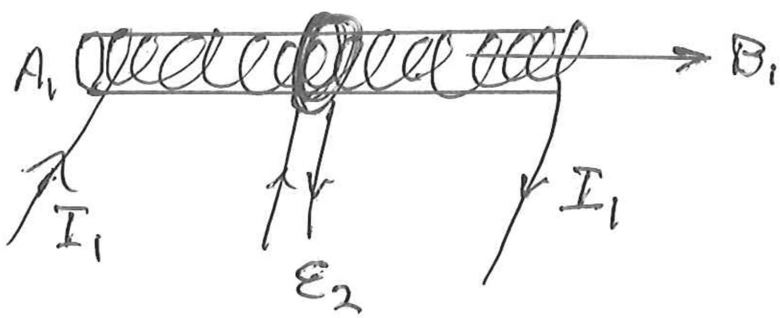
## Inductance

We previously discussed how the changing current and magnetic field in a solenoid induces an electric field throughout space and in the presence of secondary coil can induce an emf in that coil.




$$B_1 = \mu_0 \frac{N_1}{l_1} I_1$$

$N_1$ = total # of loops in the solenoid

$l_1$ = the solenoid length

$$\Phi_1 = B_1 A_1 = \frac{\mu_0 A_1 N_1}{l_1} I_1$$

The induced emf in the secondary coil with $N_2$ wraps is given by

$$\mathcal{E}_2 = -N_2 \frac{d}{dt} \Phi_1$$

$$= -\frac{N_2 \mu_0 A_1 N_1}{l_1} \frac{d}{dt} I_1 = -M \frac{dI_1}{dt}$$

$\Rightarrow$ the changing current $I_1$ and flux $\Phi_1$ induces an emf in the $N_2$ coil. The constant

$$M \equiv \frac{N_1 N_2 \mu_0 A_1}{l_1}$$

is the <u>mutual inductance</u> of the two coils. It depends only on the geometry of the two coils.

More generally,

$$M \equiv \frac{N_2 \Phi_1}{I_1}$$

with $\Phi_1$ the flux produced by the current $I_1$ that links $N_2$. A perhaps surprise is that

$$M = \frac{N_1 \Phi_2}{I_2}$$

$\Rightarrow$ with $\Phi_2$ the flux produced by $I_2$ that links $N_1$.

units: the measure of inductance is the Henry

$$1\,H \equiv \frac{1\,V\,s}{A}$$

$\Rightarrow$ typically in the range of $\mu H$

<u>example</u>: $\ell = 0.5\,m$, $A = 10\,cm^2$, $N_1 = 10^3$, $N_2 = 10$

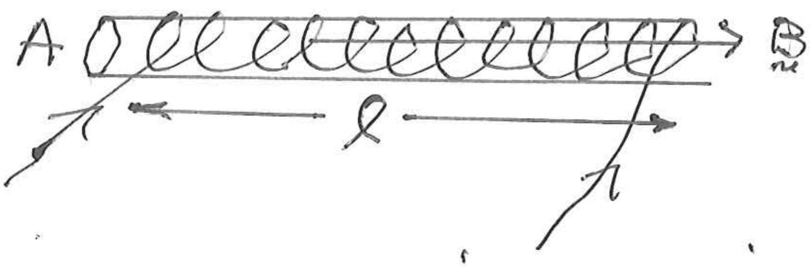
$$M = 4\pi \times 10^{-7} \frac{T\,m}{A} \frac{10^{-3}\,m^2\ 10^3 (10)}{0.5\,m}$$

$$= \frac{4\pi}{0.5}\,\mu H = 25\,\mu H$$

## Self-inductance

A single solenoid whose current is changing in time also induces an emf in itself due to its changing magnetic flux. This is called self-inductance and plays an important role in the time behavior of circuits.

Consider a long solenoid of area $A$ and length $\ell$ wound with $N$ turns of wire. Current $I$.




The magnetic field is

$$B = \mu_0 \frac{NI}{\ell}$$

The flux through the solenoid is

$$\Phi = BA = \frac{\mu_0 N I}{\ell} A$$

The induced emf across the solenoid is given by

$$\mathcal{E} = -N \frac{d\Phi}{dt} = -N \frac{\mu_0 N A}{\ell} \frac{dI}{dt}$$

$$= -\frac{\mu_0 N^2 A}{\ell} \frac{dI}{dt}$$

Define
$$L = \frac{\mu_0 N^2 A}{l} = \text{self-inductance}$$
or $L = \dfrac{N \Phi}{I}$

$$\Rightarrow \quad \mathcal{E} = -L \frac{dI}{dt}$$

The direction of the emf is so as to resist the change in current

units: $L = 1$ Henry $= 1 \dfrac{\text{volt-sec}}{\text{amp}}$

## Energy stored in an inductor

As you increase the current in an inductor, you must overcome the emf in the inductor which opposes the change in $I$. You are injecting energy into the inductor. As you move charge $\Delta Q$ along the inductor you change its energy by $\Delta Q \, \mathcal{E}$. The rate of injection of power into the inductor is therefore

$$P = -\mathcal{E} \frac{\Delta Q}{\Delta t} = -\mathcal{E} I$$

or
$$P = -\left(-L \frac{dI}{dt}\right) I = L I \frac{dI}{dt}$$
$$= \frac{d}{dt}\left(\frac{1}{2} L I^2\right)$$

$W = \frac{1}{2} L I^2$ = energy stored in the inductor

The energy stored is magnetic energy

⇒ an inductor is a magnetic energy storage device

⇒ just like a capacitor is an electric field energy storage device

<u>magnetic energy</u>

For a solenoid

$$L = \frac{\mu_0 N^2}{\ell} A \,, \quad B = \frac{\mu_0 N I}{\ell}$$

so

$$W = \tfrac{1}{2} L I^2 = \tfrac{1}{2} \frac{\mu_0 N^2 A}{\ell} I^2$$

$$= \tfrac{1}{2} \left( \frac{\mu_0^2 N^2 I^2}{\ell^2} \right) \frac{\ell A}{\mu_0}$$

$$= \left( \frac{B^2}{2\mu_0} \right) A\ell = \frac{B^2}{2\mu_0} \text{Vol}$$

⇒ $V = A\ell$ = volume of solenoid

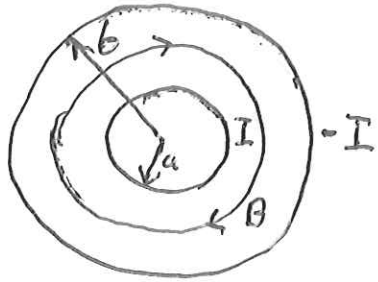
$$u = \frac{B^2}{2\mu_0} = \frac{\text{energy}}{\text{vol}}$$ ⇒ energy density of magnetic field

⇒ compare this to $u = \frac{1}{2}\epsilon_0 E^2$ for the electric field energy density

⇒ with magnetic materials

$$u = \frac{B^2}{2\mu}$$

example: Energy storage and inductance of a co-axial cable $\Rightarrow$ length $l$ and radii $a, b$ with $a < b$.




A co-axial cable has a current $I$ flowing into the page (assume current on surface) ~~and~~ on the inner conductor and a current $-I$ flowing out of the page on the outer conductor.

Use Ampere's Law to solve for $B$:

$B = 0$ for $r < a$, $r > b$ since zero current ($r < a$, $I = 0$; $r > b$, $I - I = 0$)

$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = \mu_0 \text{(current through loop)}$$

For $a < r < b$

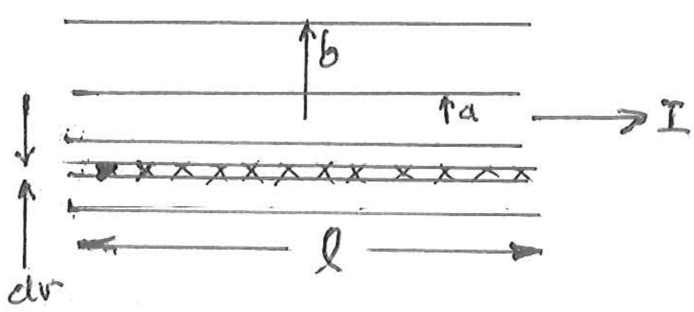
$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = B\, 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

Calculating inductance

$$L = \frac{N\Phi}{I} = \frac{\Phi}{I} \Rightarrow N = 1$$

single loop
$\Rightarrow$ down and back




magnetic flux $d\Phi$ in area $dr\, l$

$$d\Phi = \frac{\mu_0 I}{2\pi r}\, l\, dr$$

Integrate the magnetic flux from "a" to "b".

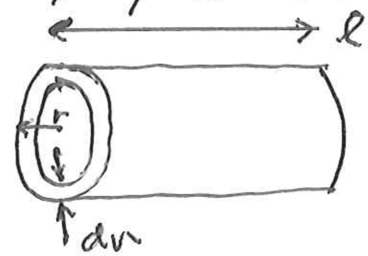
$$\Phi = \int_a^b d\Phi = \frac{\mu_0 I}{2\pi} \ell \int_a^b \frac{dr}{r} = \frac{\mu_0 I \ell}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$L = \frac{\Phi}{I} = \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{b}{a}\right)$$

Stored magnetic energy

$$U = \int dV \frac{B^2}{2\mu_0} \quad , \quad dV = 2\pi r\, dr\, \ell$$

$\Rightarrow$ cylindrical shell

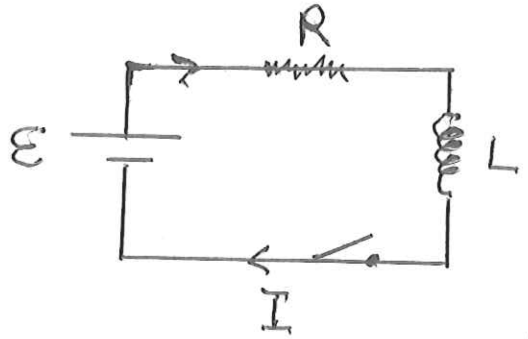



$$= \int_a^b 2\pi r\, dr\, \ell \left(\frac{\mu_0^2 I^2}{4\pi^2 r^2}\right) \frac{1}{2\mu_0}$$

$$= \frac{\mu_0 I^2 \ell}{4\pi} \int_a^b dr \frac{1}{r}$$

$$= \frac{\mu_0 \ell}{4\pi} \ln\left(\frac{b}{a}\right) I^2 = \frac{1}{2} L I^2$$

# Circuits with inductors: R-L circuits




Consider an inductor L and resistor R in a circuit.

$\Rightarrow$ close switch at $t=0$

How does the current evolve in time? Without the inductor

$$\varepsilon_L = -L\frac{dI}{dt}$$

$$I_0 \equiv \frac{\varepsilon}{R}$$

However the current can't instantaneously increase to this value since the emf needs to supply magnetic energy to the inductor for the current to build up

$\Rightarrow$ requires some time

$\Rightarrow$ after a long time?

Once the magnetic energy in the inductor has built up, the inductor no longer produces an emf so it doesn't affect the final current

$$I = \frac{\varepsilon}{R}$$

Use Kirchoff's loop rule

$$\varepsilon - IR - L\frac{dI}{dt} = 0$$

$\Rightarrow$ divide by L

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{\mathcal{E}}{L} = \frac{\mathcal{E}}{R}\frac{R}{L} = I_0 \frac{R}{L}$$

$\Rightarrow$ find time constant using units of the equation

$$\Longrightarrow \frac{\text{current}}{\text{time}}$$

$$\Longrightarrow \frac{1}{\tau} = \frac{R}{L} \Longrightarrow \tau = \frac{L}{R}$$

$\tau = \frac{L}{R}$ = time constant of a L-R circuit

units: $\tau = \frac{\text{Henry}}{\text{Ohms}} = \left(\frac{V\,s}{A}\right)\left(\frac{\text{amp}}{V}\right) = \text{sec}$

The time constant tells you how long it takes the magnetic energy in the inductor to build up, allowing the current to reach its full value.

Solution for time dependence:

$$\frac{dI}{dt} + \frac{1}{\tau} I = \frac{1}{\tau} I_0$$

$$\frac{dI}{dt} = \frac{1}{\tau}(I_0 - I)$$

$$\int_0^I \frac{dI}{I_0 - I} = \int_0^t \frac{dt}{\tau} = \frac{t}{\tau}$$

$$-\ln(I_0 - I)\Big|_0^I = \frac{t}{\tau}$$

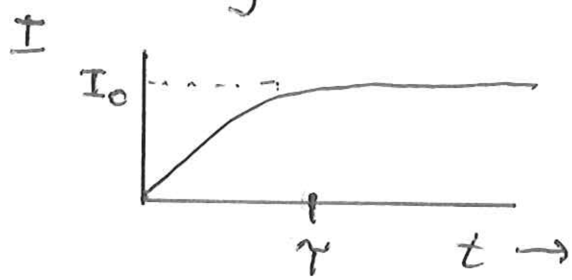
$$\ln\left(\frac{I_0 - I}{I_0}\right) = -\frac{t}{\tau}$$

Take the exponential of both sides

$$\frac{I_0 - I}{I_0} = e^{-\frac{t}{\tau}} \Rightarrow I = I_0\left(1 - e^{-\frac{t}{\tau}}\right)$$

As $t \to \infty$, $e^{-\frac{t}{\tau}} \to 0 \Rightarrow I = I_0$

At $t = 0$, $I = 0$.

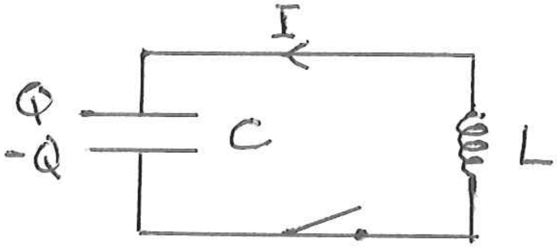



The current reaches a steady value in a time $\tau = \frac{L}{R}$

Note that for $t \ll \tau$, most of the potential drop is across the inductor since $IR \ll L\frac{dI}{dt}$ and for $t \gg \tau$ most of the potential drop is across the resistor.

## L-C circuit (no resistance)

Consider a capacitor in series with an inductor. We initially have the capacitor charged with $Q_0$. Close the switch and follow the behavior in time




Choose current direction so $I = \frac{dQ}{dt}$

$\Rightarrow$ positive I means charge flowing into C.

Use Kirchoff's loop rule in direction of I.

$$-L\frac{dI}{dt} - \frac{Q}{C} = 0$$

$$-L\frac{d^2Q}{dt^2} - \frac{Q}{C} = 0$$

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0 \quad \Rightarrow \omega^2 \equiv \frac{1}{LC}$$

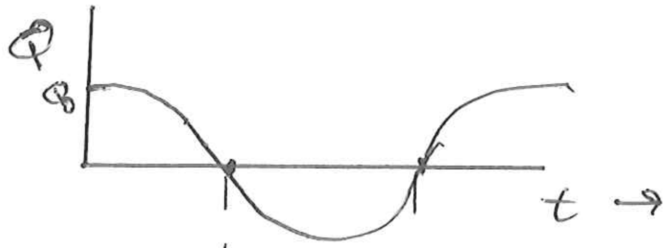
$$\frac{d^2Q}{dt^2} + \omega^2 Q = 0 \qquad \sim \frac{1}{sec^2}$$

$\Rightarrow$ same as eqn for pendulum or spring/mass system

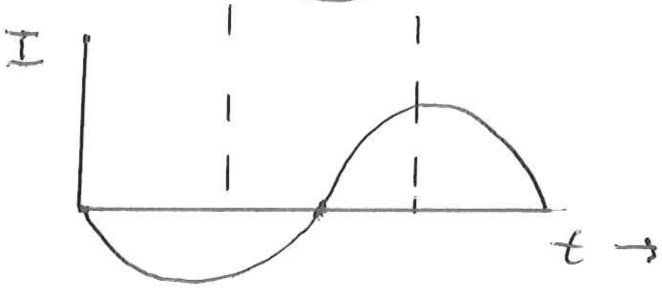
$$Q = Q_0 \cos\omega t$$

$$I = \frac{dQ}{dt} = -\omega Q_0 \sin(\omega t)$$

$\Rightarrow$ both Q and I are oscillatory




$\Rightarrow$ Q decreases in time initially




$\Rightarrow$ $I < 0$ at early time as current flows out of C.

It is useful to look at how energy is exchanged between C (electric field energy) and L (magnetic field energy).

$$L\frac{dI}{dt} + \frac{Q}{C} = 0$$

Multiply by $I$,

$$LI\frac{dI}{dt} + I\frac{Q}{C} = 0$$

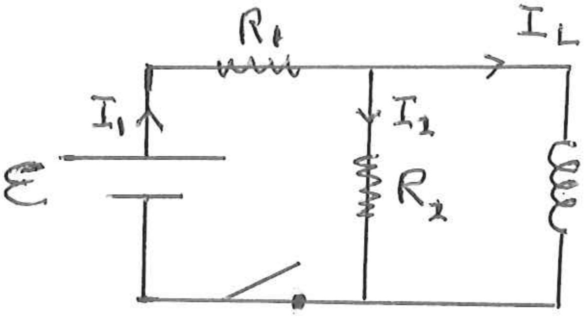
$$\frac{d}{dt}\left(\frac{1}{2}LI^2\right) + \frac{1}{C}Q\frac{dQ}{dt} = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}LI^2\right) + \frac{d}{dt}\left(\frac{Q^2}{2C}\right) = 0 \qquad V = \frac{Q}{C}$$

$$\frac{d}{dt}\left(\underbrace{\frac{1}{2}LI^2}_{\text{magnetic energy in inductor}} + \underbrace{\frac{1}{2}CV^2}_{\text{electric field energy in capacitor}}\right) = 0$$

$\Rightarrow$ energy exchanged between capacitor and inductor

## L-R Circuit




At ~~t~~ $t=0$ close switch

Find $I_1$, $I_L$ and $I_2$ just after the switch is closed.

$\Rightarrow I_L = 0 \Rightarrow$ no time to build magnetic energy in L

Loop rule (left loop)

$$\mathcal{E} - I_1 R_1 - I_2 R_2 = 0$$

point rule

$$I_1 = I_2 + I_L$$

$$\Rightarrow I_1 = I_2$$

$$I_1 = I_2 = \frac{\mathcal{E}}{R_1 + R_2}$$

Late time $\Rightarrow$ steady state $dI_L/dt = 0$

$\Rightarrow$ emf across inductor is zero

Loop rule right loop

$$-L \cancelto{0}{\frac{dI_L}{dt}} + I_2 R_2 = 0 \Rightarrow I_2 = 0$$

$\Rightarrow$ all current flows through inductor

Large loop

$$\mathcal{E} - I_1 R_1 = 0 \qquad I_1 = \frac{\mathcal{E}}{R_1} = I_L$$

Time constant for circuit

Looprule for large loop

$$\mathcal{E} - I_1 R_1 - L\frac{dI_L}{dt} = 0$$

$$I_1 = \frac{\mathcal{E} - L\frac{dI_L}{dt}}{R_1}$$

Right loop

$$-I_2 R_2 + L\frac{dI_L}{dt} = 0$$

$$I_2 = \frac{L}{R_2}\frac{d}{dt}I_L$$

point rule

$$I_1 = I_2 + I_L$$

eliminate $I_1, I_2$

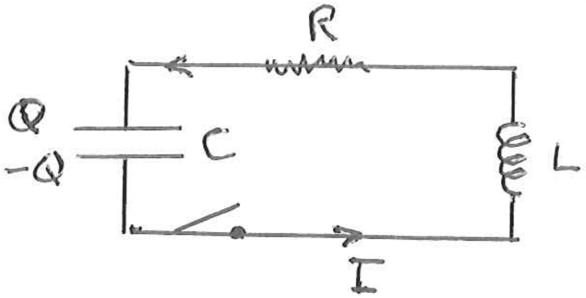
$$\frac{\mathcal{E} - L\frac{dI_L}{dt}}{R_1} = \frac{L}{R_2}\frac{dI_L}{dt} + I_L$$

$$L\left(\frac{1}{R_2} + \frac{1}{R_1}\right)\frac{dI_L}{dt} + I_L = \frac{\mathcal{E}}{R_1}$$

$$\tau = L\left(\frac{1}{R_2} + \frac{1}{R_1}\right)$$

$$\tau = \frac{L(R_1 + R_2)}{R_1 R_2}$$

## LRC Circuit




Loop rule

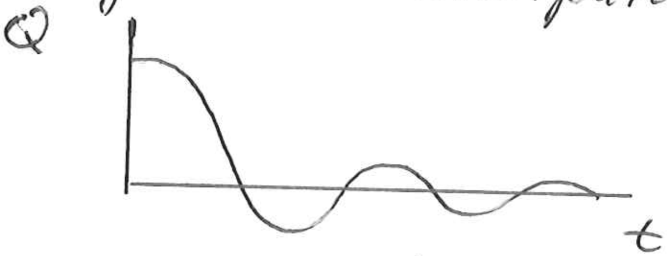
$$-L\frac{dI}{dt} - IR - \frac{Q}{C} = 0$$

Look at energy

$\Rightarrow$ multiply by $I$

$$\frac{d}{dt}\left(\frac{1}{2}LI^2 + \frac{1}{2}CV^2\right) = -I^2R$$

The energy stored in the inductor and capacitor is dissipated in the resistor




Since $I = \frac{dQ}{dt}$

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0$$

Solution

$$Q = Q_0 e^{-t/\tau} \cos \omega t$$

$$\tau = \frac{2L}{R}, \quad \omega = \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{1/2}$$

valid for

$$\frac{R^2}{4L^2} < \frac{1}{LC} \Rightarrow \text{small resistance}$$

Oscillation with damping as shown above

$\Rightarrow$ called an <u>under-damped</u> oscillator

For $\frac{R^2}{4L^2} > \frac{1}{LC}$ $\Rightarrow$ large resistance

$$Q = Q_0 e^{-t/\tau}\left(e^{t/\tau_1} + e^{-t/\tau_1}\right)\frac{1}{2}$$

$$\frac{1}{\tau_1} = \left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)^{1/2}$$

$\Rightarrow$ overdamped oscillator

$\Rightarrow$ no oscillations