Generation of magnetic fields

Magnetic field of a moving charge

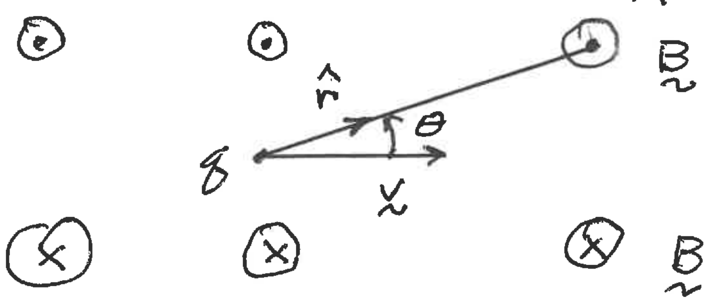
A charge q produces an electric field. If that charge is moving, it also produces a magnetic field

$\Rightarrow$ a moving charge produces a current and it is the current that is the source of the magnetic field

$$\underset{\sim}{B} = \frac{\mu_0}{4\pi} \, q \, \frac{\underset{\sim}{v} \times \hat{r}}{r^2}$$

$$\mu_0 = 4\pi \times 10^{-7} \, \frac{Tm}{A}$$

units: $B \sim \frac{Tm}{A} \, C \left(\frac{m}{s}\right) \frac{1}{m^2} \sim T$

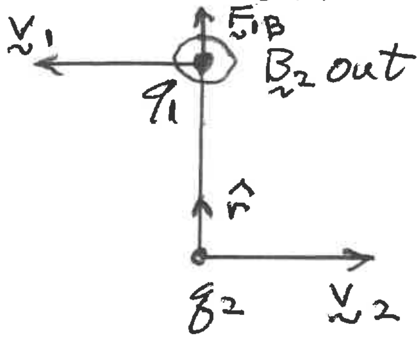


$$|\underset{\sim}{B}| = \frac{\mu_0}{4\pi} \frac{|q| |\underset{\sim}{v}| \sin\theta}{r^2}$$

The magnetic field forms closed loops circling around the trajectory of the charge. The largest field is $\perp$ to the particle location ($\theta \sim \pi/2$ where $\sin\theta \sim 1$)

Right hand rule gives direction of $\underset{\sim}{B}$: thumb along current and fingers wrap around to give $\underset{\sim}{B}$.

Compare magnetic force and electric force between moving changes.



$$B_2 = \frac{q_2 v_2}{r^2} \frac{\mu_0}{4\pi}$$ = magnetic field from q_2 at location of q_1

$\underset{\sim}{B}_2$ is pointing out of the page

$$\underset{\sim}{F}_{1B} = q_1 \underset{\sim}{v}_1 \times \underset{\sim}{B}_2 \Rightarrow \text{points up}$$

$$F_{1B} = q_1 v_1 B_2 = q_1 v_1 \frac{q_2 v_2}{r^2} \frac{\mu_0}{4\pi}$$

F_{1E} = force on q_1 due to q_2

$$= \frac{q_1 q_2}{4\pi \epsilon_0 r^2}$$

$$\frac{F_{1B}}{F_{1E}} = \frac{q_1 q_2 v_1 v_2}{r^2} \frac{\mu_0}{4\pi} \frac{4\pi \epsilon_0 r^2}{q_1 q_2}$$

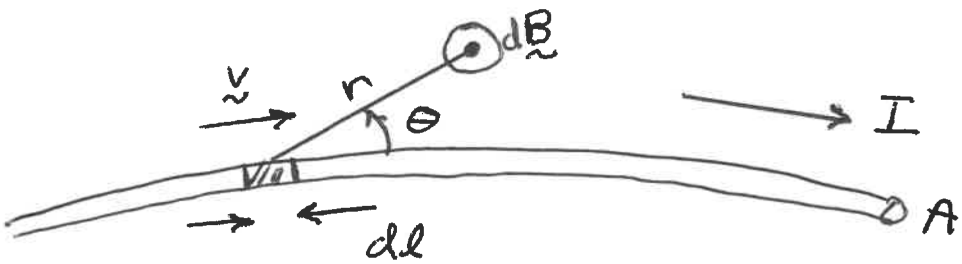
$$= v_1 v_2 \mu_0 \epsilon_0$$

$$\frac{1}{(\mu_0 \epsilon_0)^{1/2}} = 3 \times 10^8 \frac{m}{s} = c = \text{velocity of light}$$

$$\frac{F_{1B}}{F_{1E}} = \frac{v_1 v_2}{c^2}$$ $\Rightarrow$ for $v_1, v_2 \ll c$ the electric force exceeds the magnetic force

Magnetic field of a current element

Want to calculate the magnetic field from a segment of wire of length $d\ell$ carrying a current I.



dQ = charge in the segment $d\ell$

$= \underbrace{(n A d\ell)}_{\text{total \# of charges in } d\ell} q$

$I = nAqv$

$$|d\underset{\sim}{B}| = \frac{dQ\, v \sin\theta}{r^2} \frac{\mu_0}{4\pi} = \frac{n A d\ell\, q\, v \sin\theta}{r^2} \frac{\mu_0}{4\pi}$$

$$|d\underset{\sim}{B}| = \frac{I\, d\ell \sin\theta}{r^2} \frac{\mu_0}{4\pi}$$

$\Rightarrow$ define the vector $d\underset{\sim}{\ell}$ with a direction along the wire in the direction of the current

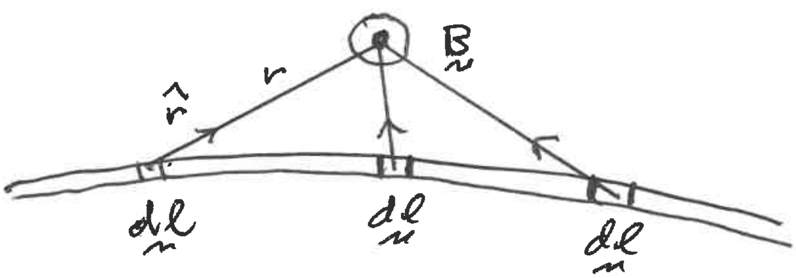
$$d\underset{\sim}{B} = \frac{\mu_0}{4\pi} I \frac{d\underset{\sim}{\ell} \times \hat{r}}{r^2}$$

$\Rightarrow$ again $\hat{r}$ points from $d\ell$ to where $d\underset{\sim}{B}$ is evaluated

Magnetic field from the entire wire

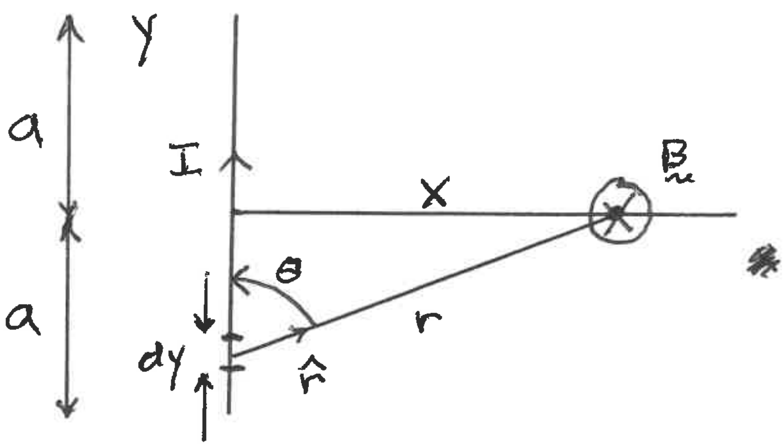
$\Rightarrow$ integrate the contribution to $d\underset{\sim}{B}$ from the entire wire

$$\underset{\sim}{B} = \frac{\mu_0}{4\pi} \int_{\text{wire}} I \frac{d\underset{\sim}{l} \times \hat{r}}{r^2}$$



Note that both r and $\hat{r}$ and $d\underset{\sim}{l}$ may vary along the wire

Magnetic field from a finite length wire :



$dl = dy$

$d\underset{\sim}{l} = \hat{j}\, dy$

$\hat{i} \times \hat{j} = \hat{k}$

$\Rightarrow \hat{k}$ out of page

$\Rightarrow \underset{\sim}{B}$ points into the page along x axis

$\Rightarrow \underset{\sim}{B}$ in $-\hat{k}$ direction

$\Rightarrow -z$ direction

$$B_z = -\frac{\mu_0}{4\pi} I \int \frac{dy \sin\theta}{r^2}$$

$r^2 = x^2 + y^2$

$\sin\theta = x/r$

$$B_z = -\frac{\mu_0 I}{4\pi} \int_{-a}^{a} dy \frac{x}{(x^2+y^2)^{1/2}}$$

$$\int dy \frac{1}{(x^2+y^2)^{3/2}} = \frac{1}{x^2} \frac{y}{(x^2+y^2)^{1/2}}$$

$$B_z = -\frac{\mu_0 I}{4\pi} \frac{2a}{x(x^2+a^2)^{1/2}}$$

large x ($x \gg a$)

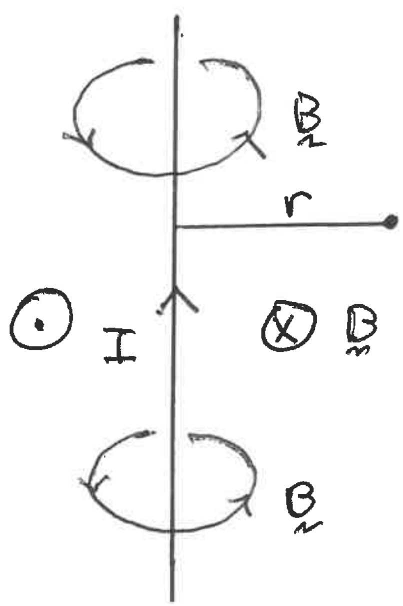
$$B_z \simeq -\frac{\mu_0 I}{4\pi} \frac{2a}{x^2}$$

$\Rightarrow$ acts like a short current segment

small x ($x \ll a$)

$$B_z = -\frac{\mu_0 I}{2\pi x}$$

Note that "a" drops out.
$\Rightarrow$ magnetic field from infinite wire



$$B = \frac{\mu_0 I}{2\pi r}$$ $\Rightarrow$ infinite wire

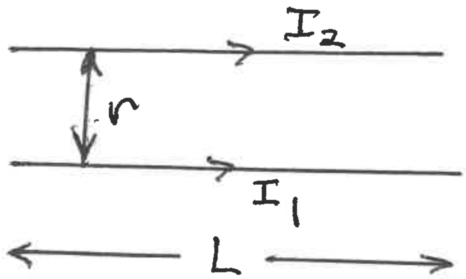
example: what is B at 1cm from a 10A current?

$$B = 4\pi \times 10^{-7} \frac{Tm}{A} \frac{10A}{2\pi 10^{-2} m}$$

$$= 2 \times 10^{-4} T = 2 \text{ gauss}$$

Note Earth's field is around 0.5 gauss

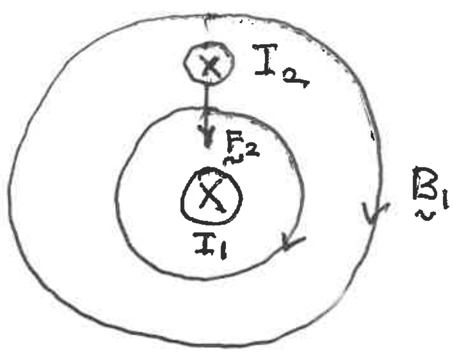
Force between parallel wires



wires of length L and separation r

$\Rightarrow$ take $r \ll L$

$\Rightarrow$ magnetic field at I_2 due to I_1 is like I_1 were an infinite wire



Calculate the force on I_2 due to magnetic field $\underset{\sim}{B}_1$ of I_1

$$\Rightarrow B_1 = \frac{\mu_0 I_1}{2\pi r}$$

$$\underset{\sim}{F}_2 = I_2 \underset{\sim}{L}_2 \times \underset{\sim}{B}_1$$

$\Rightarrow$ force is attractive

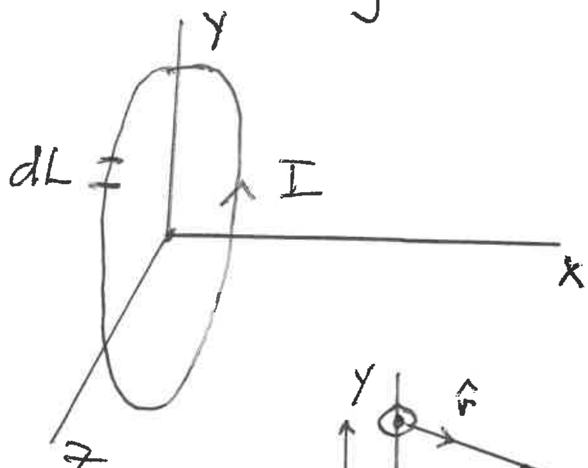
$$F_2 = I_2 L \frac{\mu_0 I_1}{2\pi r} = \frac{\mu_0}{2\pi r} L I_1 I_2$$

$\Rightarrow$ wires carrying current in the same direction attract each other

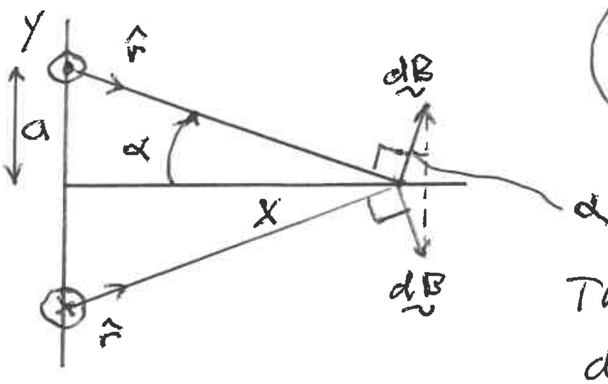
$\Rightarrow$ currents in opposite direction repell

Magnetic field from a circular current loop

Current loop in the y-z plane with the axis along x. Radius a



Consider the magnetic field dB from two segments on opposite sides of the loop.



The sum of the two $d\underset{\sim}{B}$'s points along X

$$d\underset{\sim}{B} = \frac{\mu_0}{4\pi} I \frac{d\underset{\sim}{L} \times \hat{r}}{r^2}$$

$$dB = \frac{\mu_0}{4\pi} I \frac{dL}{r^2} \quad \Rightarrow \text{ since } d\underset{\sim}{L} \text{ and } \hat{r} \text{ are } \perp.$$

$$dB_x = dB \sin\alpha, \quad r^2 = x^2 + a^2, \quad \sin\alpha = \frac{a}{r}$$

$$B_x = \frac{\mu_0}{4\pi} I \frac{a}{r^3} \int_0^L dL \quad \Rightarrow r \text{ same for all } dL.$$

$$= \frac{\mu_0}{4\pi} I \frac{a}{r^3} 2\pi a \qquad \mu \equiv I \pi a^2$$

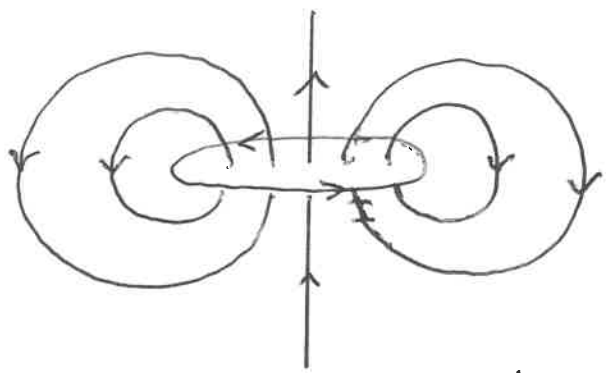
$$B_x = \frac{\mu_0 \mu}{2\pi (x^2+a^2)^{3/2}},$$

large x

$$B_x = \frac{\mu_0 \mu}{2\pi x^3}$$

$\Rightarrow$ same as electric dipole

magnetic field



Close to the wire $\underset{\sim}{B}$ are small loop around the wire $\Rightarrow$ circles

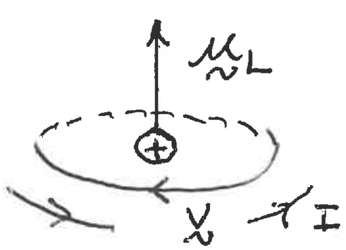
Magnetic materials

Current loops produce magnetic fields

$$B = \frac{\mu_0}{2\pi} \frac{\mu}{r^3} \Rightarrow \text{magnetic dipoles}$$

Materials can have pre-existing magnetic dipoles

$\Rightarrow$ associated with classical electron orbits in atoms



magnetic moments typically cancel pairwise as electrons fill orbitals in atoms

$\Rightarrow$ atoms with an odd # of electrons have magnetic moments

$\Rightarrow$ electrons also have intrinsic magnetic moments associated with their spin

$\Rightarrow \mu_s$

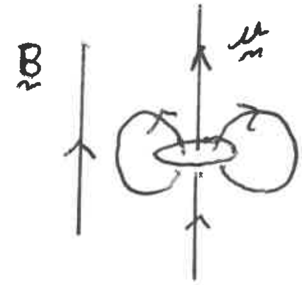
The magnetic moments are typically oriented in random directions in materials

$\Rightarrow$ magnetic field from the dipoles cancel

A magnetic field from outside can cause the magnetic moments to line up

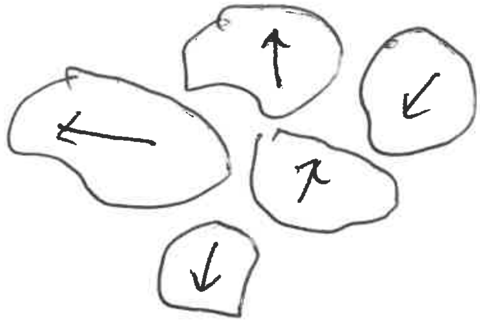
$$U = -\underset{\sim}{\mu} \cdot \underset{\sim}{B}$$

$\Rightarrow$ $\underset{\sim}{\mu}$ and $\underset{\sim}{B}$ align and cause $\underset{\sim}{B}$ to increase



In ferromagnetic materials can have a dramatic increase in $\underset{\sim}{B}$. (iron, nickel, cobalt)

The magnetic dipoles are pre-aligned in domains



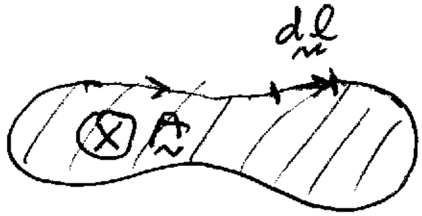
$\updownarrow \sim 0.1$ mm

$\Rightarrow$ in an external field the domains align to produce very strong magnetic fields

Ampere's Law

We know that currents produce magnetic fields. Is there a general relation between current density $\underset{\sim}{J}$ and the resulting magnetic field. ⇒ something that parallels Gauss' Law.

$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{\ell} = \mu_0 I \quad \Rightarrow \text{Ampere's Law}$$



The integral $\oint d\underset{\sim}{\ell} \cdot \underset{\sim}{B}$ is around a closed path.

I is the total current flowing through the area enclosed by the path.

What determines the sign of I?

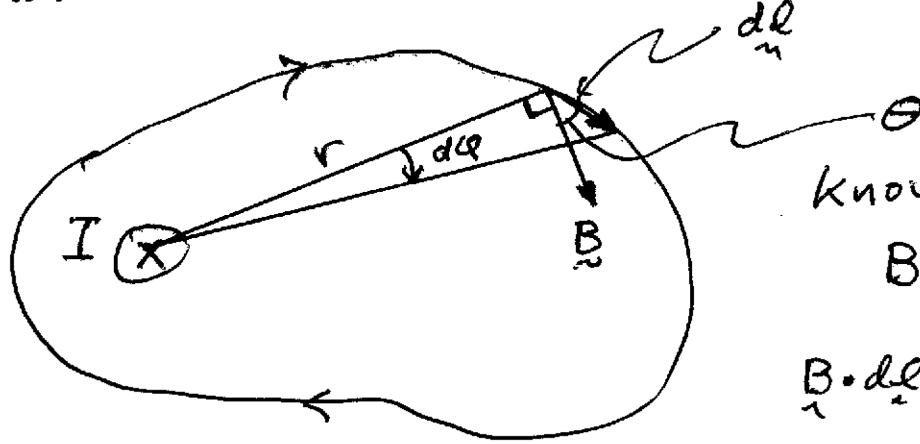
The right hand rule determines the direction of the area vector $\underset{\sim}{A}$.

⇒ fingures along $d\underset{\sim}{\ell}$ and thumb along $\underset{\sim}{A}$

⇒ I is positive in the direction of $\underset{\sim}{A}$ and negative in the opposite direction

$$I = \int_A d\underset{\sim}{A} \cdot \underset{\sim}{J}$$

Proof of Ampere's Law for a single wire:
Consider a closed path in a plane $\perp$ to a wire with current I.



know that
$$B = \frac{\mu_0 I}{2\pi r}$$
$$\underline{B}\cdot d\underline{l} = B\,dl\cos\theta$$

$$\tan(d\alpha) \simeq d\alpha = \frac{dl\cos\theta}{r} \Rightarrow dl\cos\theta = r\,d\alpha$$

$$\Downarrow$$

$$\frac{\sin(d\alpha)}{\cos(d\alpha)} \Rightarrow \left.\begin{array}{l}\sin(d\alpha) \simeq d\alpha \\ \cos(d\alpha) \simeq 1\end{array}\right\} d\alpha << 1$$

$$\oint \underline{B}\cdot d\underline{l} = \oint B\,dl\cos\theta = \oint \frac{\mu_0 I}{2\pi r}(r\,d\alpha)$$
$$= \frac{\mu_0 I}{2\pi}\oint d\alpha = \mu_0 I$$

$\Rightarrow$ valid for any path that circles I.

$\Rightarrow$ valid for any distribution of current

$$\oint \underline{B}_i\cdot d\underline{l} = \mu_0 I_i$$

$$\sum_i \oint \underline{B}_i\cdot d\underline{l} = \mu_0 \sum_i I_i \equiv \mu_0 I$$

$$\oint \left(\sum_i \underline{B}_i\right)\cdot d\underline{l} = \mu_0 I \;, \quad \sum_i \underline{B}_i \equiv \underline{B}$$

$$\oint \underline{B}\cdot d\underline{l} = \mu_0 I$$

$\Rightarrow$ Ampere's Law is always valid

$\Rightarrow$ Only useful to evaluate $\underset{\sim}{B}$ if have symmetry so

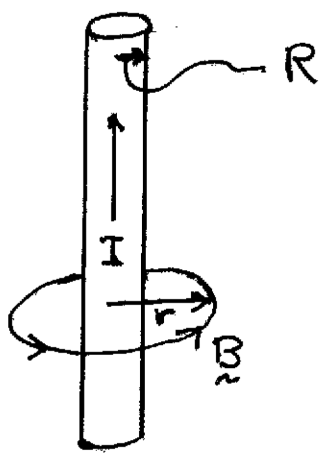
$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l}$$

can be evaluated.

$\underset{\sim}{B}$ field in a finite size conductor

Consider an infinite (long) conductor of radius R carrying a current I. Calculate $\underset{\sim}{B}$ everywhere assuming that the current density J is uniform.

$$J = \frac{I}{\pi R^2} = \frac{\text{current}}{\text{area}}$$



First consider $r > R$.
Choose a path on which $|\underset{\sim}{B}|$ is constant

$\Rightarrow$ circle of radius r

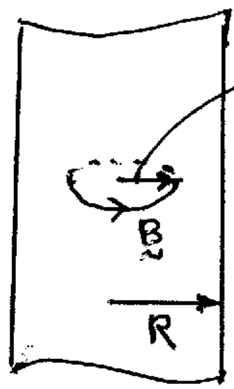
$\Rightarrow$ $\underset{\sim}{B}$ parallel to $d\underset{\sim}{l}$

$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = \int B\, dl = B \oint dl = B\, 2\pi r$$

$$= \mu_0 I$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

Now take $r < R$.



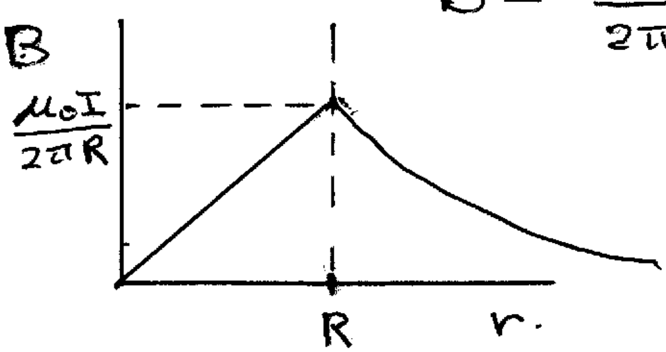
$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{\ell} = B\, 2\pi r = \mu_0 I_r$$

I_r = current cutting through area πr^2

$$I_r = J \pi r^2 = I \frac{r^2}{R^2}$$

$$B = \frac{1}{2\pi r} \mu_0 \frac{I\, r^2}{R^2}$$

$$B = \frac{\mu_0 I}{2\pi R^2} r$$

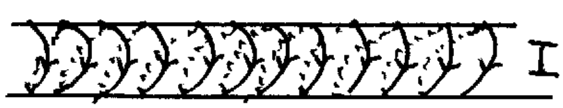


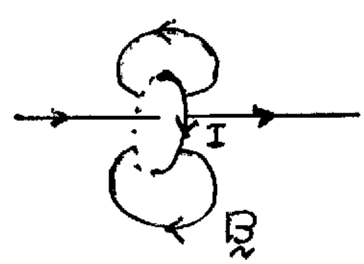
Magnetic field in a solenoid

A solenoid is a cylinder wrapped with wire.

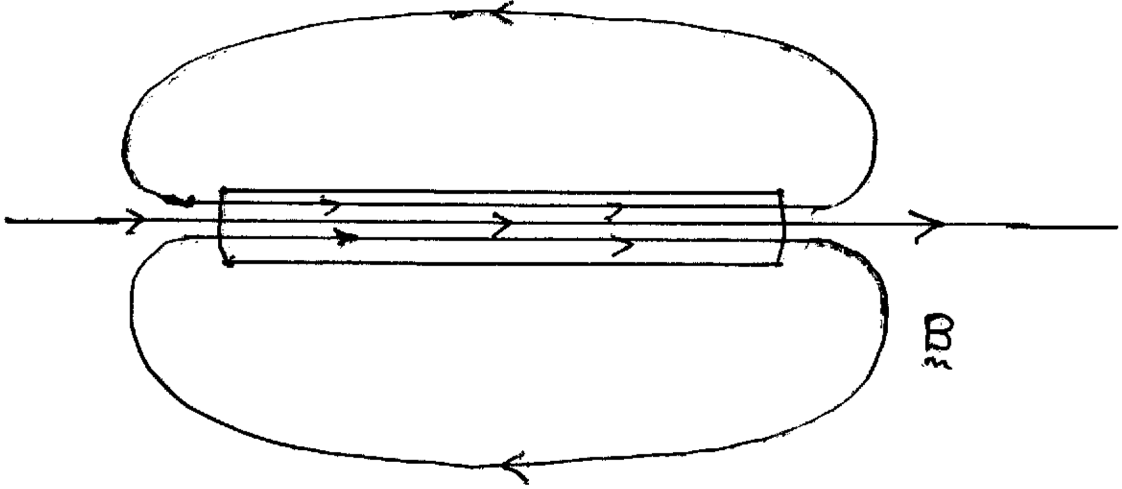
$\Rightarrow$ n = # of wraps per unit length

$\Rightarrow$ I = current in the wire

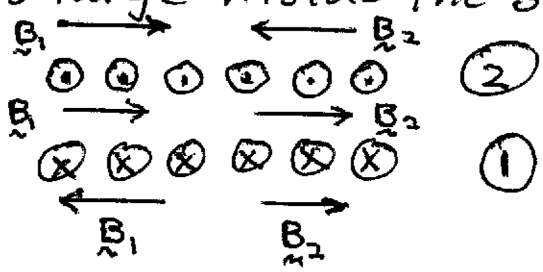




single loop

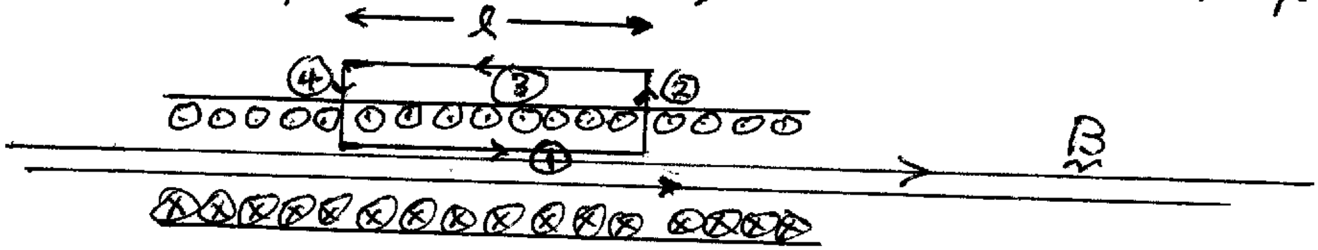


B is large inside the solenoid and small outside



$\mathbf{B}$ from the two current layers add inside and subtract outside

Use Ampere's Law $\Longrightarrow$ choose closed loop



$\Longrightarrow$ four segments of the closed loop

$$\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 I_{tot} = \mu_0 \left(\text{total current cutting through loop}\right)$$

$$I_{tot} = \underbrace{(n\ell)}_{\text{total \# of turns passing through the loop}} I$$

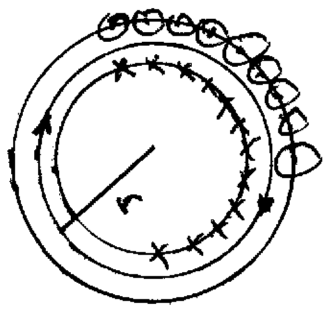
$$\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \underbrace{\int_{1} \mathbf{B}\cdot d\boldsymbol{\ell}}_{B\ell} + \underbrace{\int_{2} \mathbf{B}\cdot d\boldsymbol{\ell}}_{\mathbf{B}\cdot d\boldsymbol{\ell}=0} + \underbrace{\int_{3} \mathbf{B}\cdot d\boldsymbol{\ell}}_{\mathbf{B}=0} + \underbrace{\int_{4} \mathbf{B}\cdot d\boldsymbol{\ell}}_{\mathbf{B}\cdot d\boldsymbol{\ell}=0}$$

$$B\ell = \mu_0 n \ell I$$

$$\boxed{B = \mu_0 n I}$$

Toroidal Solenoid

$\Rightarrow$ N turns total, current I



For B inside the torus

$$\oint_{inside} \underset{\sim}{B} \cdot d\underset{\sim}{l} = \mu_0 I_{tot} = \mu_0 N I$$

$$B \, 2\pi r = \mu_0 N I$$

$$B = \frac{\mu_0 N I}{2\pi r}$$

$\Rightarrow$ B is larger at smaller radii