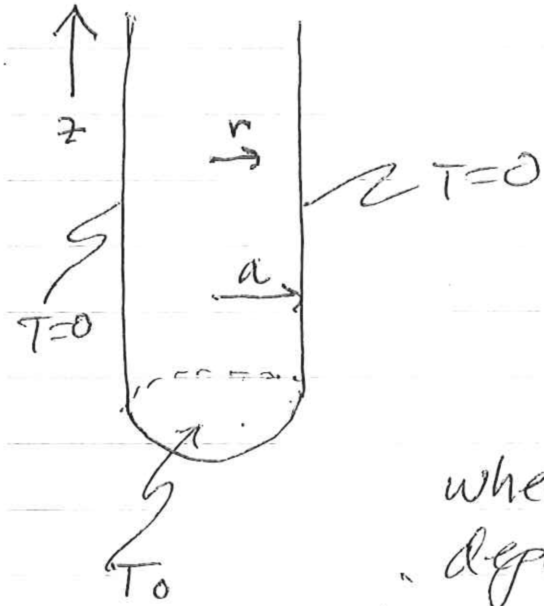
for large y.

$\Rightarrow$ note that $T(x,y)$ ~~is~~ can not be written as a function of x times a function of y

$\Rightarrow$ only each basis function G_n has this property

Laplace's Eqn: Cylindrical System

We again consider a steady-state system with heat conduction but with a cylinder of radius "a" and infinitely tall whose bottom is maintained at a temperature To and whose sides are at T=0.



Within the cylinder in steady state we again have

$$\nabla^2 T = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}$$

where there is no azimuthal dependence so $\partial/\partial\varphi = 0$. As in a rectangular system, we express T in terms of basis functions

$$T(r,z) = \sum_n c_n G_n(r,z)$$

with $G_n = R_n(r)\, Z_n(z) \implies$

$$\underbrace{\frac{1}{R_n}\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial R_n}{\partial r}}_{-k^2} + \underbrace{\frac{1}{Z_n}\frac{\partial^2}{\partial z^2}}_{k^2} Z_n = 0$$

Since the LH term is only a function of r and the RH term is only a function of z, they must each be constant and

$$\frac{\partial^2}{\partial z^2} Z_n - k^2 Z_n = 0 \Rightarrow Z_n \sim e^{kz}, e^{-kz}$$

$$\Rightarrow Z_n \sim e^{-kz} \text{ so}$$

$T \to 0$ at $z = \infty$

and

$$r^2 \frac{\partial^2}{\partial r^2} R_n + r \frac{\partial}{\partial r} R_n + k^2 r^2 R_n = 0$$

This is Bessel's eqn with $p = 0$. We transformed this eqn to Sturm-Liouville form. For $p=0$ the basis functions are

$$R_n = J_0(k_n r).$$

Requiring $R_n(r=a) = 0$, since $T(r=a) = 0$ gives $k_n a = X_{0n}$ with X_{0n} the nth zero of $J_0(x)$. The orthogonality integral is

$$\int_0^a dr\, r\, \underline{J_0(k_n r)\, J_0(k_m r)} = \delta_{nm} N_m^2$$

with

$$N_m^2 = \frac{a^2}{2} J_1^2(k_n a)$$

Matching $T(r, z=0) = T_0$,

yields

$$T_0 = \sum_n c_n J_0(k_n r)$$

since $e^{-k_n z} = 1$ at $z=0$. Multiply by

$$r J_0(k_m r)$$

and integrate r from "o" to "a". This eliminates the sum and gives an expression for c_m

$$T_0 \underbrace{\int_0^a dr\, r\, J_0(k_m r)}_{I_m} = c_m \frac{a^2}{2} J_1^2(k_m a)$$

To evaluate I_m use the recursion formula

$$\frac{d}{dx}\left(x^p J_p\right) = x^p J_{p-1}(x)$$

$$\frac{d}{dx}\left(x J_1\right) = x J_0(x)$$

$$I_m = \int_0^a dr\, r\, J_0(k_m r) = \frac{1}{k_m^2} \int_0^{x_{om}} dx\, x\, J_0(x)$$

with $x = k_{om} r$ so

$$I_m = \frac{1}{k_m^2} \int_0^{x_{om}} dx\, \frac{d}{dx}(x J_1) = \frac{1}{k_m^2} (x J_1) \Big|_0^{x_{om}}$$

$$= \frac{1}{k^2}\, x_{om} J_1(x_{om})$$

$$C_m = T_0 \frac{2}{a^2} \frac{1}{J_1^2(x_{om})} \frac{1}{k_m^2} x_{om} J_1(x_{om})$$

$$= \frac{2T_0}{x_{om}} \frac{1}{J_1(x_{om})}$$

$$T(r,z) = 2T_0 \sum_{n=1}^{\infty} \frac{1}{x_{on}} \frac{J_0(k_n r)}{J_1(x_{on})} e^{-k_n z}$$

with $k_n = \frac{x_{on}}{a}$.

What is the behavior for large z?

Can match any temperature profile $T(r, z=0)$. Write

$$T(r, z=0) = \sum_n c_n J_0(k_n r)$$

$\Rightarrow$ invert sum for c_n

What do we do if we want to specify the temperature along the surface $r=a$ of a cylindrical system?

example: Closed cylinder of radius "a" and length L with

$$T(r=a, z) = T_0$$

and

$$T(r, z=0) = 0$$
$$T(r, z=L) = 0$$

Require oscillatory functions along the z direction to match $T = T_0$ at $r = a$.

As before $\nabla^2 T = 0$ with

$$T(r,z) = \sum_n c_n R_n(r) Z_n(z)$$

$$\Rightarrow \underbrace{\frac{1}{R_n}\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} R_n}_{k^2} + \underbrace{\frac{1}{Z_n}\frac{\partial^2}{\partial z^2} Z_n}_{-k^2} = 0$$

Oscillatory functions along z

$$\frac{\partial^2}{\partial z^2} Z_n + k^2 Z_n = 0$$

$$\Rightarrow \sin kz, \cos kz$$

$\Rightarrow$ require $Z_n = 0$ at $z = 0, L$

$$Z_n = \sin(k_n z) \quad \text{with } k_n = \frac{n\pi}{L}$$

Radial direction:

$$r^2 \frac{\partial^2}{\partial r^2} R_n + r \frac{\partial}{\partial r} R_n - k_n^2 r^2 R_n = 0$$

$\Rightarrow$ modified Bessel eqn with $p = 0$

$\Rightarrow$ solutions

$$I_p(x) = i^{-p} J_p(ix)$$

$$K_p(x) = \frac{\pi}{2} i^{p+1} \left(J_p(ix) + i N_p(ix) \right)$$

K_p diverges at $x = 0$
I_p bounded at $x = 0$

Large argument

$$I_p(x) = \frac{1}{\sqrt{2\pi x}} e^{x}$$

$$K_p(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$$

Thus,

$$R_n = I_0(k_n r) / I_0(k_n a)$$

so that $R_n(r=0)$ is bounded. Note that

$$R_n(a) = 1$$

We have

$$T(r,z) = \sum_n C_n \sin(k_n z)\, I_0(k_n r) \frac{1}{I_0(k_n a)}$$

Find C_n by requiring $T = T_0$ at $r = a$

$$T_0 = \sum_n C_n \sin(k_n z)$$

$\Rightarrow$ note that this relation is simplified because we chose $R_n(a) = 1$

$\Rightarrow$ multiply by $\sin(k_m z)$ and integrate over z from 0 to L.

$$T_0 \int_0^L dz \sin(k_m z) = C_m \int_0^L dz \sin^2 k_m z$$

$$= \tfrac{1}{2} C_m L$$

$$C_m = \frac{2T_0}{L} \int_0^L dz \sin(k_m z)$$

$C_m = 0$ for m even

$$= \frac{2T_0}{L}\left(-\frac{\cos k_m z}{k_m}\right)\Bigg|_0^L$$

$$= \frac{4T_0}{k_m L} = \frac{4T_0}{m\pi} \quad \text{for } m \text{ odd}$$

$$T(r,z) = \sum_{n \text{ odd}} \frac{4T_0}{n\pi} \sin(k_n z) \frac{I_0(k_n r)}{I_0(k_n a)}$$

$n\pi$

Schroedinger Equation with Time Dependence

Consider the quantum description of a particle in a 1-D box of length L.

$\Rightarrow$ take $V = 0$

$$\Rightarrow \psi(x=0,t) = 0$$
$$\psi(x=L,t) = 0$$

$$i\hbar \frac{\partial}{\partial t}\psi + \frac{\hbar^2}{2m}\nabla^2\psi = 0$$

$\Rightarrow$ specify $\psi(x, t=0)$

$$\psi = \sum_n c_n X_n(x)\, T_n(t)$$

$$i\hbar \frac{1}{T_n}\frac{\partial}{\partial t}T_n + \frac{\hbar^2}{2m}\underbrace{\frac{1}{X_n}\frac{\partial^2}{\partial x^2}X_n}_{-k_n^2} = 0$$

$$\frac{\partial^2}{\partial x^2}X_n + k_n^2 X_n = 0$$

$$X_n = \sin(k_n x) \quad \text{with } k_n = \frac{\pi n}{L}$$

$$\Rightarrow X_n = 0 \text{ at } x = 0, L$$

$$i\hbar\frac{\partial}{\partial t}T_n = \frac{\hbar^2}{2m}k_n^2\, T_n \equiv E_n T_n$$

$$E_n = \frac{\hbar^2}{2m} k_n^2$$

$\Rightarrow$ each eigenfunction $\Chi_n$ has energy E_n

$$i\hbar \frac{\partial}{\partial t} T_n = E_n T_n$$

$\Rightarrow$ exponential solution

$$T_n \sim e^{-i\omega t}$$

$$i\hbar(-i\omega) e^{-i\omega t} = E_n e^{-i\omega_n t}$$

$$\omega_n = \frac{E_n}{\hbar} \Rightarrow T_n \sim e^{-i\frac{E_n}{\hbar}t}$$

$\Rightarrow$ each eigenfunction $\Chi_n$ oscillates at its own frequency

$$\psi(x,t) = \sum_n c_n \sin(k_n x)\, e^{-i\frac{E_n}{\hbar}t}$$

Initial Condition: Assume that the particle has equal probability of being anywhere in the box

$$\psi(x,0) = \frac{1}{L^{1/2}} \Rightarrow \int_0^L dx\, |\psi|^2 = 1$$

$$\frac{1}{L^{1/2}} = \sum_n c_n \sin(k_n x)$$

Multiply by $\sin(k_m x)$ and integrate $(0, L)$

$$\frac{1}{L^{1/2}} \int_0^L dx\, \sin(k_m x) = c_m \frac{L}{2}$$

$c_m = 0$ for n even

For m odd

$$c_m = \frac{2}{L^{3/2}} \left(-\frac{\cos(k_m x)}{k_m} \right)\Bigg|_0^L$$

$$= \frac{2}{L^{3/2}} \frac{2L}{m\pi} = \frac{4}{\sqrt{L}\, m\pi}$$

$$\psi(x,t) = \frac{4}{\pi\sqrt{L}} \sum_{n \text{ odd}} \frac{1}{n} \sin(k_n x)\, e^{-i\frac{E_n}{\hbar}t}$$

$$k_n = \frac{n\pi}{L}, \quad E_n^2 = \frac{\hbar^2}{2m} k_n^2$$

How does the total probability depend on time?

$$\int_0^L dx\, |\psi|^2 = \frac{16}{\pi^2 L} \int_0^L dx \sum_{n \text{ odd}} \sum_{m \text{ odd}} \frac{1}{nm} \sin k_n x \sin k_m$$

$$= \frac{16}{\pi^2 L} \sum_{n \text{ odd}} \frac{1}{n^2} \frac{1}{2} L$$

$$= \frac{8}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} = 1$$ see Ch 7 prob. 5.9

$\Rightarrow$ total probability conserved

$\Rightarrow$ Can prove this from original eqn.

~~Sound~~ Waves in a spherical cavity. (by popular demand)

Consider the wave equation of a spherical cavity of radius R. Take the wave amplitude to be zero at $r = R$. ~~Assume~~

$$\frac{\partial^2 u}{\partial t^2} - v^2 \nabla^2 u = 0$$

Assume no dependence on θ, φ in spherical coordinates

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} u = 0$$

Consider an initial value problem starting at $t=0$

$\Rightarrow$ specify

$$u(r, t=0) = u_0 R\, \delta(r - r_0)$$

$$\frac{\partial u}{\partial t} \equiv \dot{u}(r, t=0) = 0$$

The factor R causes u and u_0 to have the same units since $\delta(r-r_0) \sim \frac{1}{\text{length}}$

Write $u = \sum_n c_n G_n(r,t)$ and

$$G_n(r,t) = R_n(r)\, T_n(t)$$

$\Rightarrow$ input this form into the wave eqn and divide by $R_n T_n$ yields

$$\frac{1}{T_n}\frac{\partial^2}{\partial t^2} T_n - v^2 \underbrace{\frac{1}{R_n}\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} R_n}_{-k^2} = 0$$

Choose oscillator functions in r direction $\Rightarrow$ match $\delta(r-r_0)$ at $t=0$

$$\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} R_n + k^2 r^2 R_n = 0$$

$$\frac{\partial^2 T_n}{\partial t^2} + k^2 v^2 T_n = 0$$

Eqn for $R_n(r)$, is already self-adjoint. From previous calculations with Bessel's eqn (page 141 of notes) we know that the basis functions are $J_{\frac{1}{2}}(k_n r)$ where

$$J_{\frac{1}{2}}(k_n R) = 0 \quad \text{so}$$

$$k_n = \frac{x_{\frac{1}{2}n}}{R}$$

$$\int_0^R dr\, r\, J_{\frac{1}{2}}(k_n r)\, J_{\frac{1}{2}}(k_m r) = \delta_{mn} \frac{R^2}{2} J_{\frac{3}{2}}^2(x_{\frac{1}{2}}$$

Thus, T_n satisfies

$$\frac{\partial^2}{\partial t^2} T_n + \omega_n^2 T_n = 0$$

with $\omega_n = k_n V$. ~~Sout~~ Solutions are $\sin(\omega_n t)$, $\cos(\omega_n t)$ $\Rightarrow$ linear combination

So,

$$u(r,t) = \sum_n \left[C_n^s \sin(\omega_n t) + C_n^c \cos(\omega_n t)\right] J_{\frac{1}{2}}(k_n r)$$

Initial conditions at $t=0$ will produce C_n^s and C_n^c. First consider $\dot{u}(r,t=0)$

$$\dot{u}(r,t) = \sum_n \left[\omega_n C_n^s \cos(\omega_n t) - \omega_n C_n^c \sin(\omega_n t)\right] J_{\frac{1}{2}}$$

$$\dot{u}(r,t) = 0 = \sum_n \omega_n C_n^s J_{\frac{1}{2}}(k_n r) \Rightarrow \boxed{C_n^s = 0}$$

So,

$$u(r,t=0) = \sum_n C_n^c J_{\frac{1}{2}}(k_n r)$$

$$= u_0 R\, \delta(r - r_0)$$

Multiply by $\frac{1}{r} r J_{\frac{1}{2}}(k_m r)$ and integrate 0 to R.

$$u_0 R \int_0^R dr\, r\, \delta(r-r_0)\, J_{\frac{1}{2}}(k_m r) = \sum_n C_n^c\, \delta_{mn} \frac{R^2}{2} J_{\frac{3}{2}}^2(X_{\frac{1}{2}n})$$

$$= C_m^c \frac{R^2}{2} J_{\frac{3}{2}}^2(X_{\frac{1}{2}m})$$

$$C_m^c = \frac{2u_0 r_0}{R} \frac{J_{\frac{1}{2}}(k_m r_0)}{J_{\frac{3}{2}}^2(X_{\frac{1}{2}m})}$$

Finally,

$$u(r,t) = \sum_{n=1,2\ldots} \frac{2u_0 r_0}{R} \frac{J_{\frac{1}{2}}(k_n r_0)}{J_{\frac{3}{2}}^2(X_{\frac{1}{2}n})} \cos(\omega_n t)$$

$$\otimes\ J_{\frac{1}{2}}(k_n r)$$

with

$$k_n = X_{\frac{1}{2}n} \frac{1}{R}$$

$$\omega_n = k_n V$$

The cavity supports an infinite # of normal modes with distinct frequencies ω_n. The amplitude of each normal mode is controlled by the profile of $u(r,t=0)$.