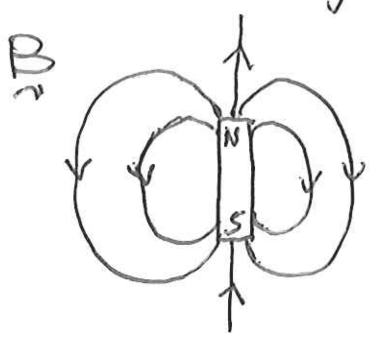
Magnetic fields

The existence of magnetic materials has been known for over 2000 years. Magnetic materials have magnetic poles called North and South poles. North poles of magnets repell each other as do South poles. N/S pairs attract each other. Certain materials (e.g. iron) though unmagnetized are attracted by a magnet. Other materials such as aluminum or copper are not attracted by magnetic materials.

Magnetic forces clearly have a different character from electric forces. We introduced the concept of an electric field that surrounds charged particles. To explore and understand magnetic forces, we want to introduce the magnetic field $\underset{\sim}{B}$ (a vector) which surrounds magnetic materials. We will then explore ① how this magnetic field acts on charged particles and ② how this magnetic field is generated.

That such a field exists can be seen by looking at how iron filings distribute around a bar magnet. We will show later that moving charged particles — currents — also generate $\underset{\sim}{B}$.



Vector product (cross product)

Before discussing magnetic forces, we review the rules on products with vectors. Consider two vectors $\mathbf{A}$, $\mathbf{B}$. In physics there are two types of products with vectors: the scalar or dot product; and the vector or cross product.

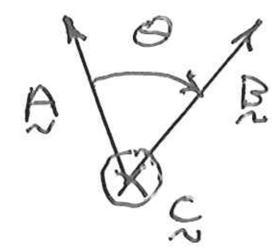
dot product: the dot product is defined by

$$C = \mathbf{A} \cdot \mathbf{B} = AB\cos\theta$$

where C is a scalar.

vector (cross) product: defined by

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$



$\mathbf{C}$ points into the page

⇒ use the right hand rule to find the direction of $\mathbf{C}$.

⇒ fingers rotate from $\mathbf{A}$ to $\mathbf{B}$ and right thumb points in the direction of $\mathbf{C}$.

$C = AB\sin\theta$

$\mathbf{C}$ is $\perp$ to $\mathbf{A}$ and $\mathbf{B}$.

In an x, y, z coordinate system.

$$\mathbf{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}\left(A_y B_z - B_y A_z\right) - \hat{j}\left(A_x B_z - B_x A_z\right) + \hat{k}\left(A_x B_y - A_y B_x\right)$$

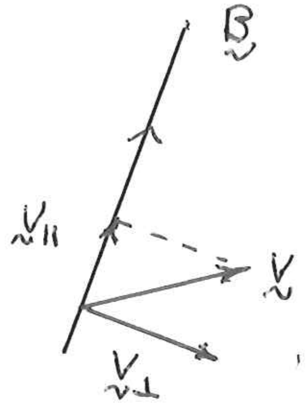
Magnetic Forces

How does a charged particle respond to a magnetic field?

observations:

① A charged particle at rest has no force acting on it.

② The force is proportional to the component of $\underset{\sim}{v}$ that is $\perp$ to $\underset{\sim}{B}$ $\Rightarrow$ $\underset{\sim}{v}_{\perp}$



③ The force is proportional to the charge q of the particle

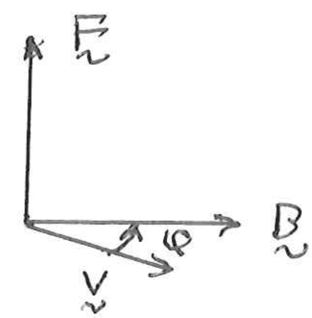
$$|\underset{\sim}{F}| = |q|\,|\underset{\sim}{v}_{\perp}|\,|\underset{\sim}{B}|$$

④ The force is $\perp$ to both $\underset{\sim}{v}_{\perp}$ and $\underset{\sim}{B}$.

$$\underset{\sim}{F} = q\,\underset{\sim}{v} \times \underset{\sim}{B}$$

$$|\underset{\sim}{F}| = q\,v_{\perp}\,B$$
$$= q\,v \sin\varphi\, B$$

with φ the angle between $\underset{\sim}{v}$ and $\underset{\sim}{B}$



$v_{\perp} = v \sin\varphi$

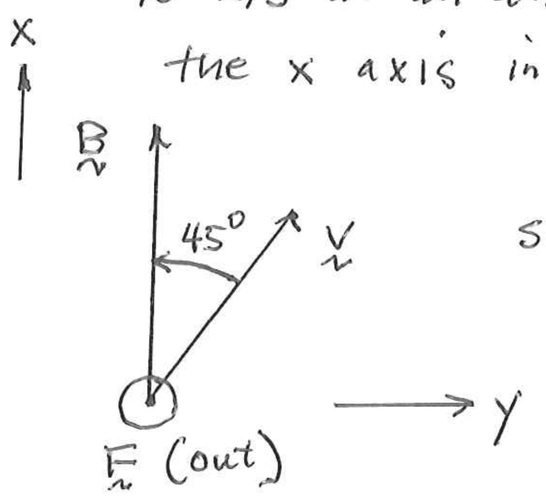
units: $\mathbf{B} \sim \dfrac{N}{C\,(m/s)} = \dfrac{N}{Am} \equiv$ tesla in SI units

In cgs units $\mathbf{B}$ is measured in gauss

$$1 \text{ gauss} = 10^{-4} \text{ Tesla}$$

<u>example</u>:

A magnetic field of 1T is in the x direction. A proton ($q = 1.6 \times 10^{-19}$ C) moves with a velocity of 10^6 m/s at an angle of 45° with respect to the x axis in the x-y plane.



$$\hat{k} = \hat{i} \times \hat{j}$$

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$F = q\, v \sin\theta\, B$$
$$= 1.6 \times 10^{-19}\,C \; \frac{10^{6}\, m/s}{\sqrt{2}} \; 1\,T$$
$$= 1.13 \times 10^{-13}\, N$$

$$F_z = -1.13 \times 10^{-13}\, N$$

$$\mathbf{F} = -1.13 \times 10^{-13}\, N\ \hat{k}$$

Motion of a charged particle in a magnetic field

Equation of motion $\quad m \frac{d}{dt} \underset{\sim}{V} = q \underset{\sim}{V} \times \underset{\sim}{B}$

To find the change in KE, take the dot product with $\underset{\sim}{V}$

$$m\, \underset{\sim}{V} \cdot \frac{d}{dt} \underset{\sim}{V} = m\left(V_x \frac{d}{dt} V_x + V_y \frac{d}{dt} V_y + V_z \frac{d}{dt} V_z \right)$$

$$= \frac{d}{dt} \frac{1}{2} m \left(V_x^2 + V_y^2 + V_z^2 \right)$$

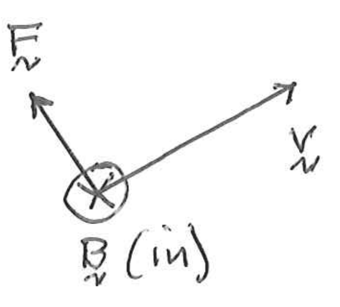
$$= \frac{d}{dt} \left(\frac{1}{2} m v^2 \right)$$

$$\frac{d}{dt}\left(\frac{1}{2} m v^2\right) = q\, \underset{\sim}{V} \cdot \left(\underset{\sim}{V} \times \underset{\sim}{B} \right) = 0$$

$\Rightarrow$ since $\underset{\sim}{V} \times \underset{\sim}{B}$ is $\perp$ to $\underset{\sim}{V}$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2} m v^2\right) = 0$$

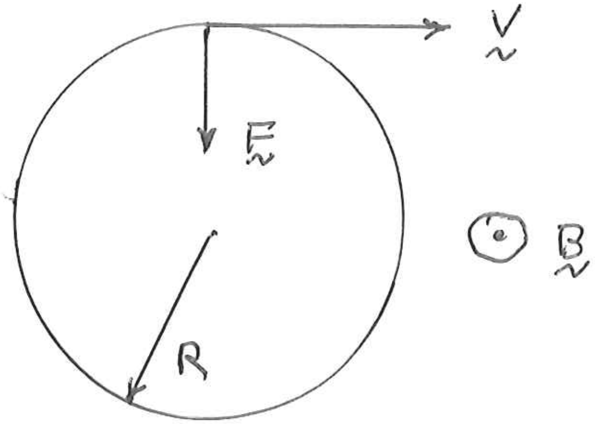
$\Rightarrow$ the magnetic field does not change the kinetic energy of a charge.



The force acts to change the direction of $\underset{\sim}{V}$ but not its magnitude.

Since there is no force parallel to $\underset{\sim}{B}$, $V_{\parallel}$ is unchanged.

In a uniform magnetic field with $\underset{\sim}{v}$ $\perp$ to $\underset{\sim}{B}$, the motion is a circle



circular motion:

$$F = ma = m\frac{v^2}{R} = qvB$$

$$\Rightarrow R = \frac{mv}{qB} = \text{radius of orbit}$$

$\Rightarrow$ increases with v

$$v = \omega R$$

$$\omega = \frac{qB}{m} = \text{gyrofrequency}$$

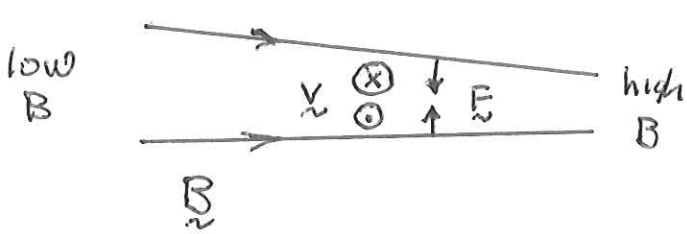
$\Rightarrow$ independent of v

What does the motion of a particle moving along and across a magnetic field look like?

$\Longrightarrow$ a spiral with $v_{\parallel}$ constant

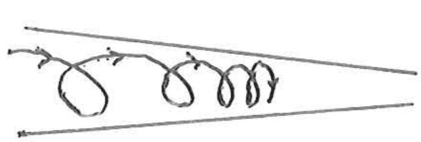
Trapping of charges in a magnetic bottle

Consider the motion of a charge in a converging magnetic field



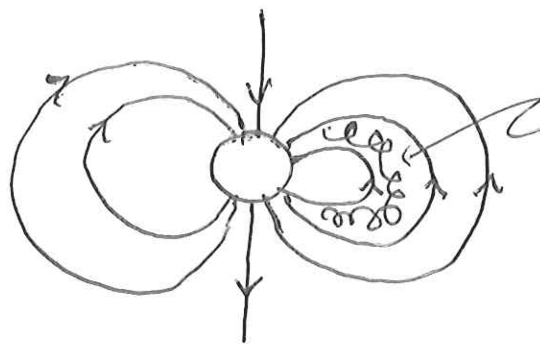
To lowest order the motion is a circle but have a small force pointing away from the high field region

$\Rightarrow$ particle is reflected



Earth's magnetic field

$\Rightarrow$ trapping in the radiation belts

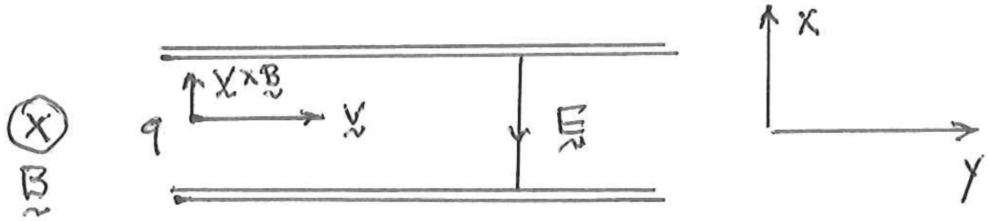


Van Allen radiation belts

electron and protons that range from 1 MeV and above.

Motion in crossed $\underset{\sim}{E}$ and $\underset{\sim}{B}$ fields

Consider a uniform magnetic field $\underset{\sim}{B}$ and a uniform $\underset{\sim}{E}$ between two plates



$\Rightarrow$ can balance the forces $\qquad (\underset{\sim}{V}\times\underset{\sim}{B})_x = V_y B_z - \cancelto{0}{V_z} B_y$

$$m\frac{d}{dt}V_x = qE_x + qV_yB_z = 0$$

$\Rightarrow$ adjust E_x so it cancels the magnetic force

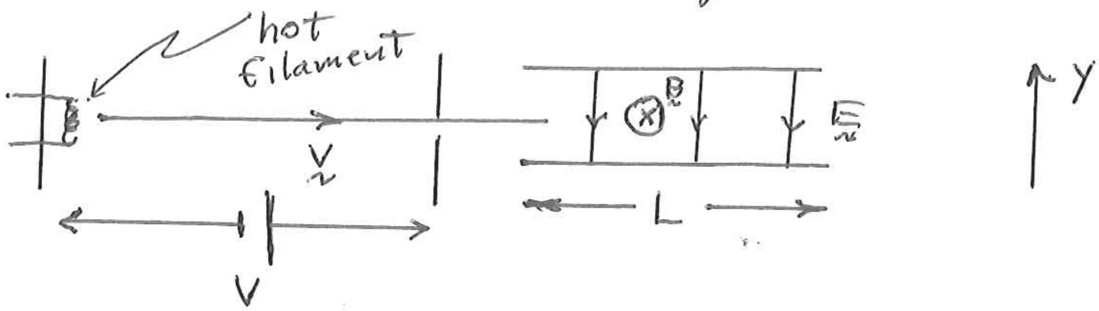
$$E_x = -V_yB_z$$

The particle will move in a straight line

$\Rightarrow$ used to measure the velocity of a charge

$$V_y = \frac{-E_x}{B_z}$$

Can use this technique to measure the ratio of charge to mass of particles



The hot filament produces electrons that are accelerated across the potential V.

first: shut off B and measure displacement in y by E

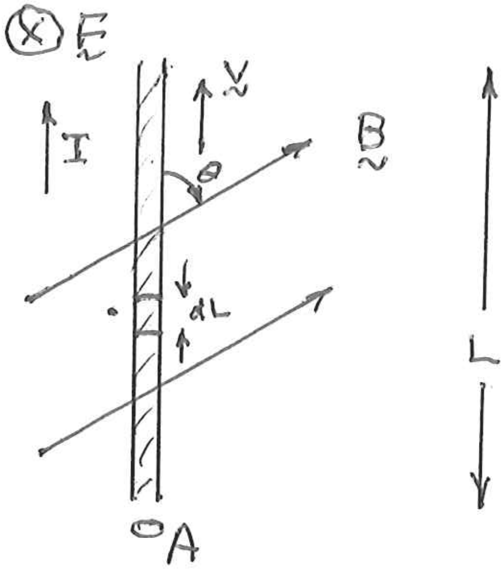
$$\Delta y = \frac{1}{2} a t^2 = \frac{1}{2} \frac{eE}{m} \left(\frac{L}{v}\right)^2$$

second: increase B until no deflection

$$E = vB \Rightarrow v = \frac{E}{B}$$

$$\frac{e}{m} = \frac{2 \Delta y \, E}{B^2 L^2} \qquad \text{JJ Thomson}$$

Force on a current carrying wire



$I = JA = nqvA$

$J = nqv$

Calculate the number of charges in a small segment of the wire dL. Find the total force on those charges and then add up all the forces on the wire of length L.

dN = # of charges in dL

$= An(dL)$

dQ = total charge in dL

$= q\,dN$

Calculate the force on dQ

$$d\underset{\sim}{F} = dQ\,\underset{\sim}{v} \times \underset{\sim}{B} \Rightarrow dF = dQ\,vB\sin\theta$$

$$= q\,dN\,vB\sin\theta$$

$$= nqv\,A(dL)\,B\sin\theta$$

$$= I\,dL\,B\sin\theta$$

F = total force $= IB\sin\theta \int dL$

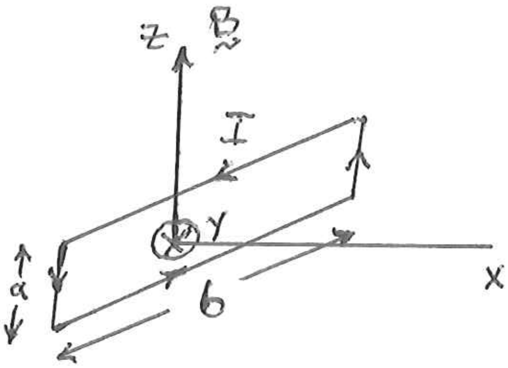
$$F = IBL\sin\theta$$

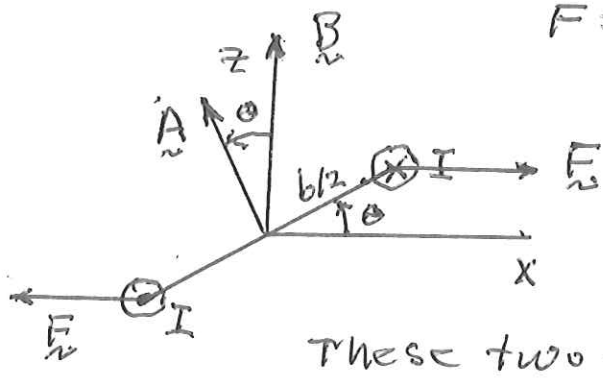
or in vector form

$$\underset{\sim}{F} = I\,\underset{\sim}{L} \times \underset{\sim}{B} \Rightarrow \underset{\sim}{F} \text{ is } \perp \text{ to } \underset{\sim}{B} \text{ and to the wire}$$

where $\underset{\sim}{L}$ is along the wire in the direction of the current

Torque on a current loop





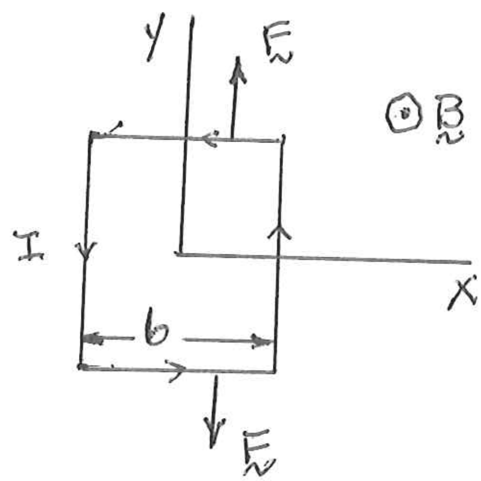
$F = BIa$

These two forces produce a torque around the y axis

$\tau = 2F\ell$

ℓ = moment arm

$= \frac{b}{2}\sin\theta$



Forces on the two sides of the loop of length "b" cancel

$\tau = 2BIa\frac{b}{2}\sin\theta$

$= BIA\sin\theta$

A = area of loop

The vector $\mathbf{A}$ is given by the RH rule with your fingers in the direction of the current and your thumb along $\mathbf{A}$.

μ = magnetic moment

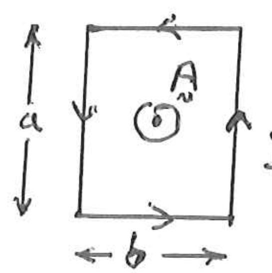
$= IA$

= product of area and current

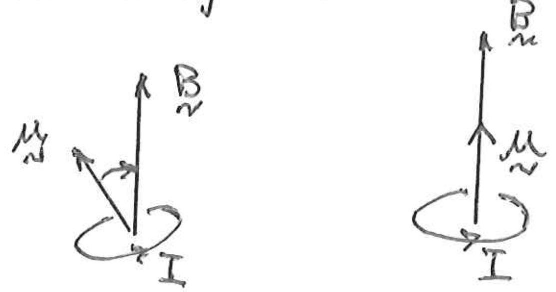
$\boldsymbol{\mu} = I\,\mathbf{A}$

$\tau = \mu B\sin\theta$

$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$



The loop wants to rotate so that $\underset{\sim}{\mu}$ and $\underset{\sim}{B}$ are aligned



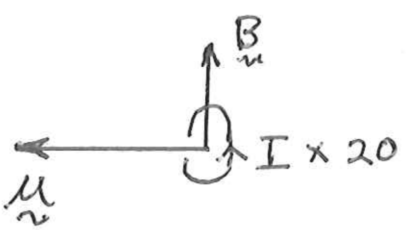
The relation
$\mu =$ area $\times$ current
is valid for any planar current loop

$$\Rightarrow \underset{\sim}{\tau} = \underset{\sim}{\mu} \times \underset{\sim}{B}$$

example:

Consider a loop of area $1\,cm^2$ with a 20 turn coil of wire, carrying a current of $5A$ in a $1T$ magnetic field

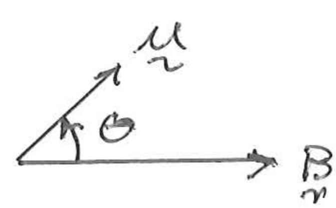
$$\mu = 10^{-4} m^2 (5 \times 20)\,A$$
$$= 10^{-2}\,Am^2$$



$$\tau = 10^{-2}\,Am^2\,1T = 10^{-2}\,Nm$$

Energy of a current loop in a magnetic field

Consider a loop with a magnetic moment μ and moment of inertia I_m



$$I_m \frac{d\omega}{dt} = \tau$$

with $\omega = \frac{d\theta}{dt}$

minus because torque acts to reduce θ

$$I_m \frac{d^2}{dt^2}\theta = -\mu B \sin\theta$$

Construct an energy equation by multiplying by $\frac{d\theta}{dt}$

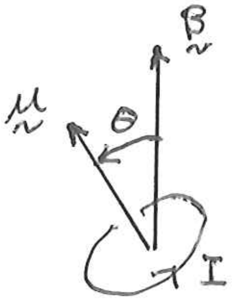
$$\underbrace{I_m \frac{d\theta}{dt}\frac{d^2\theta}{dt^2}}_{I_m\,\omega\,\frac{d\omega}{dt}} = -\mu B \frac{d\theta}{dt}\sin\theta$$

$$= \mu B \frac{d}{dt}(\cos\theta)$$

$$\frac{d}{dt}\left(\frac{\omega^2}{2}\right) = \omega\frac{d\omega}{dt}$$

$$\frac{d}{dt}\left(\frac{1}{2} I_m \omega^2\right) = \mu B \frac{d}{dt}\cos\theta$$

$$\frac{d}{dt}\Big(\underbrace{\frac{1}{2} I_m \omega^2 - \mu B\cos\theta}_{\text{energy conservation equation}}\Big) = 0$$



$\frac{1}{2} I_m \omega^2$ = rotational kinetic energy

$U = -\mu B\cos\theta$ = potential energy of the current loop in the magnetic field

$$\Downarrow$$

$$U = -\underset{\sim}{\mu}\cdot\underset{\sim}{B}$$

<u>example:</u> At $t=0$, the loop is at rest with $\theta = \pi/2$. Torque causes $\underset{\sim}{\mu}$ to align with $\underset{\sim}{B}$. What is the rate of angular rotation $\omega = d\theta/dt$ when $\theta = 0$?

$$W = \frac{1}{2} I_m \omega^2 + U = \text{const}$$

At $t=0$, $\omega = 0$ and $\theta = \frac{\pi}{2}$

$$U = -\underset{\sim}{\mu} \cdot \underset{\sim}{B} = 0$$

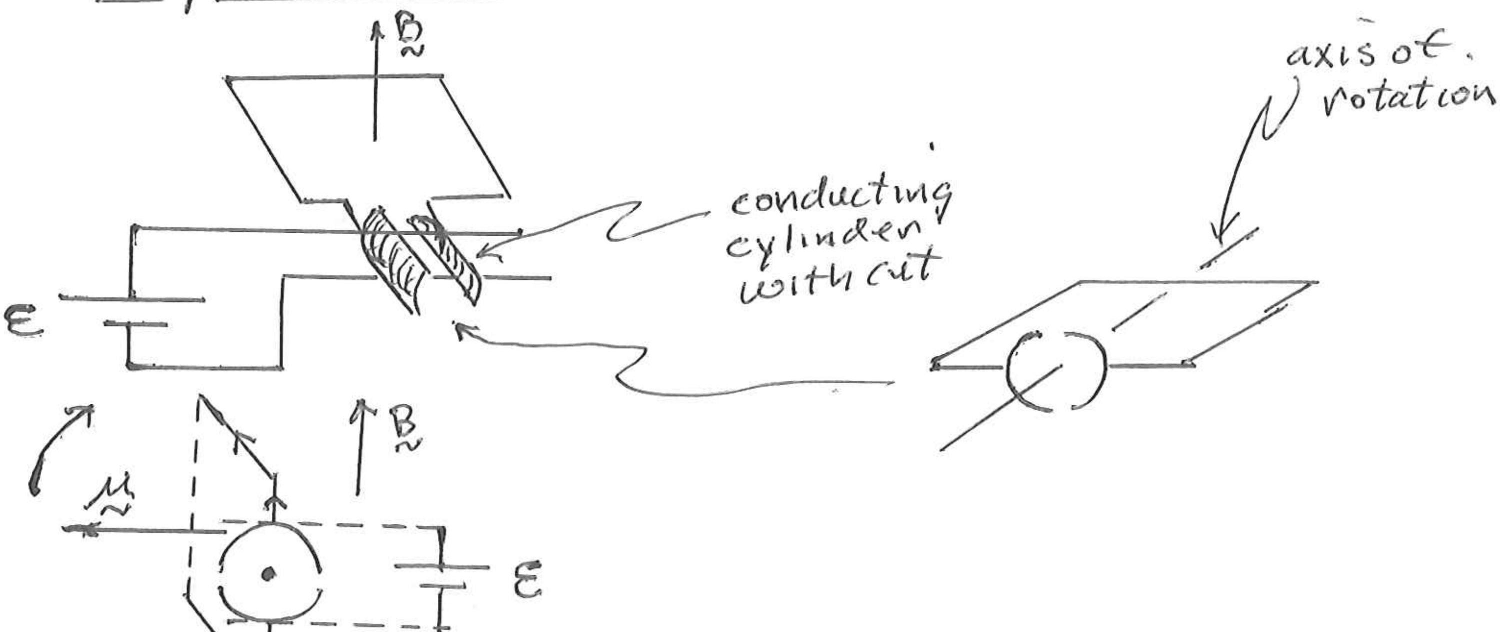
when $\theta = 0$,

$$0 = \frac{1}{2} I_m \omega^2 - \mu B$$

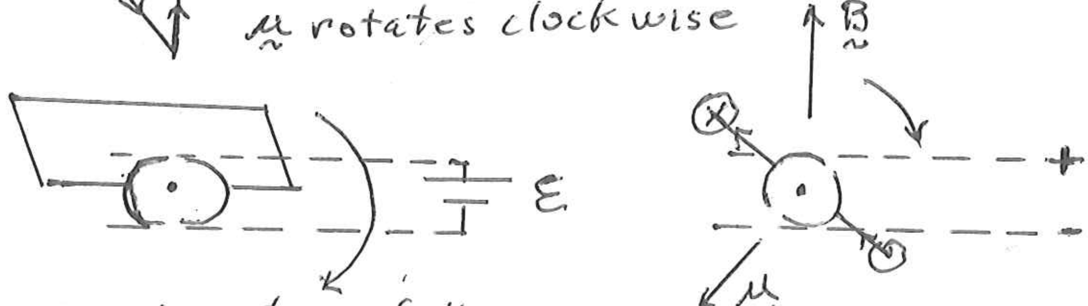
$$\omega = \left(\frac{2\mu B}{I_m}\right)^{\frac{1}{2}}$$

$\Rightarrow \omega$ has units of radians / sec

Simple electric motor.



$\underset{\sim}{\mu}$ rotates clockwise



The direction of the current switches when the $\underset{\sim}{\mu}$ aligns with $\underset{\sim}{B}$ $\Rightarrow$ reverses $\underset{\sim}{\mu}$ so opposite to $\underset{\sim}{B}$.