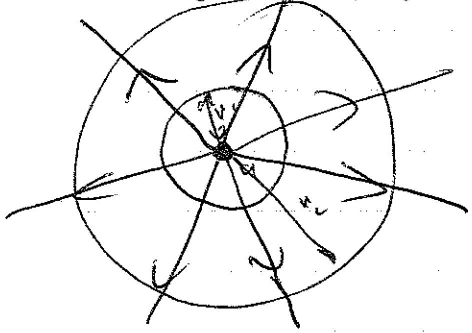
Gauss' Law

Consider a charge q



$$E = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}$$

Consider a spherical shell of radius r. The area of this shell is

$$A = 4\pi r^2$$

$$\underbrace{EA}_{\text{electric flux}} = \frac{q}{\cancel{4\pi}\epsilon_0 \cancel{r^2}}\cancel{4\pi r^2} = \frac{q}{\epsilon_0}$$

$EA = \dfrac{q}{\epsilon_0}$ is indep of r.

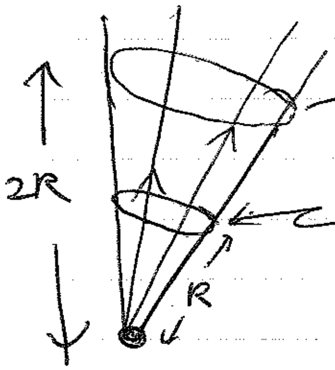
$\Rightarrow$ only depends on amount of charge q.

measure of # of field lines crossing area A.

$\Rightarrow$ can think of this as a conservation law for electric field lines

$\Rightarrow$ electric field starts and ends on charges so if some # of field lines cross through a radius r then the same # will cross through $2r$. if no other charges are present.

Same holds true for any fraction of the surface area

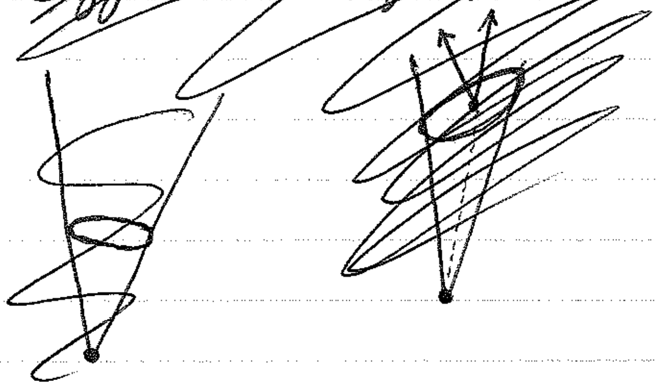


$4\Delta A$, $E = \frac{q}{4\pi\epsilon_0}\frac{1}{4R^2}$

ΔA $E = \frac{q}{4\pi\epsilon_0}\frac{1}{R^2}$

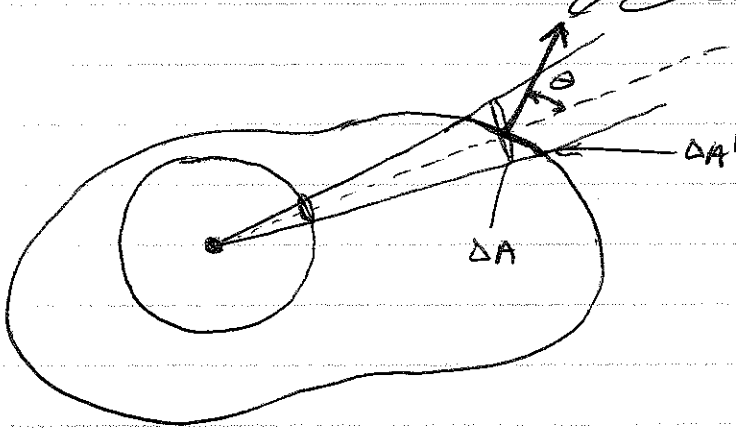
$E\Delta A$ indep of radius.

Suppose we tip the area.



$$\sum_{\substack{\text{all}\\\text{small}\\\text{areas}}} E\Delta A = \frac{q}{\epsilon_0}$$

Surround charge with a closed irregular surface

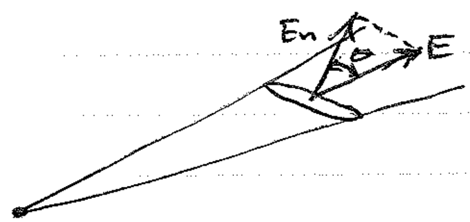


$\Delta A' \cos\theta = \Delta A$

$E\Delta A = \Delta A' E \cos\theta$

$\sum E \Delta A' \cos\theta = \frac{q}{\epsilon_0}$

$E\cos\theta = E_n =$ electric field $\perp$ to surface $\Delta A'$



$$\Sigma\, E_n \Delta A' = \frac{q}{\epsilon_0}$$

$$\oint E_n\, dA' = \frac{q}{\epsilon_0}$$

Take any ~~surface~~ closed surface around q then the ~~sum of the~~ integral of the electric field $\perp$ to the surface times the element of area is equal to the charge enclosed.

$\Rightarrow$ a ~~stat~~ statement that the # of electric field lines is conserved

~~$\Rightarrow$ the electric field field flux~~

$\Rightarrow$ electric flux is preserved.

$$E_n\, dA \equiv \underset{\sim}{E} \cdot d\underset{\sim}{A}$$

$$\boxed{\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \frac{q}{\epsilon_0}}$$

Gauss' Law.

Must be closed surface

$d\underset{\sim}{A}$ is outward and locally $\perp$ to the surface.

To extend to multiple charges eg q_1, q_2

$$\oint \underset{\sim}{E_1} \cdot d\underset{\sim}{A} = \frac{q_1}{\epsilon_0} \qquad \oint (\underbrace{\underset{\sim}{E_1} + \underset{\sim}{E_2}}_{\underset{\sim}{E}}) \cdot d\underset{\sim}{A} = \frac{q_1 + q_2}{\epsilon_0}$$

$$\oint \underset{\sim}{E_2} \cdot d\underset{\sim}{A} = \frac{q_2}{\epsilon_0} \qquad \oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \frac{q}{\epsilon_0}$$

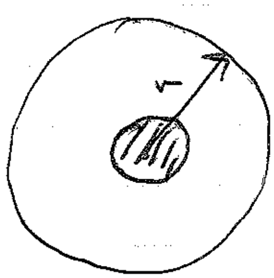
$q =$ total charge enclosed

Gauss Law can be used to solve for the electric field. It is fully equivalent to Coulombs law.

$\Rightarrow$ useful when $\underset{\sim}{E} \cdot d\underset{\sim}{A}$ is constant over a surface

<u>example</u>

Charged conducting sphere (charge Q with radius R)



What is $\underset{\sim}{E}$ inside the conductor?

$\Rightarrow \underset{\sim}{E} = 0$

On any surface inside the conductor

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = 0 = \frac{\text{Charge enclosed}}{\epsilon_0}$$

$\Rightarrow$ no excess charge inside conductor.

$\Rightarrow$ charge must lie on the surface
This is true for all conductors
irrespective of the shape.

$\Rightarrow$ excess charge wants to spread out as much as it can.

For the case of a sphere $\Rightarrow$ charge distribution on surface is uniform by symmetry.

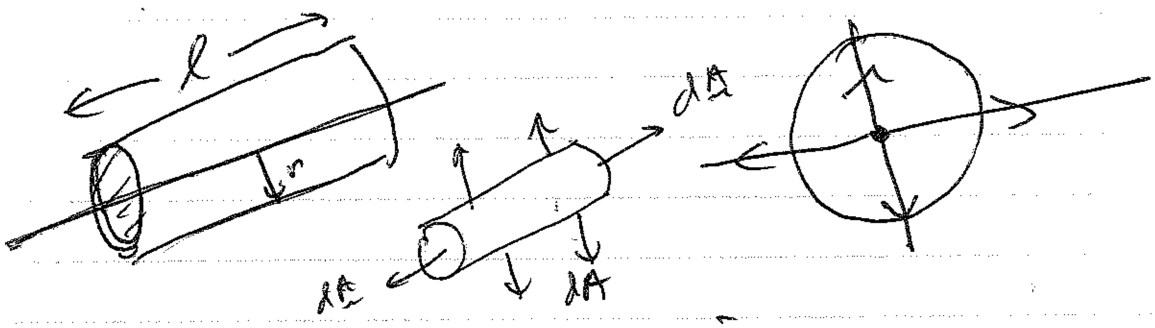
$\Rightarrow$ electric field is radial $\Rightarrow$ symmetry

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = E \, 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$ $\Rightarrow$ same as point charge.

<u>example</u>

Line charge with charge/length λ



$\underset{\sim}{E}$ outward from line

$\underset{\sim}{E} \cdot d\underset{\sim}{A} = 0$ on the ends of surface

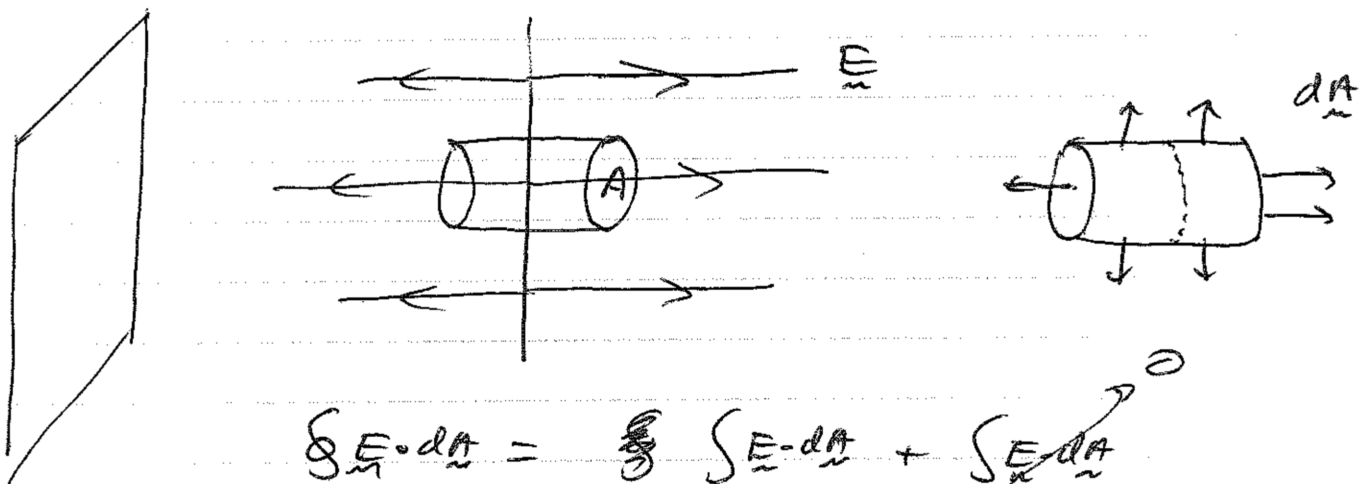
from side
$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \cancel{\int_{ends} \underset{\sim}{E} \cdot d\underset{\sim}{A}} + \int_{side} \underset{\sim}{E} \cdot d\underset{\sim}{A}$$

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = E \, 2\pi r l = \frac{\lambda l}{\epsilon_0}$$

$$\boxed{E = \frac{\lambda}{2\pi\epsilon_0 r}}$$ same as before

<u>example</u>

Consider a charged sheet with charge per unit area σ.



$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \int_{ends} \underset{\sim}{E} \cdot d\underset{\sim}{A} + \cancelto{0}{\int_{side} \underset{\sim}{E} \cdot d\underset{\sim}{A}}$$

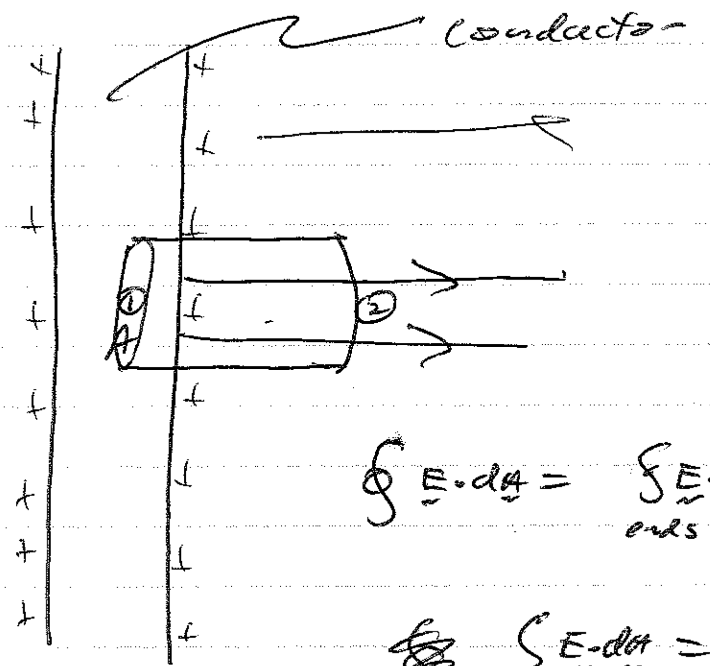
$$= 2EA = \frac{q}{\epsilon_0}$$

$$q = \sigma A$$

$$2E\cancel{A} = \frac{\sigma \cancel{A}}{\epsilon_0} \qquad \boxed{E = \frac{\sigma}{2\epsilon_0}}$$

example

Consider an infinite conducting plate. Let σ be charge per unit area on each ~~sheet~~ surface



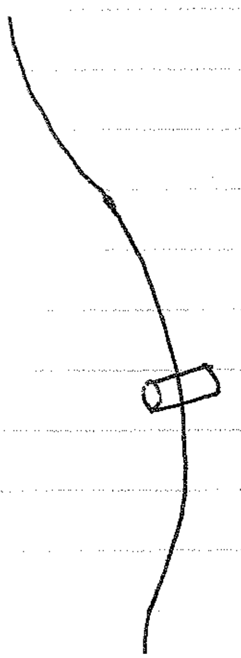
$$\oint \underset{\sim}{E}\cdot d\underset{\sim}{A} = \int_{ends} \underset{\sim}{E}\cdot dA = \cancel{\int_1 E\cdot dA}^{\,0} + \int_2 \underset{\sim}{E}\cdot dA$$

$\int_1 \underset{\sim}{E}\cdot d\underset{\sim}{A} = 0$ since $\underset{\sim}{E} = 0$ inside conductor

$$\int_2 E\cdot dA = EA = \frac{\sigma A}{\epsilon_0}$$

$$\boxed{E = \frac{\sigma}{\epsilon_0}}$$

More general shaped conductor



$\underset{\sim}{E}$ must be $\perp$ to surface or charge would move.

$\Rightarrow$ Gaussian surface over very small region near surface

$\underset{\sim}{E} \approx$ const over this surface

$$\Rightarrow \boxed{E = \frac{\sigma}{\epsilon_0}}$$

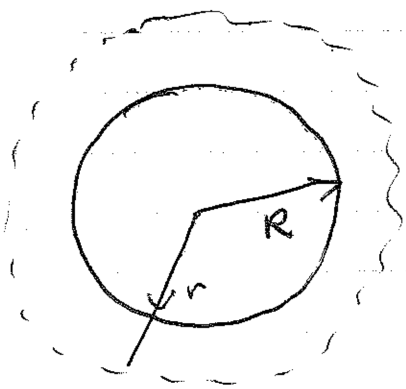
This is relation between σ and E

$\Rightarrow$ does not help you ~~...~~ calculate
E because ~~...~~ σ is not uniform
over surface and is therefore an
unknown.

~~Do this for a cylinder $\lambda = \frac{\text{charge}}{\text{unit length}}$~~

example

Sphere with uniform charge density.
Total Charge Q



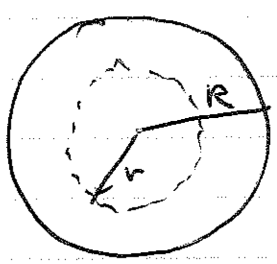
$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{\text{charge}}{\text{vol}}$$

$r > R$

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = 4\pi r^2 E = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

$r < R$

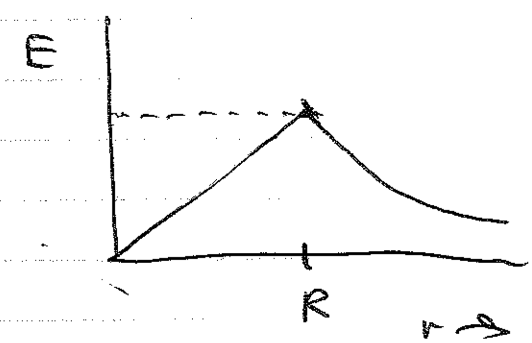


$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = 4\pi r^2 E = \frac{q}{\epsilon_0}$$

$$q = \rho \frac{4}{3}\pi r^3 = Q\left(\frac{r}{R}\right)^3$$

$$4\pi r^2 E = \frac{Q\left(\frac{r}{R}\right)^3}{\epsilon_0}$$

$$\boxed{E = \frac{Q}{4\pi\epsilon_0} \frac{r}{R^3}}$$



at $r > R$

both give $E = \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2}$