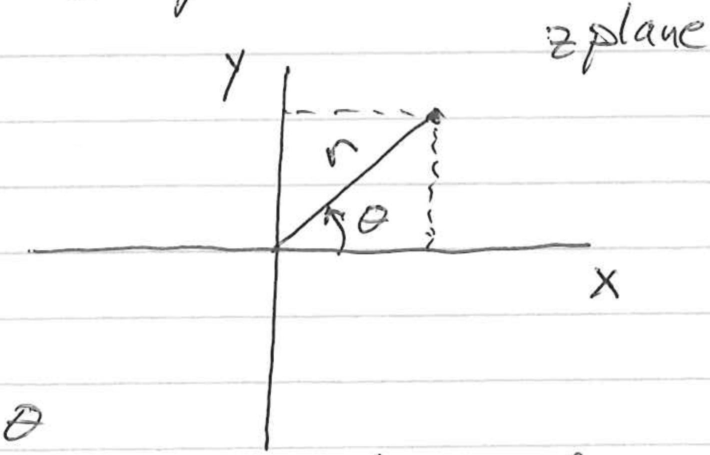
# Functions of a Complex Variable




## Basic properties

$$z = x + iy = re^{i\theta}$$
$$= r\cos\theta + ir\sin\theta$$
$$\mathrm{Arg}(z) = \theta$$
$$e^{i\theta} = \cos\theta + i\sin\theta$$

Multiplication rules same as real numbers with $i^2 = -1$

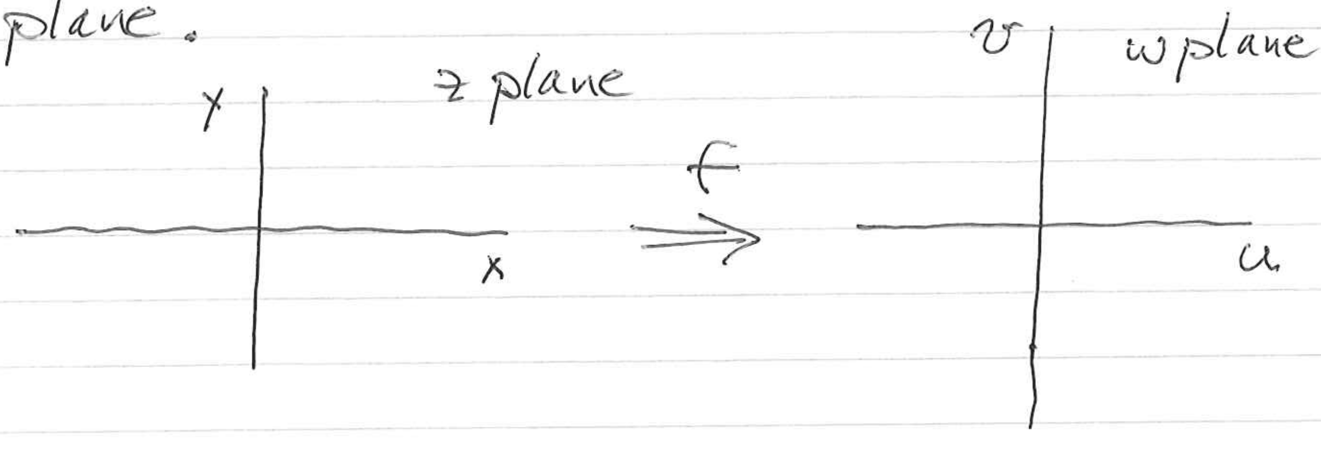
Complex conjugate $z^* = x - iy = re^{-i\theta}$

$$z^* z = r^2 \equiv |z|^2 \qquad |z| = r$$

## Maps

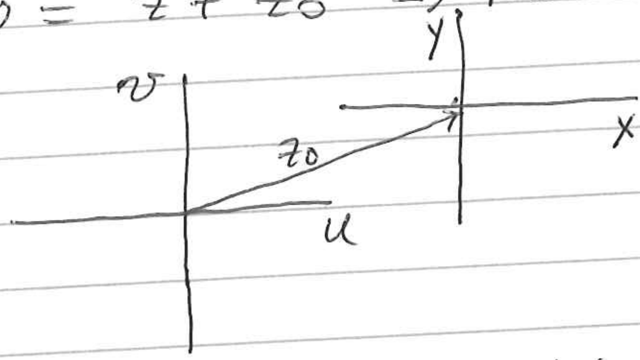
Let $w = f(z) = u + iv$
$$= u(x,y) + i\,v(x,y)$$

The function $f$ defines a mapping between the complex $z$ plane and the complex $w$ plane.



Examples

1. $w = z + z_0 \Rightarrow$ translation

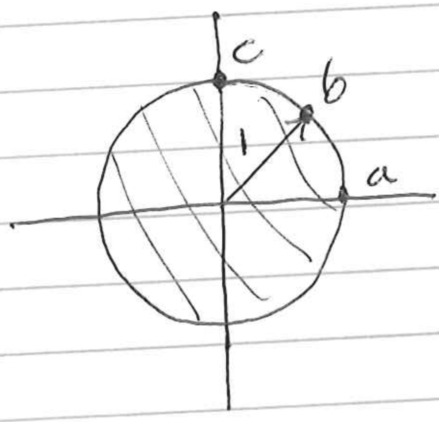


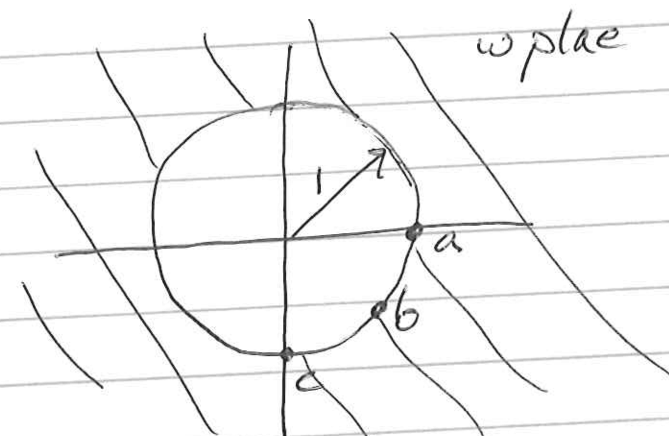

2. $w = z_0 z = r_0 r e^{i(\theta+\theta_0)}$ $\Rightarrow$ multiplicatio

$\Rightarrow$ rotation by $\theta_0$

$\Rightarrow$ amplification by $r_0$

3. $w = \frac{1}{z} = \frac{1}{r} e^{-i\theta}$ $\Rightarrow$ inversion

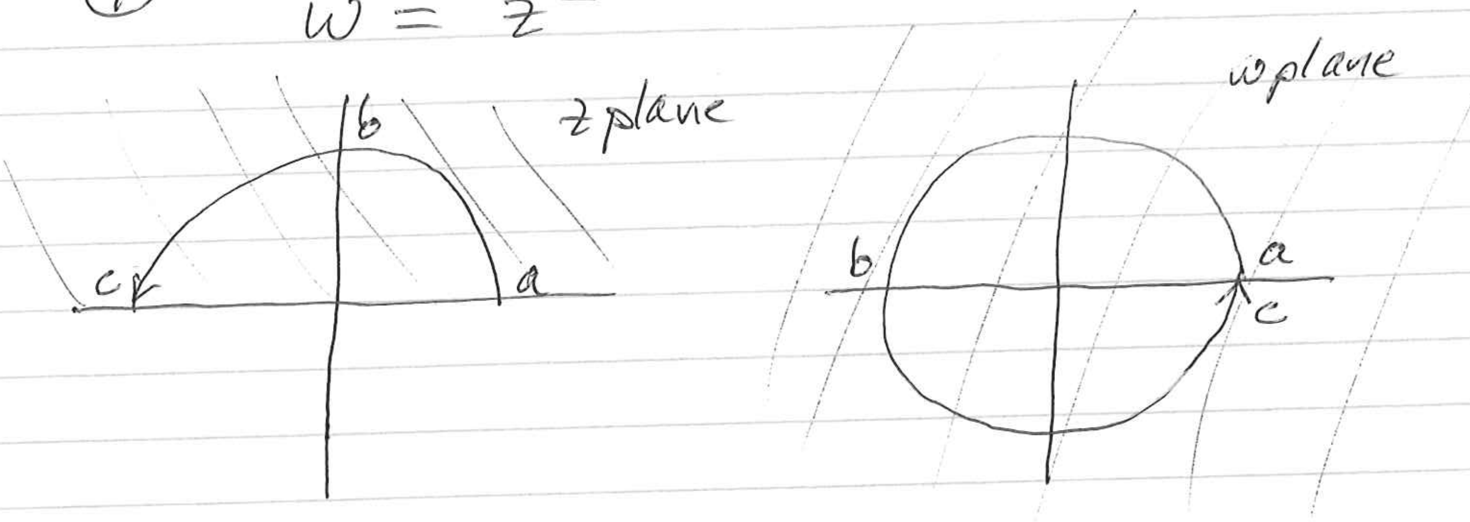



Inside of unit circle ~~maps~~ in z plane maps to outside of unit circle in w plan

These maps were all one-to-one since entire $z$ plane maps to entire $w$ plane

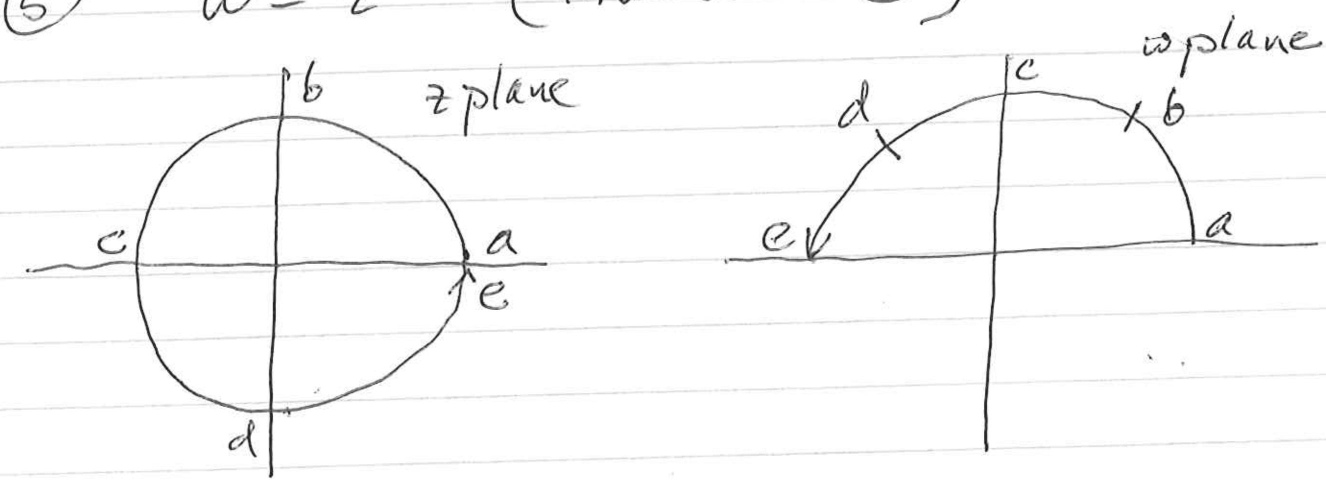
(4) $w = z^2$




Upper half $z$ plane maps to entire $w$ plane
Lower " " " " " " " "

$\Rightarrow$ any point $z_0$ and $-z_0$ map to the same point in $w$ plane

(5) $w = z^{1/2}$ (inverse of (4))



Note that $z = re^{i\theta}$ and $z = re^{i\theta + 2\pi i}$,

the same point in the $z$ plane correspond to

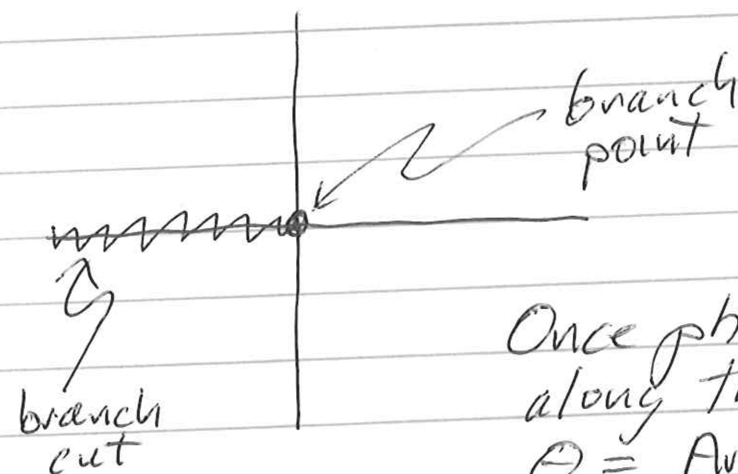
$$w = r^{1/2} e^{i\frac{\theta}{2}} \quad \text{and} \quad w = -r^{1/2} e^{i\frac{\theta}{2}},$$

two different points in the $w$ plane

$\Rightarrow$ $z^{1/2}$ is a multi-valued function

$\Rightarrow$ not good

Introduce a <u>branch</u> <u>cut</u> in the $z$ plane to limit the range of $\theta$.




Once phase of $z$ is defined along the real positive axis, $\theta = \mathrm{Arg}(z)$ can only change by $\pm\pi$ over the entire $z$ plane.

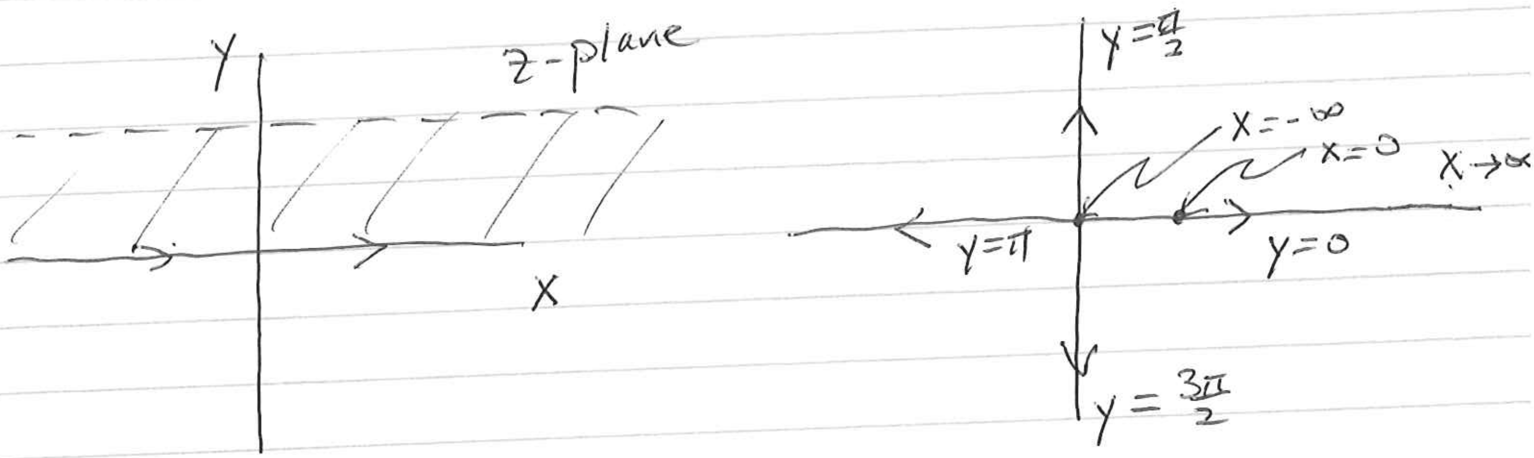
For $\theta = 0$ along the positive real axis

$$\theta \in (-\pi, \pi).$$

For $\theta = 2\pi$ along positive real axis, $\theta \in (\pi, 3$

$\Rightarrow$ $w = z^{1/2}$ is a single valued function in the cut $z$ plane.

(6) $w = e^{z} = e^{x} e^{iy}$




A horizontal line in $z$ plane maps to a radial line eminating from $w=0$ in the $w$ plane

$$w = e^{x} e^{iy_0} \quad x \to -\infty \Rightarrow w = 0$$

$y \in (0, 2\pi)$ maps to entire $w$ plane

Each succesive interval of $2\pi$ maps to entire $w$ plane.

(7) $w = \ln(z)$ (inverse of (6))

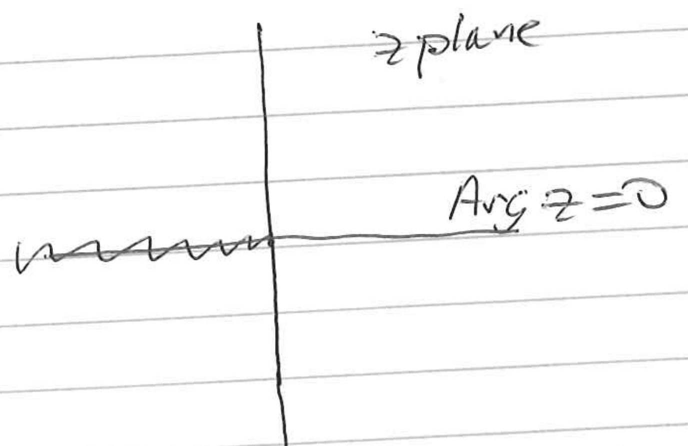
$w = \ln(re^{i\theta}) = \ln(r) + i\theta = u + iv$

In $z$ plane $\theta \to \theta + 2n\pi$

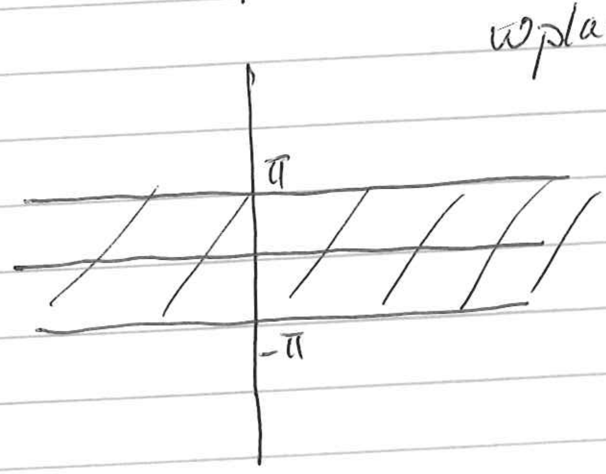
$v \to v + 2n\pi$

$\Rightarrow$ a single point in $z$ maps to many points in $w$. $\Rightarrow$ multi-valued function

$\Rightarrow$ define a branch cut in the $z$ plane




$w = \ln(z)$ is a single valued function on the cut $z$ plane



z plane


## Definition: limit

Consider a function $f(z) = w$ then

$$\lim_{z \to z_0} f(z) = w_0$$

means that as $z$ approaches $z_0$ from any direction, $f(z)$ approaches $w_0$.
By taking a small neighborhood around $z_0$ $f(z)$ can be made arbitrarily close to $w_0$.

A function is continuous at $z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

## Derivatives

$$f'(z) \equiv \frac{df}{dz} \equiv \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

From definition of limit must be same irrespective in which $\Delta z \to 0$, i.e., $\Delta z = \Delta x + i\Delta y$ and can let $\Delta x = 0$ and take $\lim_{\Delta y \to 0}$ or the reverse.

# Cauchy - Riemann Conditions

Suppose $f'(z_0)$ exists and $f(z_0) = w_0$
$$= u(x_0, y_0) + i v(x_0, y_0)$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{u(x_0+\Delta x, y_0+\Delta y) + i v(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0) - i v(x_0, y_0)}{\Delta x + i \Delta y}$$

Taylor series

$$= \lim_{\Delta z \to 0} \frac{\frac{\partial u}{\partial x_0}\Delta x + \frac{\partial u}{\partial y_0}\Delta y + i\frac{\partial v}{\partial x_0}\Delta x + i\frac{\partial v}{\partial y_0}\Delta y}{\Delta x + i \Delta y}$$

$\Delta y = 0$

$$f'(z_0) = \frac{\partial u}{\partial x_0} + i\frac{\partial v}{\partial x_0}$$

$\Delta x = 0$

$$f'(z_0) = -i\frac{\partial u}{\partial y_0} + \frac{\partial v}{\partial y_0}$$

This implies

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x_0} &= \frac{\partial v}{\partial y_0} \\ \frac{\partial v}{\partial x_0} &= -\frac{\partial u}{\partial y_0} \end{aligned}}$$

Cauchy Riemann Conditions

Theorem 1: If $f'$ exists at a value of $z$ then the C-R conditions are satisfied.

Theorem 2: If the C-R. conditions are satisfied then $f'(z_0)$ exists.

example

$$f(z) = e^z = e^x(\cos y + i\sin y)$$

$$u = e^x \cos y \qquad \Rightarrow \qquad \frac{\partial u}{\partial x} = e^x \cos y$$

$$v = e^x \sin y \qquad\qquad \frac{\partial v}{\partial y} = e^x \cos x$$

$$\frac{\partial v}{\partial x} = e^x \sin y$$

$$-\frac{\partial u}{\partial y} = -(-e^x \sin y) = e^x \sin y$$

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y$$

$$= e^x e^{iy} = e^z$$

$$\frac{d}{dz} e^z = e^z \quad \Rightarrow \text{same result as if } z \text{ were a real variable}$$

Note: $f'$ does not exist for arbitrary $u, v$ even if they are continuous and derivatives exist

Analytic function If $f'(z_0)$ exists at $z_0$ and everywhere in a neighborhood then is analytic at $z_0$.

<u>example</u>

1. $P_n(z) = a_0 + a_1 z + \cdots a_n z^n$ is analytic everywhere
   $\Rightarrow$ entire function

2. $f(z) = |z|^2 = x^2 + y^2$

   $v = 0$
   $u = x^2 + y^2$ $\Rightarrow$ require $\frac{\partial u}{\partial x} = 0$
   $\frac{\partial u}{\partial y} = 0$

   $\Rightarrow$ satisfied at $z = 0$
   $\Rightarrow$ CR conditions satisfied at $z = 0$
   $\Rightarrow$ differentiable at $z = 0$
   $\Rightarrow$ not analytic at $z = 0$

Proof of Theorem 2:

$$f'(z_0) = \lim_{z \to z_0} \frac{\frac{\partial u}{\partial x_0}\Delta x + \frac{\partial u}{\partial y_0}\Delta y + i\frac{\partial v}{\partial x_0}\Delta x + i\frac{\partial v}{\partial y_0}\Delta y}{\Delta x + i\Delta y}$$

$$= \lim_{z \to z_0} \frac{\frac{\partial v}{\partial y_0}\Delta x - \frac{\partial v}{\partial x_0}\Delta y + i\frac{\partial v}{\partial x_0}\Delta x + i\frac{\partial v}{\partial y_0}\Delta y}{\Delta x + i\Delta y}$$

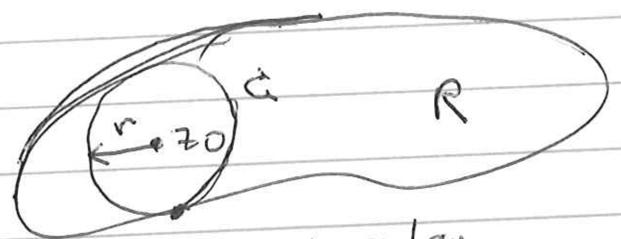
$$= \lim_{z \to z_0} \frac{\frac{\partial v}{\partial y_0}\cancel{(\Delta x + i\Delta y)} + i\frac{\partial v}{\partial x_0}\cancel{(\Delta x + i\Delta y)}}{\cancel{\Delta x + i\Delta y}}$$

$$f' = \frac{\partial v}{\partial y_0} + i\,\frac{\partial v}{\partial x_0} \Rightarrow \text{limit exists}$$

Theorem 3

If $f(z)$ is analytic in a region $R$, it has derivatives at all orders at any point in $R$.

It can be expanded in a Taylor series around any point $z_0$ in $R$. The series converges ~~inside~~ anywhere inside a circle whose radius is the distance to the closest singular point $z_1$ of $f$.




$z_1 \rightarrow$ singular point

$$r = |z_1 - z_0|$$

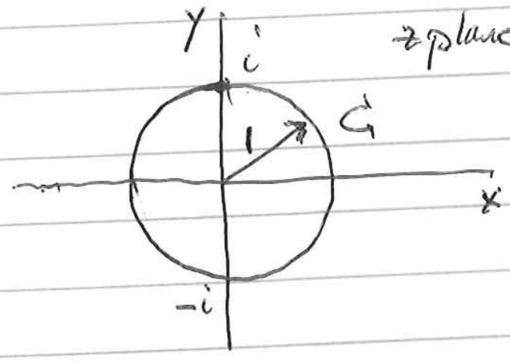
Taylor series converges within $G$

$\Rightarrow$ proofs later

example

$$f(z) = \frac{1}{1+z^2}$$

$$= 1 - z^2 + z^4 - z^6 \cdots$$




$\Rightarrow$ singularity at $\pm i$

$\Rightarrow$ radius of convergence $r = 1$

Definition

A "regular" point of $f(z)$ is a point where $f(z)$ is analytic

A singular point of $f(z)$ is a point where it is not analytic

~~Near An Isolated Singular Point~~

The singular point is isolated if $f(z)$ is analytic everywhere in a neighborhood of the singular point.

$f(z) = \frac{1}{z-1}$ has an isolated singular point at $z = 1$.

$f = \frac{1}{z^{1/2}}$ does not have an isolated sing. pt at 0 because have a branch cut.

<u>example</u> evaluating derivatives

$$f(z) = e^{z}$$

$$\frac{d}{dz} e^{z} = \lim_{\Delta z \to 0} \frac{e^{z+\Delta z} - e^{z}}{\Delta z}$$

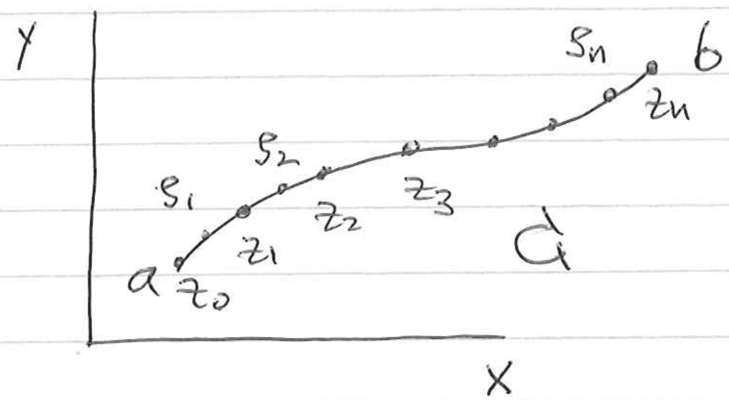
$$= \frac{e^{z}(1 + \Delta z + \cdots - 1)}{\Delta z}$$

$$= e^{z}$$

$\Rightarrow$ If $f$ can be written as a function of $z$ alone can simply take a derivative as in a real function.

~~$\frac{\partial v}{\partial y_0} + i\frac{\partial v}{\partial x_0}$ ⇒ limit exists~~

## Contour integrals




Divide contour into n segments ~~and by choosing~~ and choose n intermediate points

Let

$$S_n = \sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1})$$

where $\xi_j$ lies on $C_1$ and between $z_j$ and $z_{j-1}$. If $\lim_{n\to\infty} S_n$ exists then

$$\int_{a,\,C_1}^{b} dz\, f(z) \equiv \lim_{n\to\infty} \sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1})$$

⇒ contour integral over $C_1$.

## Cauchy - Goursat Theorem

If a function $f$ is analytic throughout a simply connected* region $R$, then for every closed contour $C$ in $R$

$$\boxed{\oint_C dz\, f(z) = 0}$$

* simply connected — every closed curve in a domain contains only points in domain

Proof that requires $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ be continuous

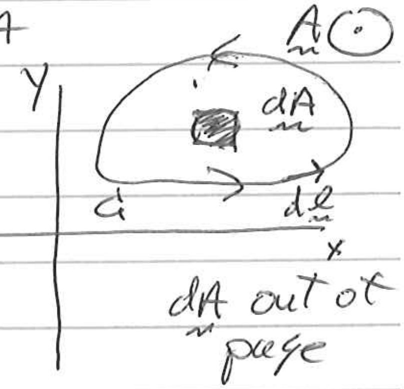
$\Rightarrow$ use Stokes Theorem for real functions

$$\oint_C dz\, f = \oint_C (u + iv)(dx + i\,dy)$$

$$= \oint_C \left[(u\,dx - v\,dy) + i(v\,dx + u\,dy)\right]$$

Stokes Theorem

$$\oint_C \underset{\sim}{B} \cdot d\underset{\sim}{l} = \int \nabla \times \underset{\sim}{B} \cdot dA$$




$$I_1 = \oint_C (u\,dx - v\,dy) = -\int_A dx\,dy \underbrace{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}_{\text{0 from C-R conditions}}$$

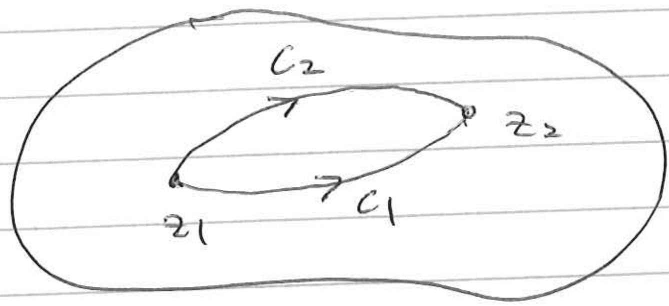
$$\underset{\sim}{B} = u\hat{x} - v\hat{y}$$

$$(\nabla \times \underset{\sim}{B})_z = \frac{\partial}{\partial x} B_y - \frac{\partial}{\partial y} B_x =$$

$$I_2 = \oint (v\,dx + u\,dy) = \int_A dx\,dy \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) = 0$$

$$\underset{\sim}{B} = v\hat{x} + u\hat{y} \quad (\nabla \times \underset{\sim}{B})_z = \frac{\partial u}{\partial x} - \frac{\partial}{\partial y} v$$

## Path independence of integrals




Consider a simply connected domain $R$ and two points $z_1$ and $z_2$ in $R$. Consider any two contours $C_1$ and $C_2$ that are entirely within $R$.

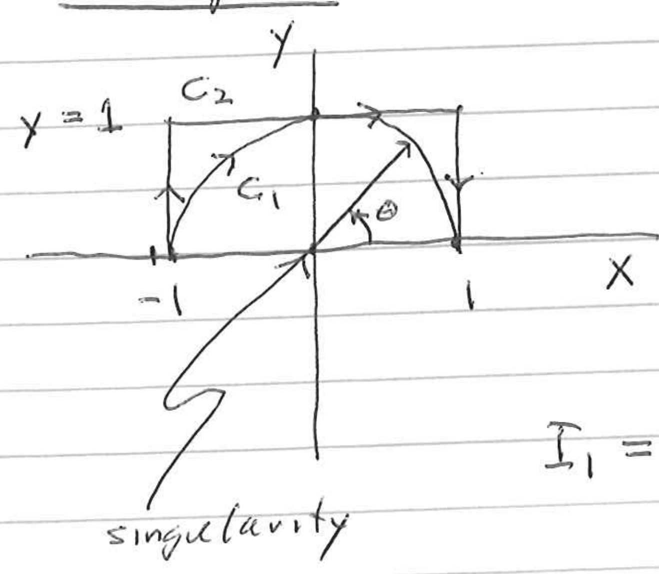
If $f(z)$ is analytic throughout $R$ then

$$\int_{C_1} dz\, f(z) = \int_{C_2} dz\, f(z)$$

$\Rightarrow$ independent of path

$\Rightarrow$ choose simplest path to do integral

**example** $\quad f(z) = \frac{1}{z}$




$$I_1 = \int_{C_1} dz\, \frac{1}{z}$$

$$z = e^{i\theta}$$

$$dz = i\,d\theta\, e^{i\theta}$$

$$I_1 = \int_{\pi}^{0} \frac{i\,d\theta\, e^{i\theta}}{e^{i\theta}} = i\int_{\pi}^{0} d\theta = -i\pi$$

$$I_2 = \int_{C_2} dz \frac{1}{z} = \int_{C_2} \frac{dx + i\,dy}{x + iy}$$

$$= \int_{C_2} \frac{(dx + i\,dy)(x - iy)}{x^2 + y^2}$$

$$= \underbrace{\int_{C_2} \frac{dx\, x}{x^2 + y^2}} + \underbrace{\int_{C_2} \frac{dy\, y}{x^2 + y^2}} + i \left[ \int_{C_2} \frac{dy\, x}{x^2 + y^2} - \int_{C_2} \frac{dx\, y}{x^2 + y^2} \right]$$

$$\underbrace{\int_{-1}^{1} \frac{dx\, x}{x^2 + 1}}_{\text{integrand odd}} \Rightarrow 0 \qquad \underbrace{\int_{0}^{1} \frac{dy\, y}{1 + y^2} + \int_{1}^{0} \frac{dy\, y}{1 + y^2}}_{\text{cancel}}$$

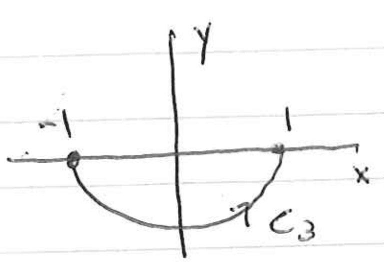
$$= i \left[ \int_{0}^{1} \frac{dy\,(-1)}{1 + y^2} + \int_{1}^{0} \frac{dy\,(1)}{1 + y^2} - \int_{-1}^{1} dx \frac{(1)}{x^2 + 1} \right]$$

$$= -2i \int_{-1}^{1} \frac{dx}{1 + x^2} = -2i \tan^{-1} x \Big|_{-1}^{1}$$

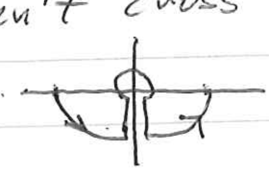
$$= -2i \left( \frac{\pi}{4} - \left( -\frac{\pi}{4} \right) \right) = -\pi i$$

$\Rightarrow$ same as $I_1$

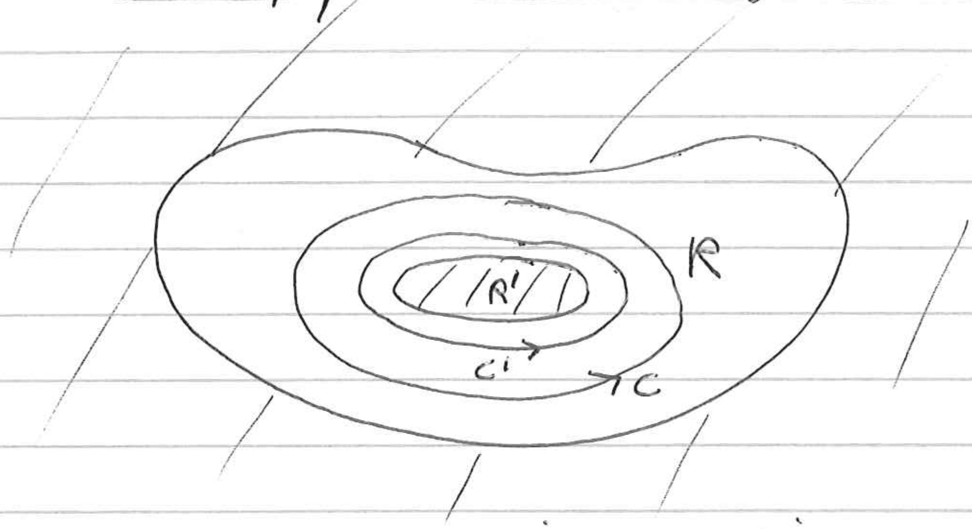
What about $I_3$? $\qquad I_3 = i\pi$



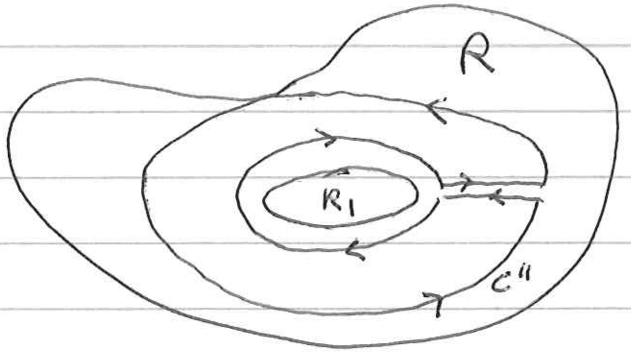

$\Rightarrow$ when moving contour can't cross singularity at $z = 0$



## Multiply Connected Regions




Suppose $f(z)$ is analytic everywhere in a simply connected region $R$ except in a region $R'$.




From C-G theorem

$$\oint_{C''} f(z)\,dz = 0$$

since $f$ analytic inside $C''$

$\Rightarrow$ two oppositely directed contours cancel as are moved close together

$$\Rightarrow \oint_{C''} f(z)\,dz = \oint_{C} f(z)\,dz - \int_{C'} dz\, f(z) = 0$$

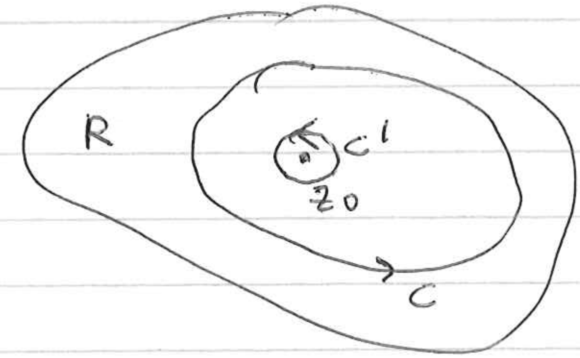
$$\Rightarrow \boxed{\oint_C f(z)\,dz = \int_{C'} dz\, f(z)}$$

$\Rightarrow$ can shrink contour

$\Rightarrow$ can't cross singularity

## Cauchy's Integral Formula

$$\oint_C \frac{f(z)\,dz}{z-z_0} = 2\pi i\, f(z_0)$$




Take $f(z)$ analytic in R.

Proof $\dfrac{f(z)}{z-z_0}$ is analytic everywhere except $z_0$

$$I = \oint_C \frac{f(z)\,dz}{z-z_0} = \int_{C'} \frac{dz\, f(z)}{z-z_0}$$

Take $c'$ very close to $z_0$ $\Rightarrow$ circle of radius $r$

$$z = z_0 + re^{i\theta}$$

$$dz = i\,d\theta\, re^{i\theta}$$

$$I = \lim_{r\to 0} \int_0^{2\pi} \frac{f(z_0+re^{i\theta})\, i\,d\theta\, re^{i\theta}}{re^{i\theta}}$$

$$= f(z_0)\, i \int_0^{2\pi} d\theta = 2\pi i\, f(z_0)$$

$\Rightarrow$ if you know a function on a boundary C then you know the function any where inside.

## General Derivatives

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z - z_0}$$

$$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i} \oint_C f\,dz \left[ \frac{1}{z - z_0 - \Delta z} - \frac{1}{z - z_0} \right]$$

$$= \frac{1}{2\pi i} \oint_C \frac{f\,dz}{(z - z_0 - \Delta z)(z - z_0)} \left[ \frac{\cancel{z - z_0} - (\cancel{z - z_0} - \Delta z)}{\Delta z} \right]$$

$$\Rightarrow \frac{1}{2\pi i} \oint \frac{f(z)\,dz}{(z - z_0)^2} \quad \text{as } \Delta z \to 0$$

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z - z_0)^2}$$

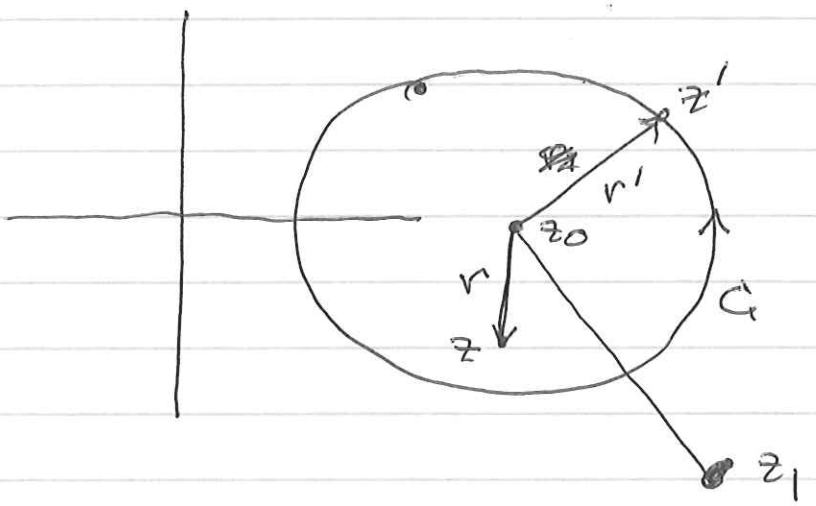
$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int dz \, \frac{f(z)}{(z - z_0)^{n+1}}$$

If $f(z)$ analytic then, derivatives of all orders exist.

$\Rightarrow$ proof of first part of Theorem 3

## Taylor Series

Expand an analytic function $f(z)$ around $z_0$ where $z_1$ is the nearest point where $f$ is not analytic (singular)




$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{z'-z}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0+z_0-z)}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)\left(1+\frac{z_0-z}{z'-z_0}\right)}$$

Note the exact relation

$$\frac{1}{1-t} = 1+t+t^2+\dots t^{N-1} + \frac{t^N}{1-t}$$

Let $t = \frac{z_0-z}{z'-z_0}$ $\Rightarrow$ note $|t| = \frac{r}{r'} < 1$

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{z'-z_0}\left[\sum_{n=0}^{N-1}\left(\frac{(z-z_0)}{(z'-z_0)}\right)^n + \frac{t^N}{1-t}\right]$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{N-1}(z-z_0)^n \oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}} + R_N$$

$$= \sum_{n=0}^{N-1}\frac{(z-z_0)^n}{n!} f^{(n)}(z_0) + R_N$$

Since $|t| < 1$,

$R_N \to 0$ as $N \to \infty$ as long as

$f(z')$ on $C'$ remains bounded.

Thus, $r' < |z_1 - z_0|$

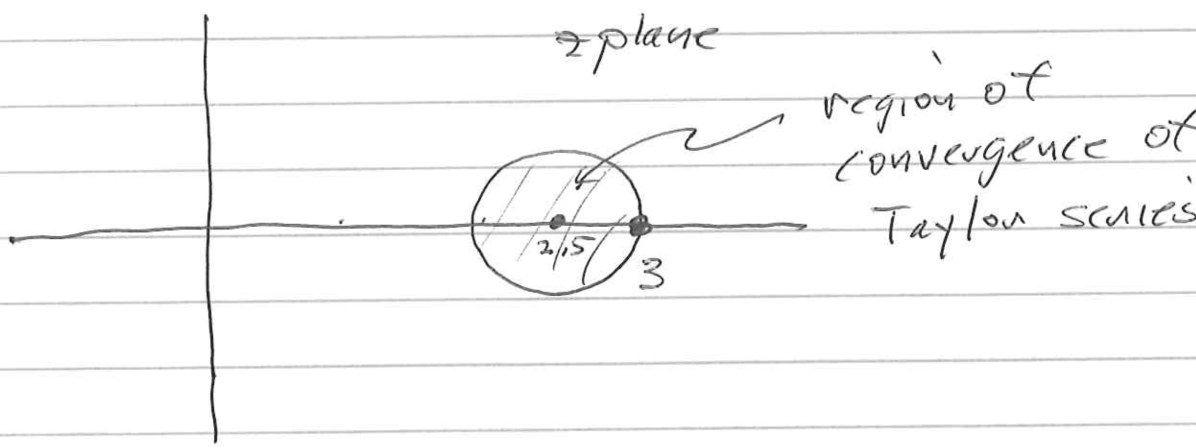
$\Rightarrow$ radius of convergence of the Taylor series is determined by the distance to the nearest singular point

$\Rightarrow$ note that series solutions are unique

example

Want to expand $\frac{1}{z-3}$ around $z_0 = 2.5$.

$\Rightarrow$ radius of convergence $|z - z_0| < 0.5$




example What is Taylor series of $f(z) = \frac{1}{1-z}$ around $z = 0$?

$$f(z) = \frac{1}{1-z} = 1 + z + z^2 + \cdots$$

Is this Taylor series? $\Rightarrow$ yes $\Rightarrow$ series representation are unique. why

## Laurent Series

Suppose that $f(z)$ has an isolated singularity at $z = z_0$, then can write $f(z)$ near $z_0$ in a series as follows

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

Can show that the $a_j$'s can be calculated by doing a closed contour integral but this is not the easiest way to find the $a_j$s.

If $a_j = 0$ for all $j < -n$, then $f(z)$ has an nth order pole at $z_0$.
If $a_j$ there are an infinite # of values of $a_j$ ~~for~~ $\neq 0$ for $j < 0$, $f(z)$ has an essential singularity at $z_0$.

The coefficient $a_{-1}$ is the residue of $f$ at $z_0$. We will see the importance of $a_{-1}$ later.

example

$$f(z) = \frac{1}{(z-2)^2}$$

has a second order pole at $z = 2$

<u>example</u> Find the Laurent series of

$$f(z) = \frac{1}{z}\,\frac{1}{1-z}$$

around $z = 0$.

$$f(z) = \frac{1}{z}\left(1 + z + z^2 + z^3 + \cdots\right)$$

$$= \frac{1}{z} + 1 + z + z^2 + \cdots$$

$\Rightarrow$ valid for $|z| < 1$

Find the Laurent series around $z = 1$.

Let $t = z - 1$

$$f(t) = -\frac{1}{t}\,\frac{1}{1+t}$$

$$= -\frac{1}{t}\left(1 - t + t^2 - t^3 \cdots\right)$$

$$= -\frac{1}{t} + 1 - t + t^2 \cdots$$

$\Rightarrow$ valid for $|t| < 1$ or $|z-1| < 1$


<u>example</u> $f(z) = e^{\frac{1}{z}}$

$$f(z) = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots$$

$\Rightarrow$ essential singularity

<u>example</u> $f(z) = \dfrac{\sin^2 z}{z^3}$

What is the order of the pole at $z=0$?

For small $z$, $\sin(z) \simeq z$ (Taylor series)

so

$$f(z) \sim \frac{z^2}{z^3} = \frac{1}{z}$$

$\Rightarrow$ first order pole.