Current and the motion of charge

In our study of electrostatics we were primarily interested in charge distributions which were stationary in time. Now we want to consider the motion of charge. Such charged motion will also produce a magnetic field. For now we assume that the motion of charge is weak so that magnetic fields can be neglected.

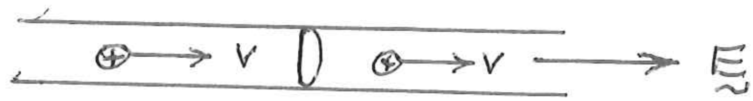
We say that charges in motion produce a current. How do we produce a current?. Suppose we place charge on a conductor in a localized region. The charge will move due to the local electric field $\underset{\sim}{E}$ and spread out until $\underset{\sim}{E} = 0$ inside the conductor. The current is a transient that flows only when $\underset{\sim}{E} \neq 0$. Once $\underset{\sim}{E} = 0$ the current dies away. Why?

$\Longrightarrow$ In a conductor the electrons can move but they are constantly bumping into the particles that make up the crystal lattice of the metal. They lose energy as they bump into the stationary lattice particles. That is, friction with the lattice can't be neglected and the current dies away when $\underset{\sim}{E} = 0$.

⇒ To maintain the current, we must maintain $\underset{\sim}{E} \neq 0$.

⇒ For now we will assume that there is some mechanism for maintaining $\underset{\sim}{E} \neq 0$ in a conductor so that electrons remain in motion.

Consider a thin wire made of a conductor along which we have $\underset{\sim}{E} \neq 0$.



A = cross sectional area

For simplicity we take the moving charges to be positive and are moving along $\underset{\sim}{E}$ (Actually the moving charges are negative and move in the $-\underset{\sim}{E}$ direction) with a velocity v.

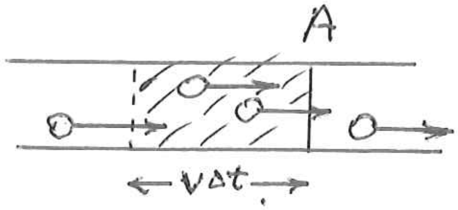
$$I = \text{current} = \frac{\Delta Q}{\Delta t}$$

ΔQ = total charge which crosses the area A in a time Δt.

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

units of current: $\text{ampere} = \frac{\text{coulomb}}{\text{sec}}$

I is a scalar that gives the rate at which charge flows in the wire.



ΔQ gives all the charge ~~moving~~ in the shaded volume. All of this charge crosses A in a time Δt.

$$\Delta Q = \underbrace{A v \Delta t}_{\Delta V = \text{volume}}\, n\, q \qquad n = \frac{\#\text{ of charges}}{\text{volume}}$$

q = single charge

$$= \Delta N q \quad \text{with } \Delta N = A v \Delta t\, n$$

$= \#$ of charges in shaded area.

$$I = \frac{\Delta Q}{\Delta t} = \frac{q A v \Delta t\, n}{\Delta t}$$

$$= q A n v$$

Define the current per unit area J as

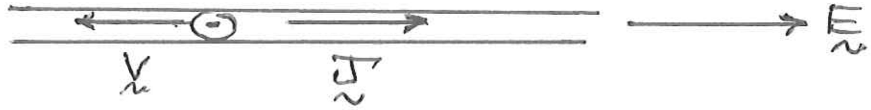
$$J = \frac{I}{A} = q n v$$

Can also define the current density as a vector

$$\underset{\sim}{J} = q n \underset{\sim}{v}$$

where $\underset{\sim}{v}$ is in the direction of $\underset{\sim}{E}$.

In a real conductor, electrons carry the current



$$\underset{\sim}{J} = n q \underset{\sim}{v} = -n e \underset{\sim}{v}$$

$\Rightarrow$ $\underset{\sim}{J}$ is in the direction of $\underset{\sim}{E}$ even though $\underset{\sim}{v}$ is in the $-\underset{\sim}{E}$ direction.

Drift velocity of electrons

In Cu, $n = 8.5 \times 10^{28}/m^3$

Take $A = 1\,mm^2$ and $I = 1\,A$

$$J = neV \Rightarrow V = \frac{J}{ne}$$

$$V = \frac{1\,A/(10^{-3}m)^2}{1.6\times10^{-19}\,C\;8.5\times10^{28}/m^3}$$

$$= \frac{C}{s}\,\frac{1}{10^{-6}m^2}\,\frac{m^3}{1.6\times10^{-19}C\;8.5\times10^{28}}$$

$$= \frac{10^{-3}}{(1.6)(8.5)}\,\frac{m}{s} = 7.4\times10^{-5}\,\frac{m}{s}$$

$\Rightarrow$ very slow

$\Rightarrow$ Why does a light turn on so quickly once a switch is closed?

Resistivity

How do electrons respond to $\underline{E}$ in a conductor?

$$m\frac{d}{dt}\underline{V} = -e\underline{E} - m_e \nu_e \underline{V}$$

ν_e = rate at which electrons collide with the lattice of a metal

$\Rightarrow$ similar to the friction force of an object moving through air.

The velocity increases until the force from $\underset{\sim}{E}$ balances the drag,

$$0 = -e\underset{\sim}{E} - m_e \nu_e \underset{\sim}{V}$$

$$\Rightarrow m_e \nu_e \underset{\sim}{V} = -e\underset{\sim}{E}$$

$$\underset{\sim}{V} = -\frac{e\underset{\sim}{E}}{m_e \nu_e}$$

$$\underset{\sim}{J} = -ne\underset{\sim}{V} = \frac{ne^2\underset{\sim}{E}}{m_e \nu_e}$$

$$\eta \equiv \frac{m\nu_e}{ne^2} = \text{resistivity}$$

η is intrinsic to the material

$\eta \underset{\sim}{J} = \underset{\sim}{E}$ $\Rightarrow$ the current density is proportional to $\underset{\sim}{E}$.

$\Rightarrow$ note that η tends to increase with the temperature $\Rightarrow$ increase of electron thermal speed V_{te}

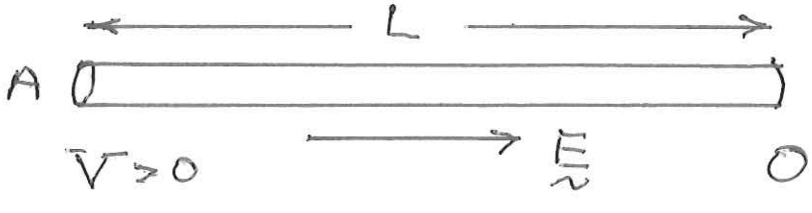
$\Rightarrow$ also $V_{te} \sim 10^5$ m/s for

$T_e \sim 300\,^{\circ}K$

$\Rightarrow V_{te} >> V$

Potential drop along a wire

Apply a potential V along a wire of length L and cross section A.



$\Rightarrow$ $\underset{\sim}{E}$ points down the potential

$\Rightarrow$ current flows along the direction of the potential drop

$$E = \frac{V}{L} \quad \Rightarrow \quad J = \frac{E}{\eta} = \frac{1}{\eta}\frac{V}{L}$$

$$I = JA = \frac{A}{\eta L} V$$

$$R \equiv \frac{\eta L}{A} = \text{resistance}$$

$$IR = V \quad \Rightarrow \text{Ohm's Law}$$

$\Rightarrow$ Ohm's Law gives the relation between current and voltage in a wire

$\Rightarrow$ large L $\Rightarrow$ larger resistance
smaller A $\Rightarrow$ " "

units: $1 \text{ Ohm} = \dfrac{1 \text{ volt}}{1 \text{ amp}} \Rightarrow 1\,\Omega$

η has units of Ohm-meters

example:

resistivity of copper $= 1.72 \times 10^{-8}\ \Omega\text{-m}$
steel $= 20 \times 10^{-8}\ \Omega m$

How long is a wire of Cu with $A = 1\,mm^2$ and $R = 1\,\Omega$?

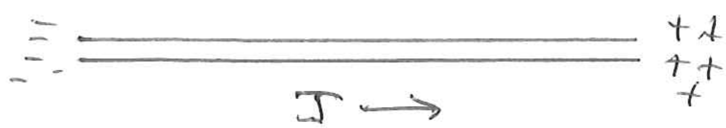
$$R = 1\,\Omega = \frac{\eta L}{A} \Rightarrow L = \frac{RA}{\eta}$$

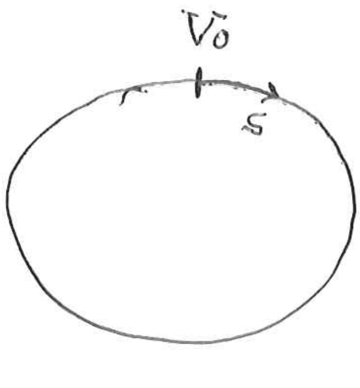
$$L = \frac{1\,\Omega\ 10^{-6}\,m^2}{1.72 \times 10^{-8}\,\Omega m} = 58\,m$$

Steel?
$L = 5\,m$

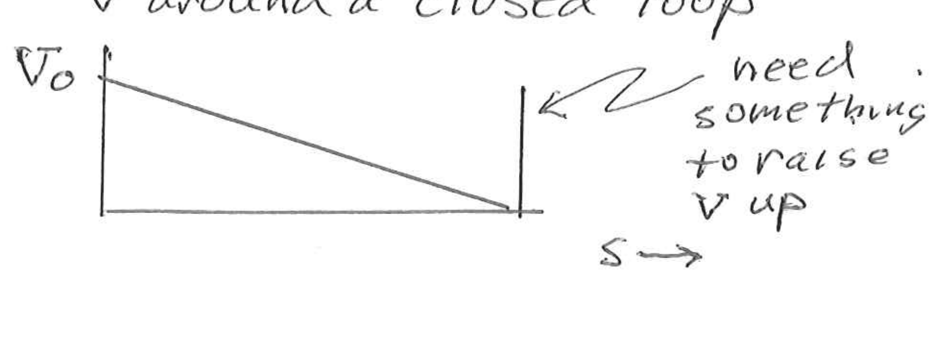
Electromotive force and circuits

An electric current always flows down a potential hill. To form a circuit the current must close on itself or charge will accumulate





V around a closed loop

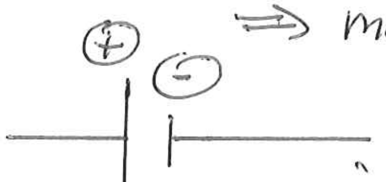


An <u>electromotive force</u> forces charge up hill to a higher potential

⇒ EMF

⇒ acts against the electric field

⇒ measured in volts



⇒ represents a source of EMF

⊕ is high voltage side

Roller coaster analogy:

⇒ chain pulls the car up the hill (EMF)

⇒ gravity accelerates the car down until it reaches the starting point (current driven by potential drop)

⇒ cycle repeats

examples: A battery ⇒ a chemical reaction forces the charge up the hill

Electric generator

Van de Graat machine ⇒ mechanical

Kirchoff's Loop rule

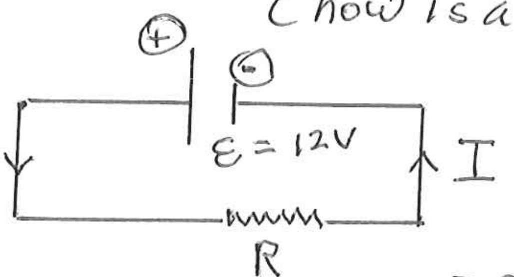
As you move completely around a current loop, you must arrive back at the potential where you~~s~~ started

$\Rightarrow$ the sum of the potential drops around a loop must equal the sum of the EMFs.

example:

Consider a circuit with an EMF of 12 volts and a resistor with $R = 1\,\Omega$.

$\Rightarrow$ neglect the resistance of the wire (how is a resistor made?)



The EMF drives current in the circuit from the $\oplus$ to the $\ominus$.

$\rightarrow \quad \leftarrow \; -IR \Rightarrow$ voltage across the resistor drops along the direction of I.

Sum of potentials: counter clockwise

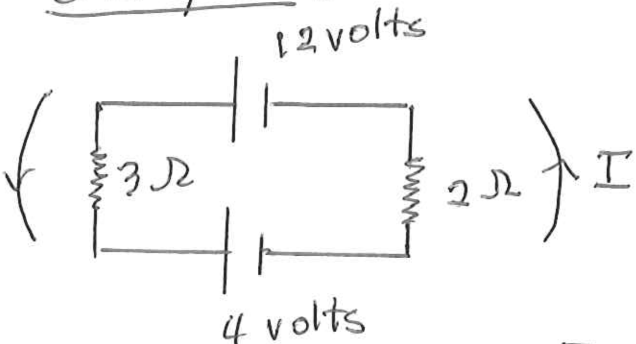
$$\mathcal{E} - IR = 0 \Rightarrow I = \frac{\mathcal{E}}{R} = \frac{12\,V}{1\,\Omega}$$

$$= 12\,A$$

Clockwise

$$-\mathcal{E} + IR = 0 \Rightarrow \text{again } I = \frac{\mathcal{E}}{R}$$

example :



Start by assuming a direction for I

counter clock wise

$$-I\,3\Omega - 4\text{ volts} - I\,2\Omega + 12\text{ volts} = 0$$

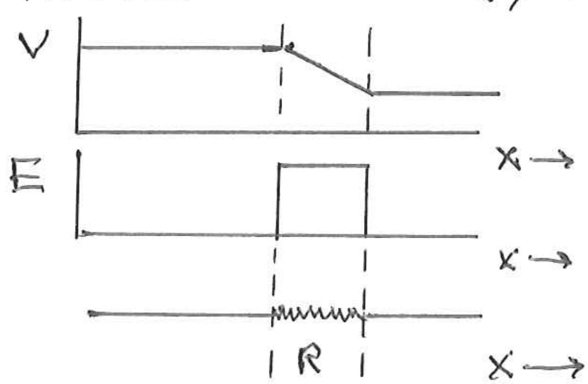
$$I = \frac{(12-4)\text{ volts}}{(3+2)\,\Omega} = \frac{8}{5}A$$

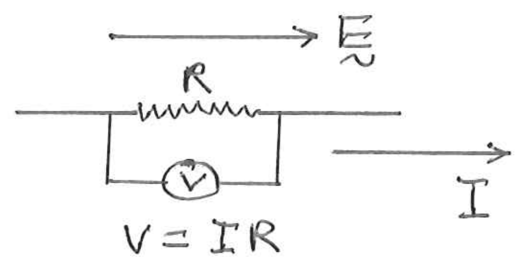
$\Rightarrow$ if choose I clockwise would find

$$I = -\frac{8}{5}A$$

Energy dissipation in a resistor

Consider a resistor R with current I. What is the rate of energy dissipation in the resistor?





Consider a charge dQ which moves down the potential $\Rightarrow$ receives energy

$$dw = dQ\,V$$

$\Rightarrow$ from E $\Rightarrow$ energy lost as heat during collisions in the resistor

Rate of energy transfer to heat

$$\frac{dW}{dt} = P = \text{power} = \frac{dQ}{dt} V = I V$$

$$P = IV = I^2 R$$

units:

$$1 \text{ watt} = \frac{C}{s}\frac{J}{C} = \frac{J}{s}$$

For $V = 120$ volts, $R = 10\,\Omega$

$$\Rightarrow I = 12\,A$$

$$P = 1440\,W$$

<u>Source of energy from EMF</u>

An EMF raises a charge dQ up the potential $\mathcal{E}_m$

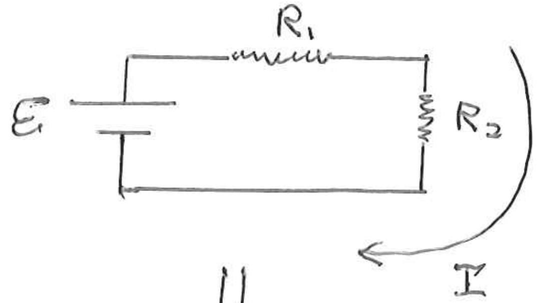
$\Rightarrow$ gives charge $dW = dQ\,\mathcal{E}$

$$\frac{dW}{dt} = I\,\mathcal{E}$$

$I\mathcal{E}$ = rate at which the EMF supplies energy to the charges through the emf.

<u>Circuits with combinations of resistors</u>

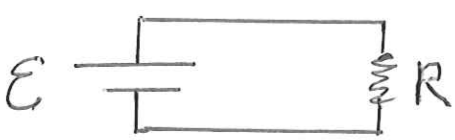
Circuits may have a more complex combination of resistors. These can often be lumped into an effective resistivity to simplify the configuration resistors in series:



$$\mathcal{E} - IR_1 - IR_2 = 0$$

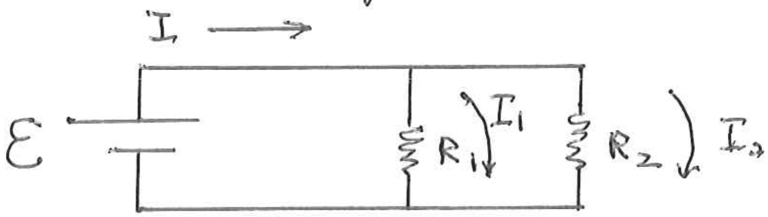
$$\mathcal{E} - I(R_1 + R_2) = 0$$

$$R \equiv R_1 + R_2$$



$\Rightarrow$ note that the current through the two resistors is the same

resistors in parallel:



$\Rightarrow$ voltages across R_1 and R_2 are equal

$$I_1 R_1 = I_2 R_2$$

loop rule clockwise around right hand loop:

$$-I_2 R_2 + I_1 R_1 = 0$$

$$\Rightarrow I_1 = I_2 \frac{R_2}{R_1}$$

$$I = I_1 + I_2$$

$$= I_2 \frac{R_2}{R_1} + I_2$$

$$= I_2 \left(1 + \frac{R_2}{R_1}\right)$$

$$I_2 = \frac{I}{1 + \frac{R_2}{R_1}}$$

loop rule for ~~[illegible]~~ large loop:

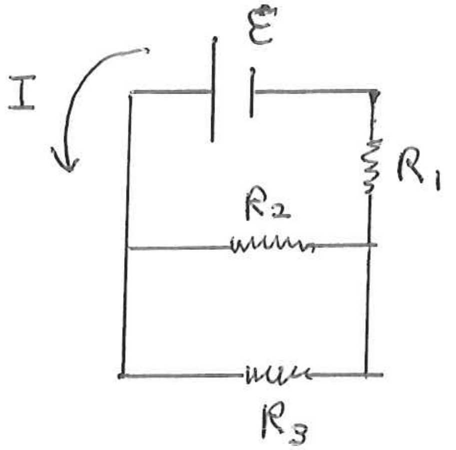
$$\mathcal{E} - I_2 R_2 = 0 \quad \Rightarrow \quad \mathcal{E} = \frac{I}{1 + \frac{R_2}{R_1}} R_2$$

$$I = \mathcal{E}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

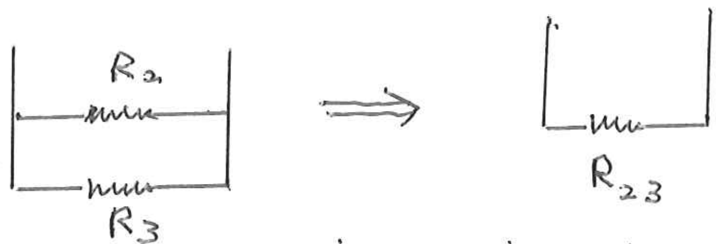
$$\frac{1}{R} \equiv \frac{1}{R_1} + \frac{1}{R_2} \quad \Longrightarrow \quad IR = \mathcal{E}$$

$\Rightarrow$ R is equivalent resistance of resistors in parallel

mixed circuits :

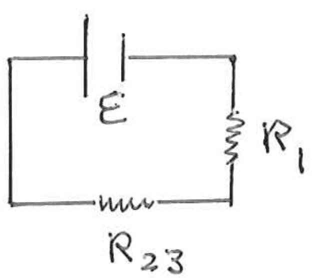


First consider

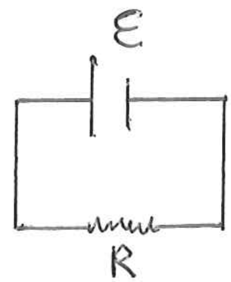


$$\frac{1}{R_{23}} = \frac{1}{R_2} + \frac{1}{R_3}$$

We are left with



$\Longrightarrow$

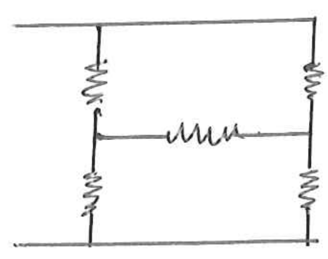


$$R = R_1 + R_{23}$$

$$= R_1 + \frac{R_2 R_3}{R_2 + R_3}$$

Kirchoff's rules

Some circuits can't be collapsed into parallel or series resistors $\Rightarrow$



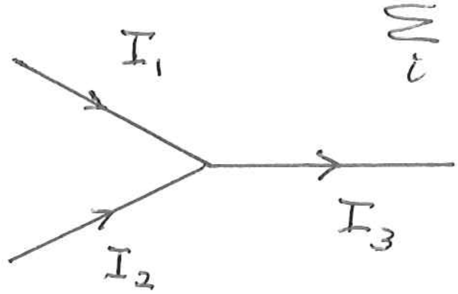
$\Rightarrow$ need general rules to proceed

A <u>branch</u> point is where three or more conductors come together.
A <u>loop</u> is any closed path in the circuit.

$\Longrightarrow$ have two rules to address complex circuits

① <u>point rule</u>: the sum of all the currents toward a branch point is zero

$$\sum_i I_i = 0$$



The currents toward the BP are
$I_1, I_2, -I_3$

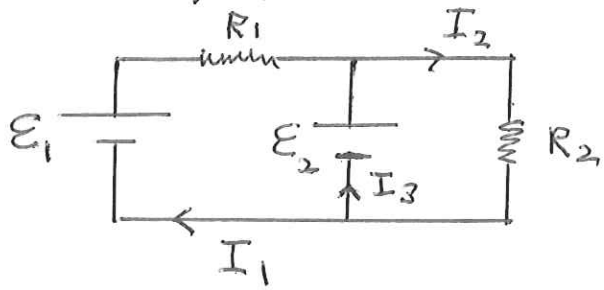
so

$$I_1 + I_2 - I_3 = 0$$

② The sum of the potential differences around any loop must equal zero.

$\Longrightarrow$ must include EMFs

<u>example</u>:



$\mathcal{E}_1 = 13\,V$, $\mathcal{E}_2 = 4\,V$

$R_1 = 3\,\Omega$, $R_2 = 2\,\Omega$

Choice of current directions is arbitrary point rule (top junction)

$$I_1 + I_3 - I_2 = 0$$

loop rule (left loop) clockwise

$$\mathcal{E}_1 - I_1 R_1 - \mathcal{E}_2 = 0 \quad \Rightarrow \quad I_1 = \frac{\mathcal{E}_1 - \mathcal{E}_2}{R_1}$$

$\Rightarrow$ backwards through $\mathcal{E}_2$

$$= \frac{9V}{3\Omega} = 3A$$

looprule (right loop clockwise)

$$\mathcal{E}_2 - I_2 R_2 = 0$$

$$I_2 = \frac{\mathcal{E}_2}{R_2} = \frac{4V}{2\Omega} = 2A$$

$$I_3 = I_2 - I_1 = 2A - 3A = -1A$$

$\Rightarrow$ current I_3 flows down rather than up

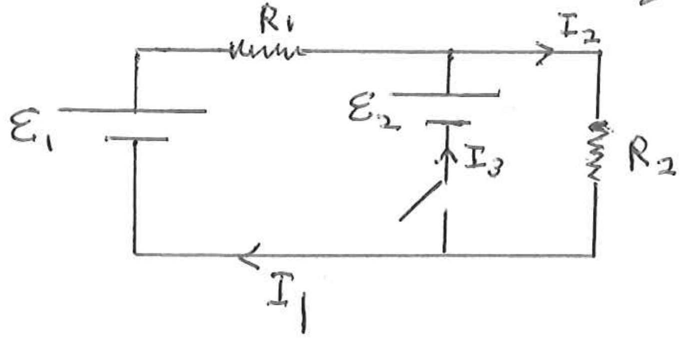
voltage drop across R_1?

$$I_1 R_1 = 9V$$

voltage across R_2

$$I_2 R_2 = 4V$$

Add a switch to the I_3 wire



$$I_3 = 0$$

$$\Rightarrow I_1 = I_2$$

$$\mathcal{E}_1 - I_1 R_1 - I_2 R_2 = 0$$

$$I_1 = I_2 = \frac{\mathcal{E}_1}{R_1 + R_2} = \frac{13}{5} A$$

What is potential drop across the switch

$\Rightarrow \Delta V$

left loop counter clockwise

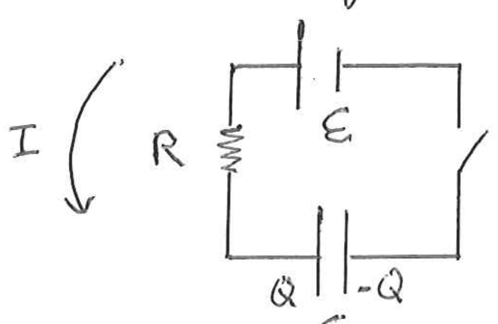
$$\Delta V + \mathcal{E}_2 + I_1 R_1 - \mathcal{E}_1 = 0$$

$$\Delta V = \mathcal{E}_1 - \mathcal{E}_2 - I_1 R_1$$

$$= 9V - \frac{13}{5} 3V = \left(9 - \frac{39}{5}\right) V = \frac{6}{5} V$$

RC circuit

Consider a circuit with an emf $\mathcal{E}$, a resistor R and a capacitor C.



$V_c = \frac{Q}{C}$

At $t=0$ close the switch. A current will flow. Charge will build up on the capacitor until the potential drop across the capacitor equals $\mathcal{E}$. At this point the current flow stops. To show this add the potential jumps around the loop $\Rightarrow$ counter clock wise

$$\mathcal{E} - IR - \frac{Q}{C} = 0$$

The current I equals the rate of increase of Q

$$I = \frac{dQ}{dt}$$

$$\mathcal{E} - R\frac{dQ}{dt} - \frac{Q}{C} = 0$$

$$\frac{dQ}{dt} + \frac{1}{RC}Q = \frac{\mathcal{E}}{R} = \frac{\mathcal{E}C}{\tau}$$

$\Rightarrow$ RC has units of time $\Rightarrow \tau \equiv RC$

$\Rightarrow$ can see this by comparing the units of $\frac{dQ}{dt}$ and $\frac{Q}{RC}$

$\Rightarrow$ $\tau = RC$ is the time constant for the RC circuit.

$\Rightarrow$ time required to build up charge on the capacitor

$\Rightarrow$ larger resistance to current flow $\Rightarrow$ longer τ

$\Rightarrow$ larger capacitance $\Rightarrow$ longer τ

$$\frac{dQ}{dt} + \frac{1}{\tau}Q = \frac{1}{\tau}\mathcal{E}C$$

Solve for $Q(t)$ by integration

$$\frac{dQ}{dt} = \frac{1}{\tau}(\mathcal{E}C - Q)$$

$$\frac{dQ}{\mathcal{E}C - Q} = \frac{dt}{\tau}$$

$$\int_0^Q \frac{dQ'}{\mathcal{E}C - Q'} = \int_0^t \frac{dt}{\tau} = \frac{t}{\tau}$$

$$-\ln(\mathcal{E}C - Q')\Big|_0^Q = \frac{t}{\tau}$$

$$-\ln\left(\frac{\mathcal{E}C - Q}{\mathcal{E}C}\right) = \frac{t}{\tau} \Rightarrow \ln\left(\frac{\mathcal{E}C - Q}{\mathcal{E}C}\right) = -\frac{t}{\tau}$$

Take the exponential of both sides

$$\frac{\mathcal{E}C - Q}{\mathcal{E}C} = e^{-\frac{t}{\tau}}$$

$$Q = \mathcal{E}C\left(1 - e^{-t/\tau}\right)$$

$$I = \frac{dQ}{dt} = -\mathcal{E}C\frac{d}{dt}e^{-t/\tau}$$

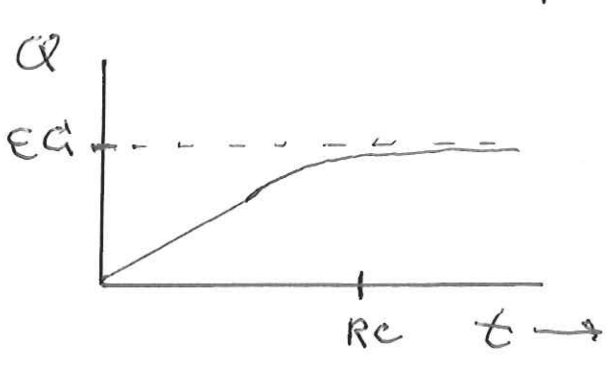
$$= \frac{\mathcal{E}C}{\tau}e^{-t/\tau}$$

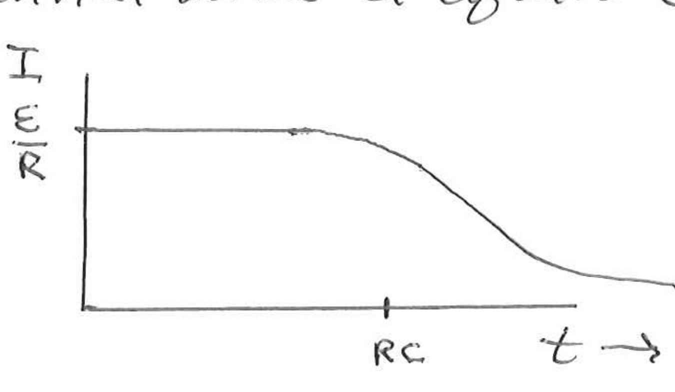
For $t \ll \tau$, $\quad Q \simeq 0$, $I = \frac{\mathcal{E}C}{\tau} = \frac{\mathcal{E}}{R}$

$\Rightarrow$ current independent of C

For $t \gg \tau$, $\quad Q = \mathcal{E}C$, $I = 0$

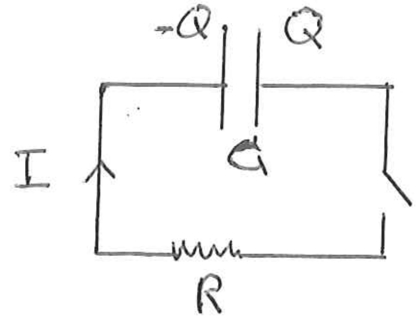
$\Rightarrow$ potential across C equals $\mathcal{E}$





Discharging a capacitor

Suppose a capacitor C in a circuit with a resistor R has an initial charge Q_0. At $t=0$ the switch is closed.



loop rule clockwise

$$\frac{Q}{C} - IR = 0$$

$\Rightarrow$ note that $\frac{dQ}{dt} = -I$

$$\frac{dQ}{dt} + \frac{1}{\tau} Q = 0$$

$$\frac{dQ}{Q} = -\frac{dt}{\tau} \Rightarrow \int_{Q_0}^{Q} \frac{dQ'}{Q'} = -\int_0^t \frac{dt}{\tau} = -\frac{t}{\tau}$$

$$\ln\left(\frac{Q}{Q_0}\right) = -\frac{t}{\tau}$$

$$Q = Q_0 e^{-t/\tau}$$

$\Rightarrow$ charge decays with time

$\Rightarrow$ energy stored in C $\left(\frac{1}{2} C V_0^2 = \frac{1}{2}\frac{Q_0^2}{C}\right)$ is dissipated in the resistor

dissipation rate

$$P = I^2 R, \quad \text{with } I = -\frac{dQ}{dt} = \frac{Q_0}{\tau} e^{-t/\tau}$$

$$P = \frac{Q_0^2}{\tau^2} R e^{-\frac{2t}{\tau}}$$

W_{diss} = total power dissipated

$$W_{diss} = \int_0^\infty dt\, P = \frac{Q_0^2}{\tau^2} R \int_0^\infty dt\, e^{-\frac{2t}{\tau}}$$

$$= \frac{Q_0^2}{\tau^2} R \left. \frac{e^{-\frac{2t}{\tau}}}{-\frac{2}{\tau}} \right|_0^\infty$$

$$= \frac{Q_0^2 R}{2\tau} = \frac{Q_0^2}{2C}$$

= initial energy stored in C