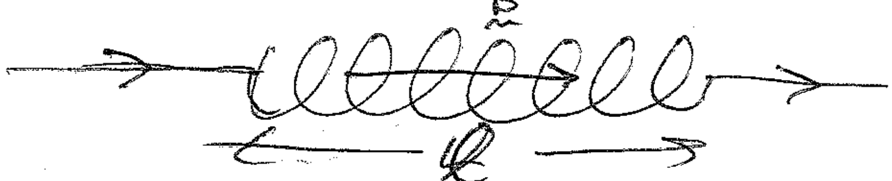
Self-Inductance

Consider a long solenoid of area A and length ℓ wound with N turns of wire with current I.



magnetic field is $B = \frac{\mu_0 N I}{\ell}$

magnetic flux going through the solenoid is

$$\Phi = BA = \frac{\mu_0 N I}{\ell} A$$

What is the EMF induced ~~in the loop~~ across the solenoid as the current changes

$$\mathcal{E} = -N \frac{d\Phi}{dt} = -N \frac{\mu_0 N A}{\ell} \frac{dI}{dt}$$

$$= -\frac{\mu_0 N^2 A}{\ell} \frac{dI}{dt}$$

$$\mathcal{E} = -N \frac{d\Phi}{dt}$$

$$\boxed{L = \frac{N\Phi}{I}}$$

$L = \frac{\mu_0 N^2 A}{\ell}$ = self-inductance

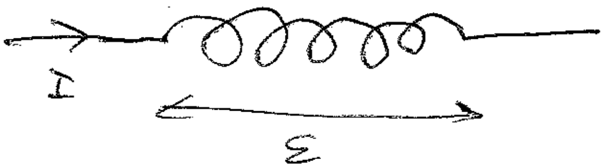
$$\boxed{\mathcal{E} = -L \frac{dI}{dt}}$$

direction of EMF is so as to ~~maintain~~ resist the change in current.

units $L = 1 \text{ Henry} = \frac{1 \text{ volt-sec}}{\text{Amp}}$

Energy Stored in an Inductor

As you increase the current in an inductor, you must ~~push~~ overcome the EMF in the inductor which opposes the change. Basically you are ~~supplying~~ injecting energy into the inductor.



$$\mathcal{E} = -L\frac{dI}{dt}$$

power into the inductor

$$\frac{dW}{dt} = P = -\mathcal{E}I = \cancel{\mathcal{E}}\, I\left(L\frac{dI}{dt}\right)$$

$$\frac{dW}{dt} = \cancel{\tfrac{1}{2}}\,\frac{d}{dt}\,\frac{1}{2}LI^2$$

$W = \frac{1}{2}LI^2$ = total energy stored in the inductor.

$\Rightarrow$ as the current builds up, the magnetic field of the inductor increases

$\Rightarrow$ energy ~~stored~~ is stored as magnetic energy.

an inductor is a mag. energy storage device __example__ __solenoid__

$$L = \frac{\mu_0 N^2}{l} A \qquad B = \frac{\mu_0 N I}{l}$$

$$w = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\mu_0 N^2 A}{l} I^2 \Rightarrow$$

$$= \frac{1}{2} \frac{1}{\mu_0} \left(\frac{\mu_0^2 N^2 I^2}{l^2} \right) l A$$

$$= \left(\frac{1}{2} \frac{B^2}{\mu_0} \right) l A \qquad V = vol = lA$$

$$= \frac{1}{2} \frac{B^2}{\mu_0} V$$

$$\boxed{u = \frac{B^2}{2\mu_0} = \frac{\text{energy}}{\text{unit vol.}}}$$

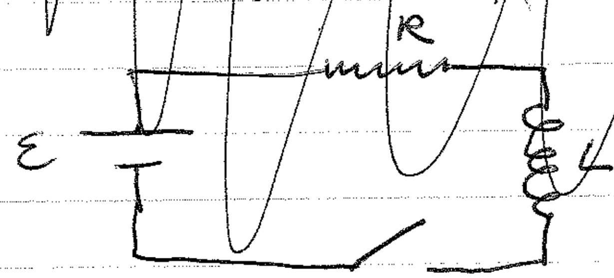
Energy in the magnetic field.

with magnetic material $u = \frac{B^2}{2\mu}$

$\Longrightarrow$ Compare this to $u = \frac{1}{2}\epsilon_0 E^2$ associated with E field.

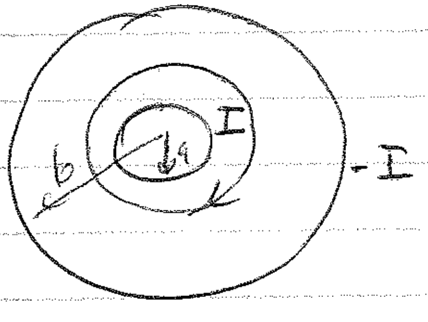
__R-L Circuit__

~~We now want to include an inductor in a circuit. Consider the following circuit~~



Example

Energy storage and inductance of a co-axial cable. (Length l)



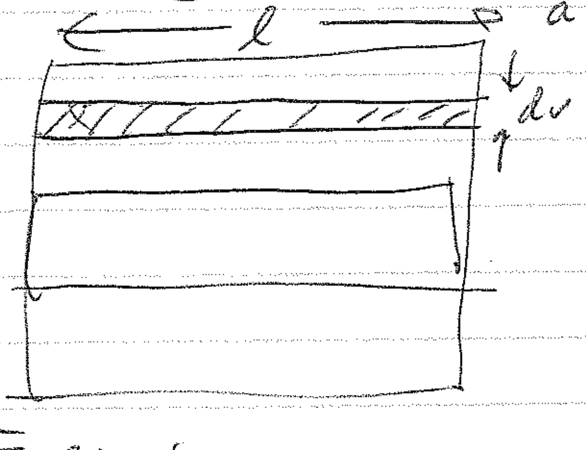
$$\oint \underline{B}\cdot d\underline{l} = \mu_0 I$$

$$B\, 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

$$L = \frac{N\Phi}{I} = \frac{\Phi}{I}$$

$$\Phi = \oint \underline{B}\cdot d\underline{A} = l\int_a^b \frac{\mu_0 I}{2\pi r}\,dr = \frac{l\mu_0 I}{2\pi}\ln\left(\frac{b}{a}\right)$$



$$L = \frac{l\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)$$

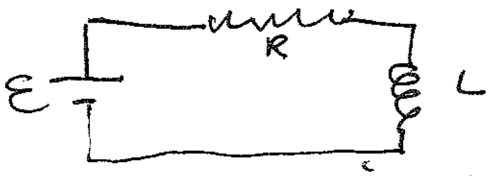
Energy

$$U = \int dV\,\frac{B^2}{2\mu_0} = \int_a^b \frac{\mu_0^2 I^2}{4\pi^2 r^2}\,\frac{(2\pi r\,dr)\,l}{2\mu_0}$$

$$= \frac{\mu_0 I^2 l}{4\pi}\ln\left(\frac{b}{a}\right) = \frac{1}{2}LI^2$$

~~Can calculate L by calculating~~

R-L Circuit. Include inductor in a circuit. Consider



At $t=0$, we close the switch. How does the current evolve in time?

Without the inductor $I_0 = \frac{\mathcal{E}}{R}$.

Can the current instantly increase to this value?

$\Rightarrow$ must supply energy to the inductor until the magnetic field is built up.

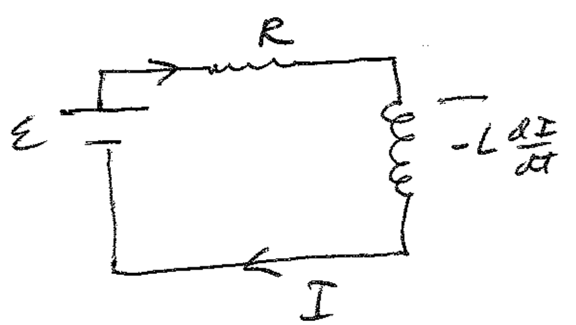
$\Rightarrow$ requires some time.

What is the current after the switch has been closed for a long time?

$$I = \frac{\mathcal{E}}{R}$$

Once the magnetic energy in the inductor has been built up, the inductor no long produces an EMF so ~~drops~~ does not affect final current.

Use Kirchoff's loop rule.



$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{\mathcal{E}}{L} = \frac{\mathcal{E}}{R}\frac{R}{L} = I_0 \frac{R}{L}$$

$\tau = \frac{L}{R}$ = time constant of ~~the~~ L-R circuit

$$= \frac{\text{Henry}}{\text{Ohm}} = \frac{\text{volt sec}\ \text{amp}}{\text{Amp}\ \text{volt}} = \text{sec}$$

$$\frac{dI}{dt} + \frac{I}{\tau} = \frac{I_0}{\tau}$$

$$\frac{dI}{dt} = \frac{I_0 - I}{\tau}$$

$$\int_0^I \frac{dI}{I_0 - I} = \int_0^t \frac{dt}{\tau} = \frac{t}{\tau}$$

$$-\ln(I_0 - I)\Big|_0^I = \frac{t}{\tau}$$

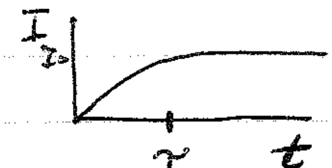
$$+\left[\ln(I_0 - I) - \ln I_0\right] = -\frac{t}{\tau}$$

$$\ln\left(\frac{I_0 - I}{I_0}\right) = -\frac{t}{\tau}$$

$$\frac{I_0 - I}{I_0} = e^{-\frac{t}{\tau}} \qquad I_0 - I = I_0 e^{-\frac{t}{\tau}}$$

$$\boxed{I = I_0(1 - e^{-t/\tau})}$$

$t \to \infty$	$t = 0$
$I = I_0$	$I = 0$

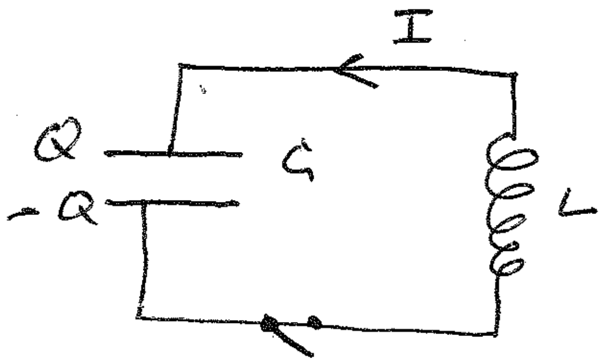


Current reaches its steady value in a time $\tau = L/R$

for $t \ll \tau$ most of potential drop is across the inductor since $IR \ll L\frac{dI}{dt}$. for $t \gg \tau$ most of drop is across resistor since $IR \gg L\frac{dI}{dt}$

LC Circuit (no resistance)

Consider a capacitor in series with an inductor. We initially charge the capacitor with a charge Q_0



Close the switch and follow the behavior in time

$$\frac{dQ}{dt} = I$$

Using Kirchoffs Loop rule in direction of I

$$-L\frac{dI}{dt} - \frac{Q}{C} = 0$$

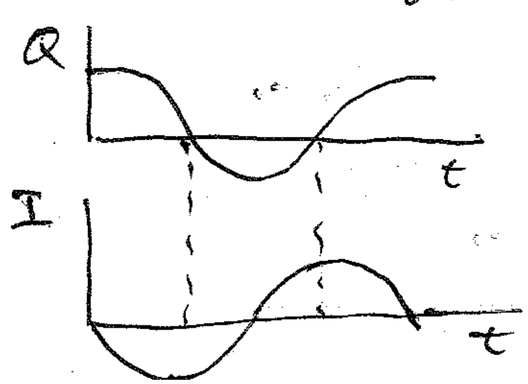
$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0 \qquad \omega^2 = \frac{1}{LC}$$

$$\boxed{\frac{d^2}{dt^2}Q + \omega^2 Q = 0}$$

same as equation for pendulum on spring/mass system,

$$Q = Q_0 \cos\omega t$$

~~Thus~~ $I = \frac{dQ}{dt} = -\omega Q_0 \sin\omega t$



Energy

$$L\frac{dI}{dt} + \frac{Q}{C} = 0$$

multiply by I

$$LI\frac{dI}{dt} + \frac{Q}{C}I = 0$$

$$\frac{d}{dt}\frac{1}{2}LI^2 + \frac{d}{dt}\frac{Q^2}{2C} = 0$$

$$V = \frac{Q}{C}$$

$$\frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}LI^2 + \tfrac{1}{2}CV^2}\Big) = 0$$

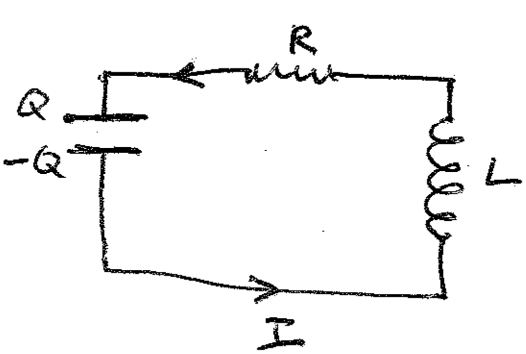
energy in inductor

energy in capacitor.

$\Rightarrow$ total energy conserved.

$\Rightarrow$ energy is exchanged between the inductor and capacitor.

LRC Circuit



Loop rule

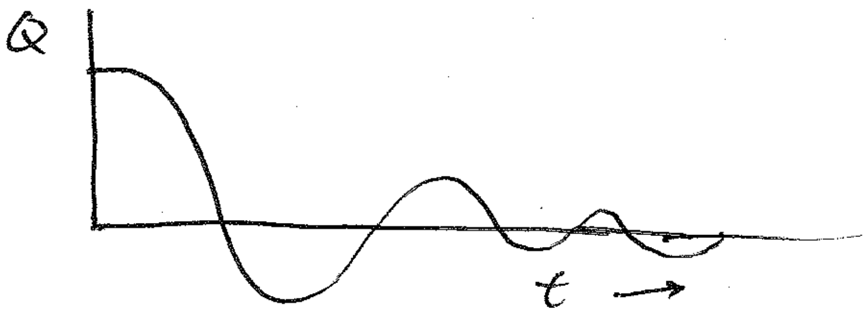
$$-L\frac{dI}{dt} - IR - \frac{Q}{C} = 0$$

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0$$

energy

$$\frac{d}{dt}\left(\frac{1}{2}LI^2 + \frac{1}{2}CV^2\right) = -I^2R$$

The energy stored in the inductor and ~~capacitor~~ capacitor is dissipated by the resistor.



Solution

$$Q = Q_0 e^{-t/\tau} \cos\omega t$$

$$\tau = \frac{2L}{R}$$

$$\omega = \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{\frac{1}{2}}$$

for $\boxed{\frac{R^2}{4L^2} < \frac{1}{LC}}$

$\Rightarrow$ R reduces the frequency of oscillation

$\Rightarrow$ charge damps with time.

$\Rightarrow$ oscillation with damping called <u>underdamped oscillator</u>

for $\frac{R^2}{4L^2} > \frac{1}{LC}$ $\longrightarrow$ large resistance

$$Q = Q_0 e^{-t/\tau}\left(\frac{e^{t/\tau_1} + e^{-t/\tau_1}}{2}\right)$$

$$\frac{1}{\tau_1} = \left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)^{\frac{1}{2}}$$

$\Rightarrow$ overdamped

$\Rightarrow$ no oscillatory behavior

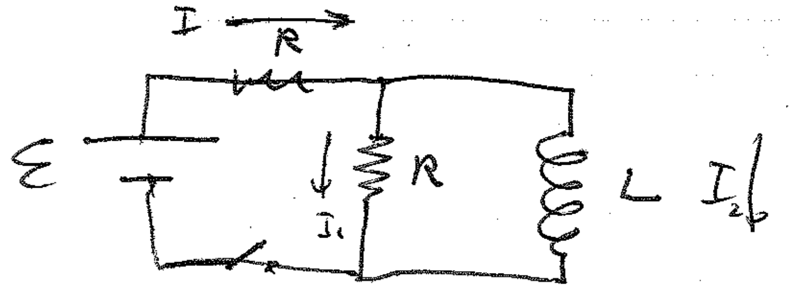
Mutual Inductance

Consider a coil of wire (coil 1) which has a current I_1 ~~and~~. This coil produces a magnetic field.

Suppose we have a second coil (coil 2) with N_2 turns.

The flux due to coil 1 which links through coil two is Φ_2.

<u>example</u> circuit



At $t=0$ close switch.

Find I_1, I_2, I just after switch is closed.

Current through I_2 is zero.

$$I = I_1$$

<u>Loop rule</u>

$$\varepsilon - IR - I_1 R = 0$$

$$\varepsilon - IR - IR = 0$$

$$\Rightarrow \quad I = \frac{\varepsilon}{2R}$$

After long time $\Rightarrow$ steady state.

$\frac{dI_2}{dt} = 0 \implies$ emf across inductor is zero

loop rule

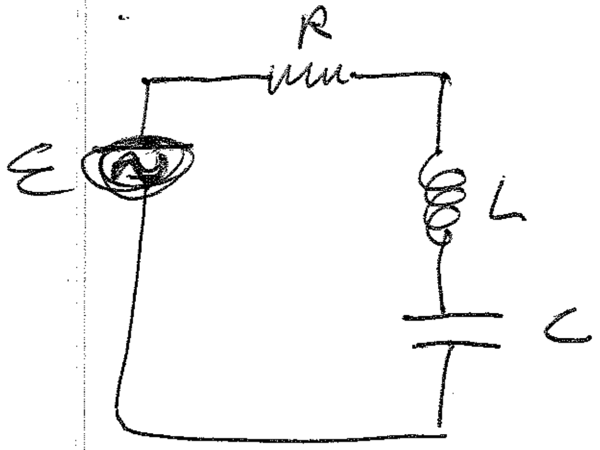
$$-I_1 R + L \frac{dI_2}{dt} = 0$$

$$\Rightarrow I_1 = 0$$

loop rule

$$\varepsilon - IR = 0 \qquad I = \frac{\varepsilon}{R}$$

AC Circuit



$$\mathcal{E} = \mathcal{E}_0 \sin\omega t$$

if $L=0$, $C=0$

$$\mathcal{E}_0 \sin\omega t - IR = 0$$

$$I = \frac{\mathcal{E}_0}{R}\sin\omega t$$

Current and emf in phase

$$\mathcal{E}_0 \sin\omega t - IR - L\frac{dI}{dt} - \frac{Q}{C} = 0$$

$\Rightarrow$ note that because of L and C, I not in phase with $\mathcal{E}_0$

Let $Q = Q_s \sin\omega t + Q_c \cos\omega t$

$$I = \omega Q_s \cos\omega t - \omega Q_c \sin\omega t$$

$$\frac{dI}{dt} = -\omega^2 Q_s \sin\omega t - \omega^2 Q_c \cos\omega t$$

gather $\sin\omega t$ and $\cos\omega t$ together separately.

$$[\ ]_1 \sin\omega t + [\ ]_2 \cos\omega t = 0$$

$$\mathcal{E}_0 \cancel{\sin\omega t} + \omega Q_c R + L\omega^2 Q_s - \frac{Q_s}{C} = 0$$

$$\Rightarrow \quad Q_c = Q_s\left(\frac{1}{C} - L\omega^2\right)\frac{1}{\omega R} - \frac{\mathcal{E}_0}{\omega R}$$

$$-\omega Q_s R + L\omega^2 Q_c - \frac{Q_c}{C} = 0$$

$$\Rightarrow \quad Q_S = \left(\omega^2 L - \frac{1}{C}\right)\frac{Q_C}{\omega R}$$

$$Q_C = -\left(-\frac{1}{CL} + \omega^2 \cancel{L}\right)\left(\omega^2 \cancel{L} - \frac{1}{CL}\right)\frac{Q_C L^2}{\omega R^2} - \frac{\mathcal{E}_0}{\omega R}$$

$$\omega^2 Q_C + \left(\omega^2 - \frac{1}{LC}\right)^2 \frac{L^2}{\cancel{\omega^2} R^2} Q_C = -\frac{\mathcal{E}_0}{\cancel{\omega}}\frac{\omega^{\cancel{2}}}{R}$$

$$\tau = \frac{L}{R}$$

$$Q_C\left[\omega^2 + \left(\omega^2 - \frac{1}{LC}\right)^2 \tau^2\right] = -\omega\frac{\mathcal{E}_0}{R}$$

$$Q_C = \frac{-\omega\frac{\mathcal{E}_0}{R}}{\left[\omega^2 + \left(\omega^2 - \frac{1}{LC}\right)^2 \tau^2\right]}$$

also can find Q_S

$\Rightarrow$ suppose R is very small $\Rightarrow$ τ large.

Q_C small if $\omega^2 \neq \frac{1}{LC}$

At $\omega^2 = \frac{1}{LC}$ $\quad Q_S = 0$

$$Q_C = -\omega\frac{\mathcal{E}_0}{R\omega^2} = -\left(\frac{\mathcal{E}_0}{R}\right)\frac{1}{\omega}$$

$$I = \cancel{\omega Q_C \cos\omega t} + \frac{\mathcal{E}_0}{R}\sin\omega t \quad \Uparrow$$

as if no L, C

$\Rightarrow$ large current. $\Rightarrow$ resonance.

Maxwells Eqns

Have found that if have a magnetic field which varies in time, we generate an inductive electric field.

What if we have an electric field which varies in time

$\Rightarrow$ produces a magnetic field

Maxwells Eqns

closed surface

$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{Q}{\epsilon_0}$$

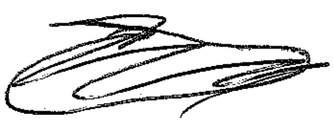
$$\oint \mathbf{B}\cdot d\mathbf{A} = 0$$

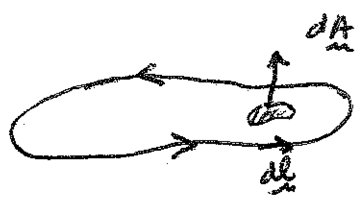
~~no start~~ magnetic field lines don't start or stop.

surface with bounding curve.

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I + \underbrace{\mu_0 \epsilon_0 \frac{d}{dt}\oint \mathbf{E}\cdot d\mathbf{A}}_{\text{Displacement current}}$$

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\oint \mathbf{B}\cdot d\mathbf{A}$$

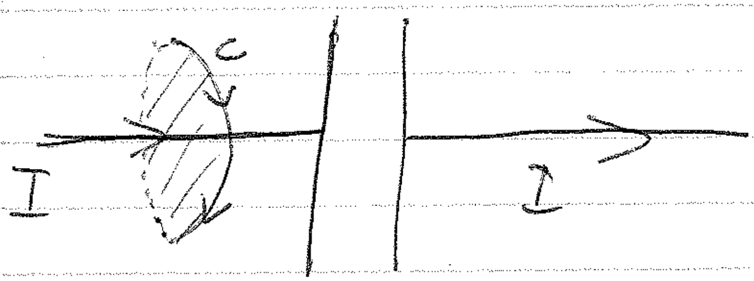




~~Addition~~

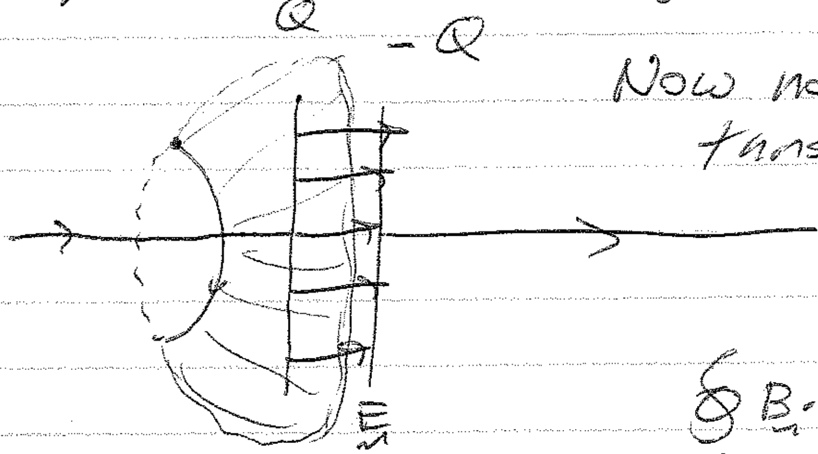
Induced magnetic fields

Consider a current carrying wire entering a parallel plate capacitor



$$\oint_C \underline{B} \cdot d\underline{l} = \mu_0 I$$

Now distort the surface enclosing C to pass through the capacitor



Now no current cuts through the surface so Ampere's law says

$$\oint_C \underline{B} \cdot d\underline{l} = 0$$

$\Longrightarrow$ this is not possible since know that $\underline{B}$ has not changed.

$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = \mu_0 \epsilon_0 \frac{d}{dt} \oint \underset{\sim}{E} \cdot d\underset{\sim}{A}$$

In the capacitor

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$$

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \oint E \, dA$$ ($d\underset{\sim}{A}$ points to right)

$$= EA$$

$$= \frac{Q}{\epsilon_0}$$

$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = \mu_0 \epsilon_0 \frac{d}{dt}\left(\frac{Q}{\epsilon_0}\right) = \mu_0 I$$

$\Rightarrow$ same as before.

$\Rightarrow$ This completes Maxwell's Eqns

Changing $\underset{\sim}{B} \Longrightarrow \underset{\sim}{E}$

Changing $\underset{\sim}{E} \Longrightarrow \underset{\sim}{B}$

$\Rightarrow$ electromagnetic waves (light).