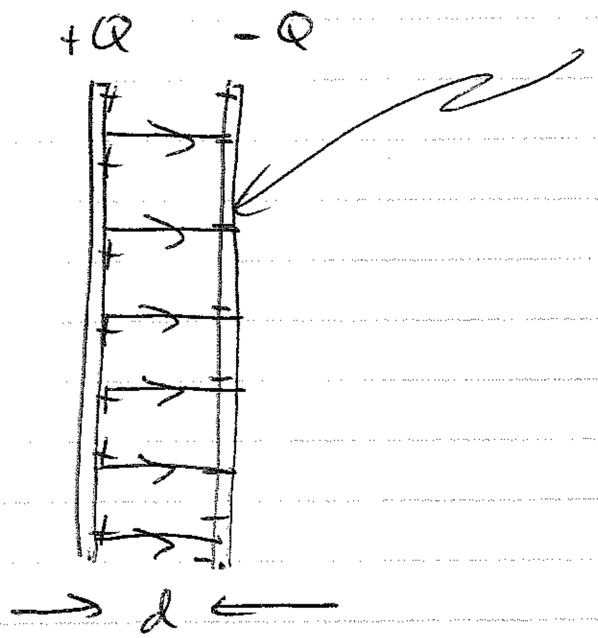
Capacitors

A capacitor is a ~~[illegible]~~ device consisting of two conductors separated by an insulating material. A capacitor can store charge and therefore the energy associated with that charge.

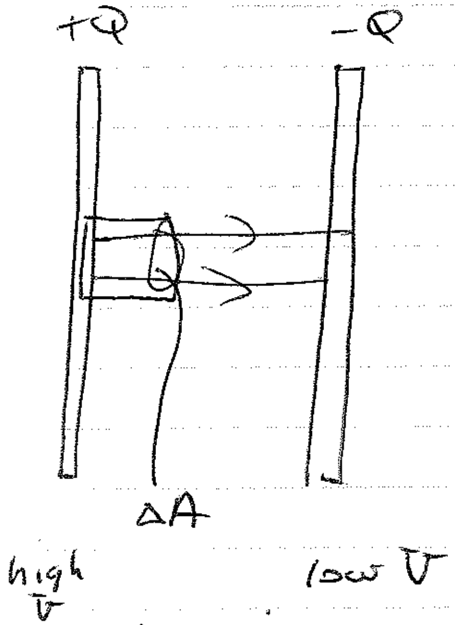
parallel plate capacitor



Assume that $d \ll L$ where L is the dimension transverse to d.

$\Rightarrow$ can calculate the electric field by assuming that the plates are effectively of infinite extent.

$$\sigma = \frac{Q}{A} = \frac{\text{charge}}{\text{area}}$$



$$\oint \mathbf{E}\cdot d\mathbf{A} = E\,\Delta A = \frac{\sigma \Delta A}{\epsilon_0}$$

$$\boxed{E = \frac{\sigma}{\epsilon_0}}$$

$$\boxed{E = \frac{Q}{A\epsilon_0}}$$

What is the potential drop across the capacitor.

$$\Delta V = Ed = \frac{Qd}{A\epsilon_0}$$

Define the capacitance as the ability of the capacitor to store charge

$$C = \frac{Q}{\Delta V} = \frac{\cancel{Q}\, A\epsilon_0}{\cancel{Q}\, d}$$

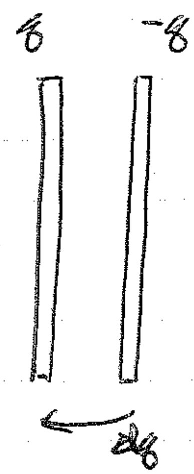
$$\boxed{C = \frac{A\epsilon_0}{d}}$$

Note that C is a function of the geometry of the capacitor and not charge

A capacitor with C large can store a lot of charge with only a small voltage drop.

units C in farads $= \dfrac{1\,C^2}{\text{Joule}}$ over

⇒ Energy Stored in a Capacitor



Consider a capacitor which at some time has a charge q and voltage difference

$$v = \frac{1}{\epsilon_0}\frac{q d}{A} = \frac{q}{C}$$

We take a small amount of charge dq and move it from $-q$ to $+q$. The work required to do this is

$$dW = v\,dq = \frac{q d}{\epsilon_0 A}\,dq$$

To find the total work must sum dW as we build up the charge from $q=0$ to $q=Q$

$$W = \int dW = \int_0^Q \frac{q d}{\epsilon_0 A}\,dq$$

$$= \frac{d}{\epsilon_0 A}\frac{q^2}{2}\Big|_0^Q = \frac{dQ^2}{2\epsilon_0 A} = \frac{Q^2}{2C}$$

$$= \frac{QV}{2}$$

$$U = \frac{Q^2}{2C}$$

U is the energy stored in the capacitor.

Can write U as

$$U = \frac{1}{2} C V^2 = \frac{1}{2} Q V$$ ← good form for energy!

example

Consider a capacitor with $d = 1\,mm$ and $A = 1\,cm^2$ charged to 100 volts. How much energy is stored?

$$\frac{A\epsilon_0}{d} = C = 8.85\times10^{-12} \frac{C^2}{Nm^2} \frac{\cancel{1cm^2}}{\cancel{.1cm}} = 8.85\times10^{-13}\ F$$

$$U = \frac{1}{2}\, 8.85\times10^{-13} \frac{\cancel{C^2}}{\cancel{Nm}}\, 10^4 \frac{J^2}{\cancel{C^2}} \qquad 1V = \frac{J}{C}$$

$$\boxed{U = 4.4\times10^{-9}\ J}$$

Energy in Electric Field

$$U = \frac{1}{2} C V^2 = \frac{1}{2}\epsilon_0 \frac{Ad}{d^2} V^2 = \frac{1}{2}\epsilon_0 \underbrace{(Ad)}_{\text{vol.}} E^2$$

$$u = \frac{U}{\text{vol}} = \frac{1}{2}\epsilon_0 E^2$$ = energy density of the local electric field.

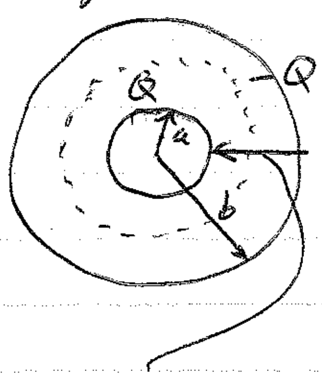
Calculating the capacitance

Calculate the electric field
$\Rightarrow$ usually with Gauss' law

$\Rightarrow$ calculate the potential difference

$$V_2 - V_1 = -\int_{x_1}^{x_2} \underset{\sim}{E} \cdot d\underset{\sim}{x}$$

example Cylindrical capacitor
Length L



path integral

$$\int \underset{\sim}{E} \cdot d\underset{\sim}{A} = \frac{1}{\epsilon_0} q$$

$$EA = \frac{Q}{\epsilon_0}$$

$$E\, 2\pi r L = \frac{Q}{\epsilon_0}$$

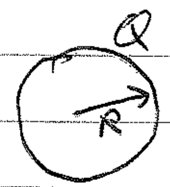
$$E = \frac{Q}{2\pi r L \epsilon_0}$$

$$V = V_a - V_b = -\int_b^a E\, dr$$

$$= -\frac{Q}{2\pi L \epsilon_0} \int_b^a \frac{dr}{r} = -\frac{Q}{2\pi L \epsilon_0} \ln\left(\frac{a}{b}\right)$$

$$V = \frac{Q}{2\pi L \epsilon_0} \ln\left(\frac{b}{a}\right) \qquad C = \frac{Q}{V} = \frac{2\pi L \epsilon_0}{\ln\left(\frac{b}{a}\right)}$$

<u>example</u> capacitance of an isolated sphere

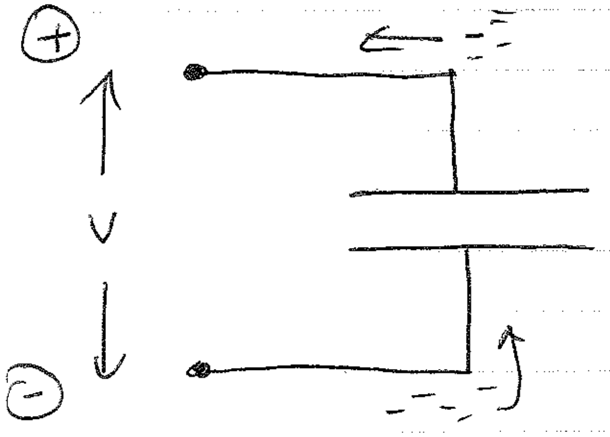


$$V = \frac{Q}{4\pi\epsilon_0 R}$$

$$C = 4\pi\epsilon_0 R$$

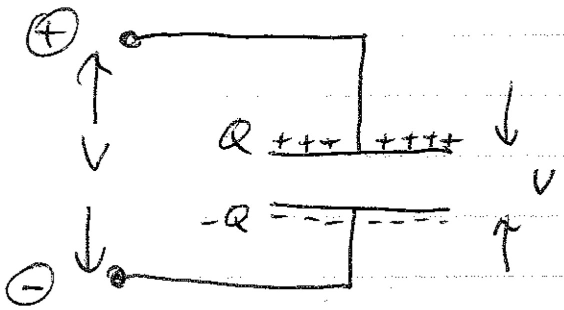
$\Rightarrow$ can think of the negative plate as being at ∞.

Charging a Capacitor



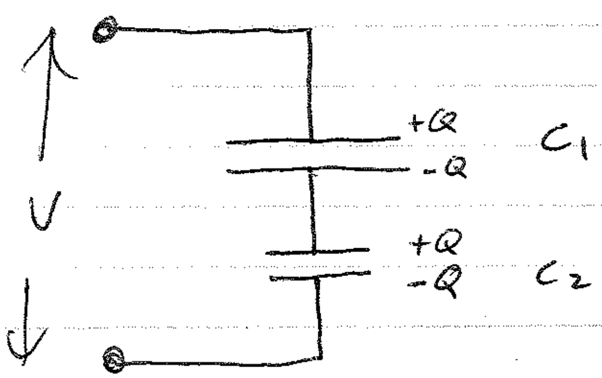
Apply a potential V across the wires connected to the capacitor.

Electrons in plate flow toward ⊕ and away from ⊖ until charge in capacitor builds up



$$\boxed{Q = CV}$$

Series ~~Parallel~~ Capacitors



$$V_1 = \frac{Q}{C_1}$$

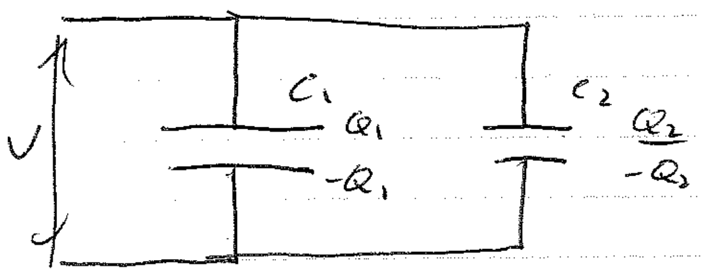
$$V_2 = \frac{Q}{C_2}$$

$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$= Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

$$\boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}} \implies \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$$

~~Ser~~
Parallel Capacitors



$Q_1 = C_1 V$

$Q_2 = C_2 V$

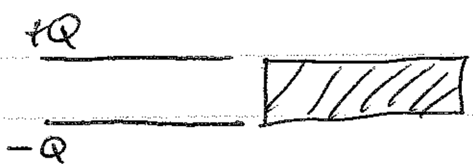
$Q = Q_1 + Q_2 = (C_1 + C_2) V$

$$\boxed{C = C_1 + C_2}$$

Polarization of Dielectrics

A dielectric is a non-conducting material which is placed between the plates of a capacitor $\Longrightarrow$ mechanically holds the plates apart.

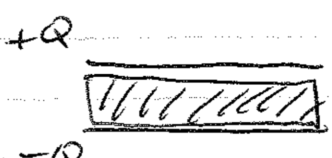
$V_0 = \frac{C_0}{Q}$



$V = \frac{V_0}{K}$ K = dielectric constant

$K \geq 1$

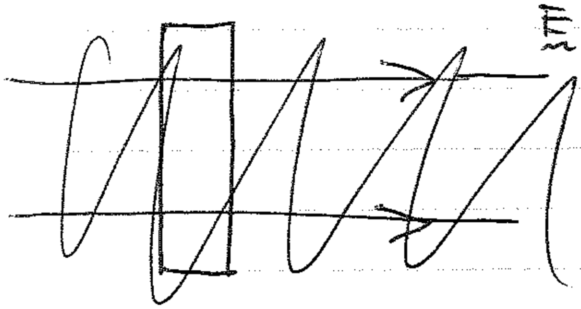
why is $V < V_0$?



$V < V_0$

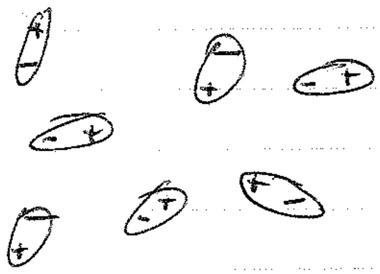
$Q = CV = C\,\frac{V_0}{\ldots} = C_0 V_0$

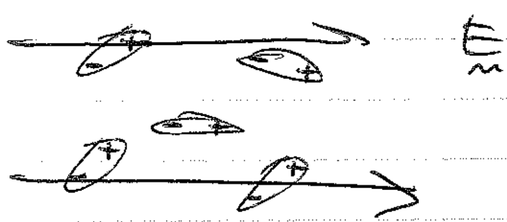
$C_0 = \frac{C_1}{K}$ $\boxed{C_1 = K C_0}$ capacitance increased.



why?

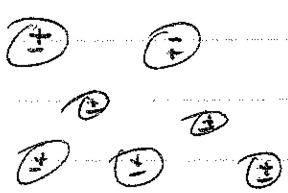
① polar materials have molecules in which charge is not evenly distributed



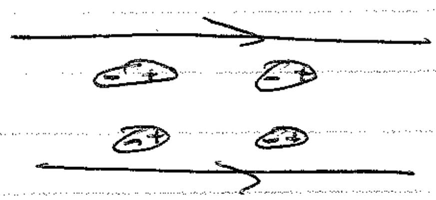


molecules tend to align with $\underset{\sim}{E}$ field

② Non polar molecules

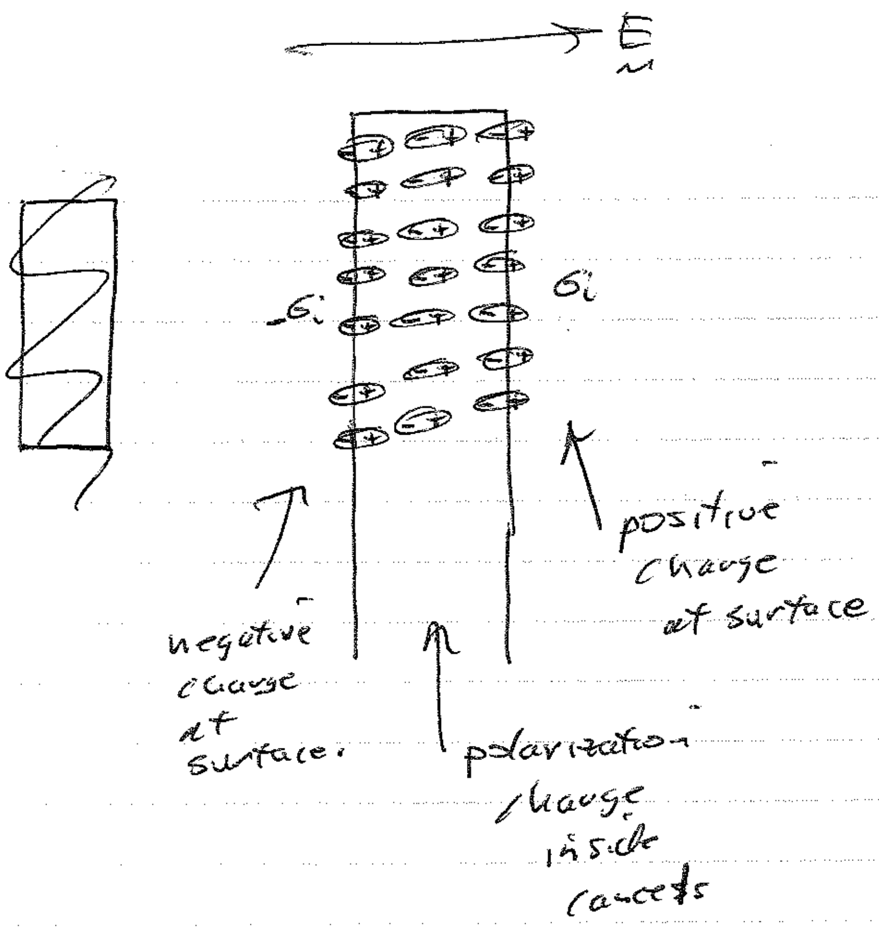


even charge distribution

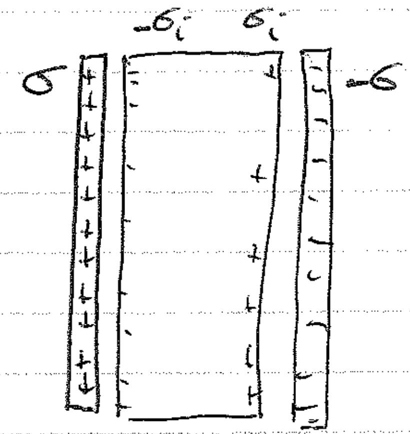


electrons pulled to left + to the right.

$\Longrightarrow$ say that the dielectric is polarized by $\underset{\sim}{E}$



In the capacitor



$$\text{Total } \frac{\text{charge}}{\text{area}} = \sigma - \sigma_i$$

$$E = \frac{\sigma - \sigma_i}{\epsilon_0} \qquad E_0 = \frac{\sigma}{\epsilon_0}$$

$$\left.\begin{aligned} \frac{E}{E_0} &= \frac{Ed}{E_0 d} = \frac{V}{V_0} = \frac{1}{K} \Rightarrow \\ \frac{E}{E_0} &= \frac{\sigma - \sigma_i}{\epsilon_0} \frac{1}{\frac{\sigma}{\epsilon_0}} = \frac{\sigma - \sigma_i}{\sigma} \end{aligned}\right\} \quad \begin{aligned} \frac{\sigma - \sigma_i}{\sigma} &= \frac{1}{K} \\ \sigma - \sigma_i &= \frac{\sigma}{K} \end{aligned}$$

$$\sigma_i = \sigma\left(1 - \frac{1}{K}\right)$$

Can write

$$E = \frac{\sigma - \sigma_i}{\epsilon_0} = \frac{\sigma}{K\epsilon_0} = \frac{\sigma}{\epsilon}$$

$$\boxed{\epsilon = \epsilon_0 K} \qquad \boxed{C = \frac{\epsilon A}{d}} \qquad \frac{\epsilon_0 A}{d}$$

ϵ = permittivity

Energy Density with dielectric

$$U = \frac{1}{2} Q V$$

V reduced from absense of dielectric $\quad VC = Q$

$$U = \frac{1}{2} C V^2 = \frac{1}{2} \epsilon E^2 (Ad)$$

$$u = \frac{1}{2} \epsilon E^2$$ energy density with dielectric.

Gauss' Law with Dielectrics

Gauss' law is given by

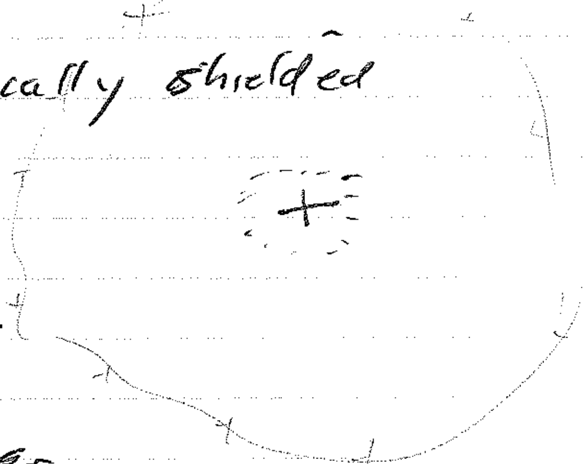
$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$ where q is the total charge with the surface

$$q = q_f + q_i$$

q_f = free charge

q_i = induced charge.

If have charge in a uniform dielectric the free charge is basically shielded by the induced charge.



$$q_f + q_i = \frac{q_f}{K}$$

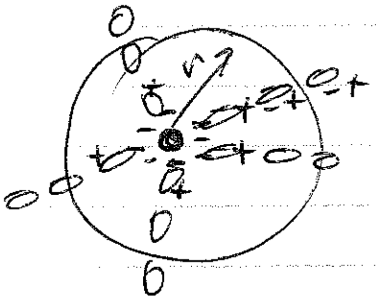
$$\oint \underset{\sim}{E}\cdot d\underset{\sim}{A} = \frac{q_f}{K\epsilon_0} = \frac{q_f}{\epsilon}$$

If the dielectric does not have a uniform ϵ then.

$$\oint \epsilon \underset{\sim}{E}\cdot d\underset{\sim}{A} = q_f$$

$$\underset{\sim}{D} \equiv \epsilon \underset{\sim}{E} = \text{electric displacement.}$$

<u>example</u> charge Q in uniform dielectric



charge is shielded by the dielectric

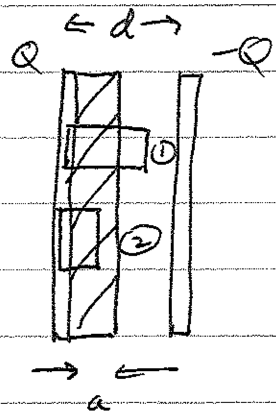
$$\int \epsilon \underset{\sim}{E}\cdot d\underset{\sim}{A} = Q$$

$$E\, 4\pi r^2 = \frac{Q}{\epsilon_0 K}$$

$$E = \frac{Q}{K}\,\frac{1}{4\pi\epsilon_0 r^2}$$

$\Rightarrow$ reduced E

example Capacitor with dielectric



$$\oint_{1} \epsilon \underset{\sim}{E} \cdot d\underset{\sim}{A} = \epsilon_0 E_1 A = Q$$

$$E_1 = \frac{Q}{\epsilon_0 A}$$

$$\oint_{2} \epsilon E \cdot dA = \epsilon E_2 A = Q$$

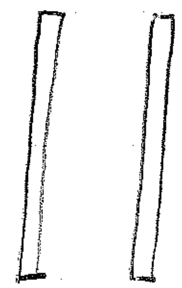
$$E_2 = \frac{Q}{\epsilon A} = \frac{Q}{KA\epsilon_0}$$

$$V = E_2 a + E_1 (d-a)$$

$$V = \frac{Q}{A\epsilon_0}\left[\frac{1}{K}a + d - a\right]$$

~~example Force on dielectric~~

<u>Example</u> force on a capacitor plate

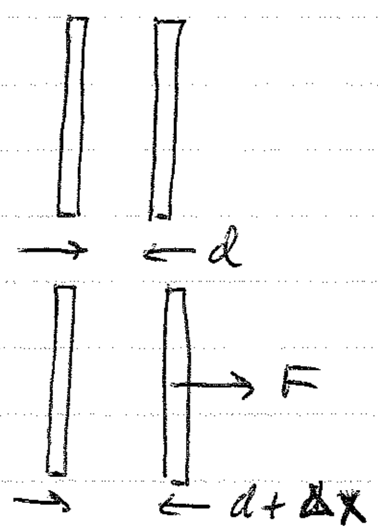


Can calculate the force on the plates in two ways

① directly evaluating
$F = qE$

② Look at the change in energy due to small change in the separation of the plates and relate to the work done.

$\Rightarrow$ do ② first



$$W_i = \left(\frac{\epsilon_0 E^2}{2}\right) A d$$

$$W_f = \frac{1}{2}\epsilon_0 E^2 A (d + \Delta x)$$

$\Rightarrow$ note E is the same

work done $= F\Delta x = W_f - W_i = \frac{1}{2}\epsilon_0 E^2 A \Delta x$

$$F = \frac{1}{2}\epsilon_0 E^2 A$$

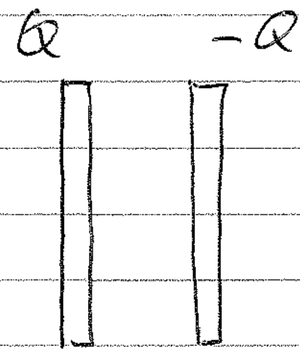
$$E = \frac{\sigma}{\epsilon_0}$$

$$F_{plate} = -\frac{1}{2}\epsilon_0 E^2 A$$

$$= -\frac{1}{2}\cancel{\epsilon_0} E \frac{\sigma}{\cancel{\epsilon_0}} A$$

$$\boxed{F_{plate} = -\frac{1}{2} E Q}$$

why factor of $\frac{1}{2}$?



$\Rightarrow$ charge can not accelerate itself. What is E in location of $-Q$ due only to $+Q$

$$E = +\frac{\sigma}{2\epsilon_0}$$

$$F = \frac{\sigma}{2\epsilon_0} Q = \frac{\sigma A}{2\epsilon_0}\frac{Q}{A}$$

$$= \frac{1}{2}\epsilon_0 E^2 A$$