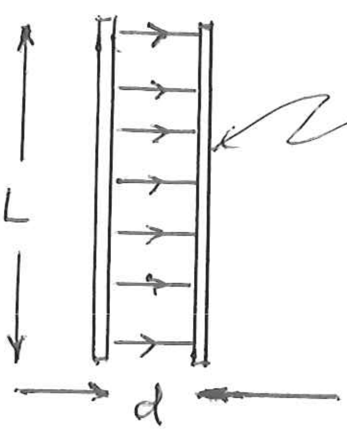
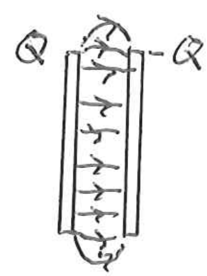
Capacitors and dielectrics

A capacitor is a device consisting of two conductors separated by an insulating material. A capacitor can store charge and therefore the energy associated with that charge.

Parallel plate capacitor

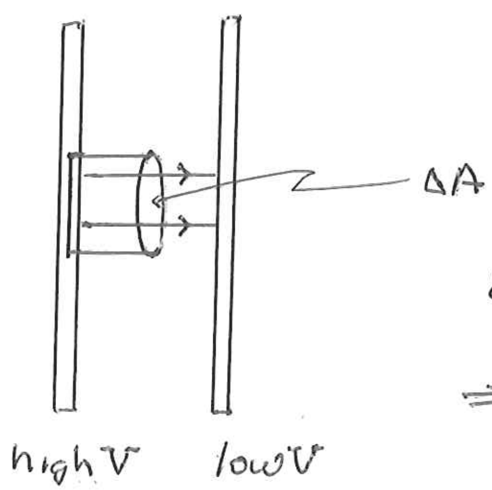


Assume that $d \ll L$
$\Rightarrow$ can calculate E by assuming that the plates are effectively of infinite extent
$\Rightarrow$ fringe field neglected



$$\sigma = \frac{Q}{A} = \frac{\text{charge}}{\text{area}}$$

Use Gauss' Law to find E:



Choose Gaussian surface with one side in conductor and other between the plates.

$$\oint \underline{E} \cdot d\underline{A} = E \Delta A = \frac{\sigma \Delta A}{\epsilon_0}$$

$\Rightarrow$ no E flux through sides or in conductor

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$$

Potential drop:

$$\Delta V = Ed = \frac{Qd}{A\epsilon_0}$$

Define "capacitance" as the ability to store charge

$$C \equiv \frac{Q}{\Delta V} = \frac{Q A \epsilon_0}{Q d} = \frac{A\epsilon_0}{d}$$

⇒ C is a function of the geometry of the capacitor and not the charge

units: C is measured in Farads

$$\text{Farad} = \frac{C}{J/C} = \frac{C^2}{J}$$

A Farad is an unrealistically large capacitance

⇒ Take $d = 1\text{cm}$ and $C = 1$ Farad

$$A = \frac{1 \text{ farad } d}{\epsilon_0} = \frac{1 \text{ farad } 10^{-2}\text{m} \; \text{Nm}^2}{8.85 \times 10^{-12} \; C^2}$$

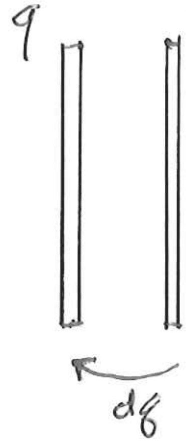
$$\approx 10^9 \frac{\cancel{C^2}}{\cancel{Nm}} \frac{\cancel{m}\,\cancel{N}m^2}{\cancel{C^2}} = 10^9 m^2$$

More typically measure capacitance in

micro farads $= 10^{-6}$ farad $= 1\mu F$

pico farads $= 10^{-12}$ farad $= 1pF$

Energy stored in a capacitor



Consider a capacitor which at some time has a charge q and a voltage

$$v = \frac{qd}{\epsilon_0 A} = \frac{q}{C}$$

Move a small charge dq from $-q$ to q. The work required is

$$dw = v\,dq = \frac{1}{C}\,q\,dq$$

To find the total work required to assemble the charge from $q=0$ to $q=Q$, integrate dW

$$W = \int dw = \int_0^Q \frac{q\,dq}{C} = \left.\frac{q^2}{2C}\right|_0^Q = \frac{Q^2}{2C}$$

$\Rightarrow$ energy stored in capacitor

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

example:

Consider a capacitor with $d = 1\,mm$ and $A = 1\,cm^2$ charged to 100 volts. How much energy is stored?

$$C = \frac{A\epsilon_0}{d} = 8.85\times10^{-12}\,\frac{C^2}{Nm^2}\,\frac{10^{-4}\,m^2}{10^{-3}\,m} = 8.85\times10^{-13}\,F$$

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\,8.85\times10^{-13}\,\frac{C^2}{Nm}\,10^4\,\frac{J^2}{C^2}$$

$$= 4.4\times10^{-9}\,J$$

Energy in the electric field

The energy stored in the capacitor to infer the energy in an electric field

$$\mathcal{U} = \frac{1}{2} C V^2 = \frac{1}{2} \frac{A\epsilon_0}{d} E^2 d^2$$

$$= \frac{1}{2} \epsilon_0 (Ad) E^2 = \underbrace{(Ad)}_{\text{volume}} \frac{1}{2} \epsilon_0 E^2$$

$$u = \frac{\mathcal{U}}{\text{Vol}} = \frac{1}{2} \epsilon_0 E^2$$

= energy density of the local electric field

$\Rightarrow$ this is a generic result

Calculating capacitance

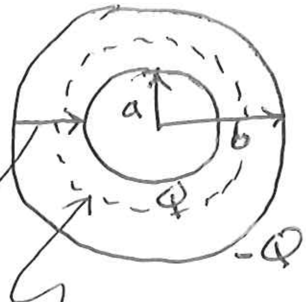
Calculate the electric field

$\Rightarrow$ Gauss' Law or another approach

$\Rightarrow$ Calculate the potential difference between conductors or between conductor and infinity

$$V_2 - V_1 = - \int_{\underset{\sim}{x}_1}^{\underset{\sim}{x}_2} \underset{\sim}{E} \cdot d\underset{\sim}{x}$$

<u>example</u>: Cylindrical capacitor
Length L
inner conductor ~~ab~~ radius a
outer conductor radius b



Gaussian surface

path integral

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \frac{Q}{\epsilon_0}$$

$\underset{\sim}{E}$, $d\underset{\sim}{A}$ are radial
end caps drop out since
$\underset{\sim}{E} \cdot d\underset{\sim}{A} \approx 0$ there

$$E\, 2\pi r L = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{2\pi r L \epsilon_0}$$

$$V = V_a - V_b = -\int_b^a E\,dr = -\frac{Q}{2\pi L \epsilon_0} \int_b^a \frac{dr}{r}$$

$$= -\frac{Q}{2\pi L \epsilon_0} \ln\left(\frac{a}{b}\right)$$

$$V = \frac{Q}{2\pi L \epsilon_0} \ln\left(\frac{b}{a}\right) \qquad \text{note: } \ln\left(\frac{b}{a}\right) = -\ln\left(\frac{a}{b}\right)$$

$$C = \frac{Q}{V} = \frac{2\pi L \epsilon_0}{\ln(b/a)}$$

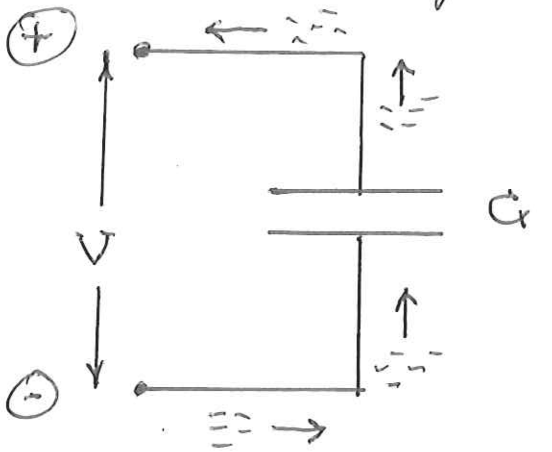
<u>example</u>: capacitance of an isolated sphere

$$V = \frac{Q}{4\pi\epsilon_0 R} \Rightarrow C = 4\pi\epsilon_0 R$$

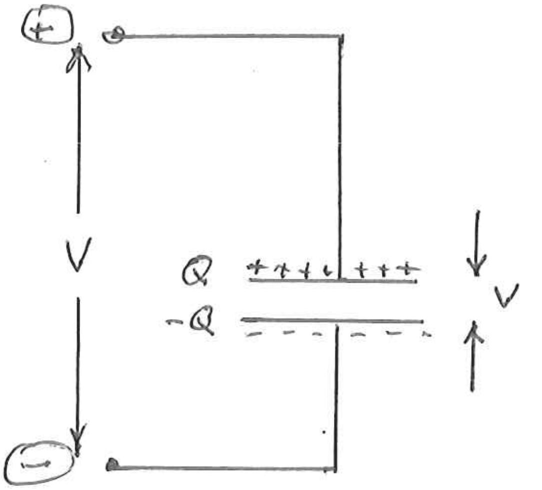
$\Rightarrow$ negative plate at ∞.

Charging a capacitor

Apply a potential V across wires connected to the capacitor plates (battery, ...)

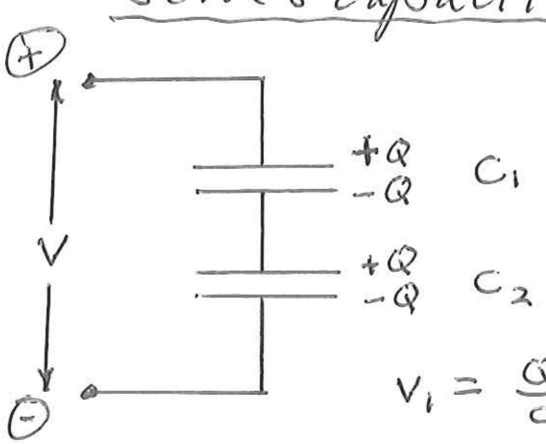


Electrons in the upper plate flow toward the ⊕.
Electron flow away from the ⊖ to the lower plate
⇒ positive charge left on upper plate and negative on lower plate
⇒ charge flows until potential across the capacitor matches the imposed potential



$$\Rightarrow \quad Q = C V$$

Series capacitors



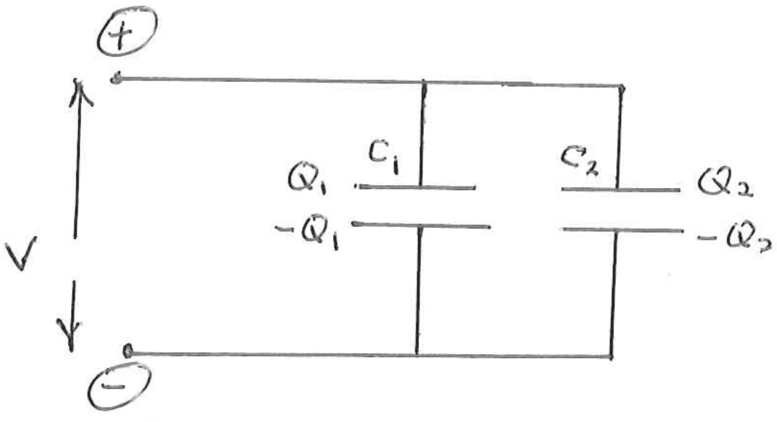
Two capacitors in series
⇒ charge on top of C_2 and bottom of C_1 are equal in magnitude but opposite sign ⇒ charge conservation

$$V_1 = \frac{Q}{C_1}, \quad V_2 = \frac{Q}{C_2}$$

$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

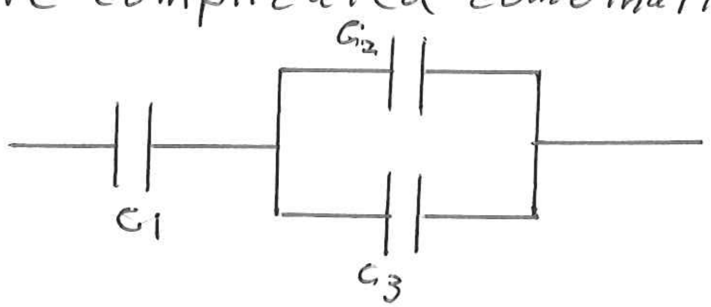
$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

Capacitors in parallel



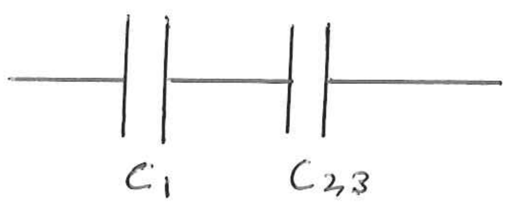
$$Q_1 = C_1 V$$
$$Q_2 = C_2 V$$
$$Q = Q_1 + Q_2 = V(C_1 + C_2)$$
$$C = C_1 + C_2$$

More complicated combinations



C_2 and C_3 are in parallel $\Rightarrow$ replace by single capacitor

$$C_{23} = C_2 + C_3$$



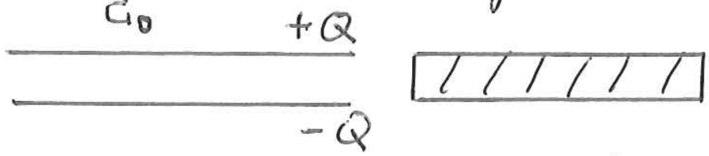
$\Rightarrow$ combine C_1 and C_{23} which are in parallel

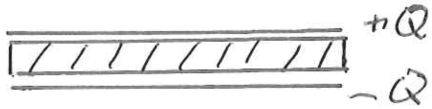
$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_{23}}$$

Polarization of dielectrics

A dielectric is a non-conduction material (insulator) that is placed between the plates of a capacitor
$\Rightarrow$ mechanically holds the plates apart

$V_0 = \frac{Q}{C_0}$





when the dielectric is inserted, the potential V across the capacitor is reduced

$$V = \frac{V_0}{K}, \quad k = \text{dielectric}$$

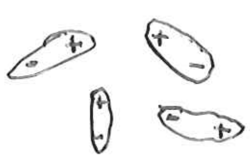
with $K > 1$

Thus, the capacitance increases

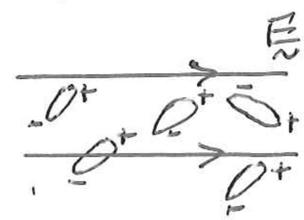
$$C = \frac{Q}{V} = \frac{Q}{V_0} k > \frac{Q}{V_0} = C_0$$

$$C = K C_0 \quad \Rightarrow \text{why?}$$

① polar materials have molecules in which the charge is not evenly distributed



molecules align
$\Longrightarrow$



$\Rightarrow$ positive portions shift in direction of $\underset{\sim}{E}$ and negative portions in the opposite direction

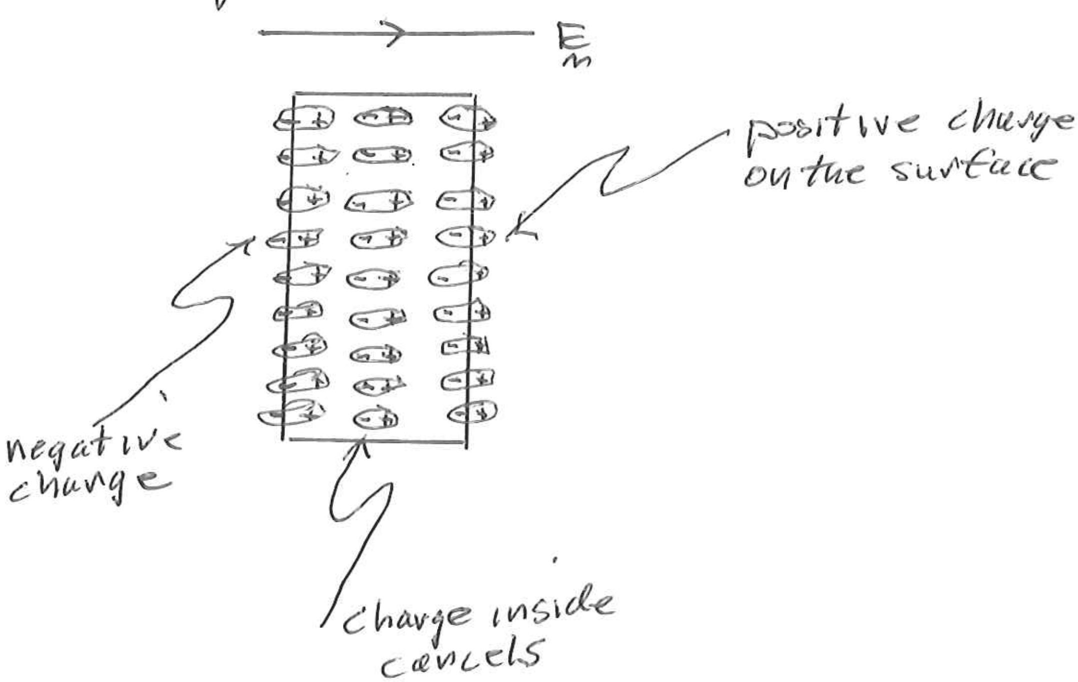
$\Rightarrow$ recall motion of a dipole in $\underset{\sim}{E}$.

(2) nonpolar molecules have an even charge distribution

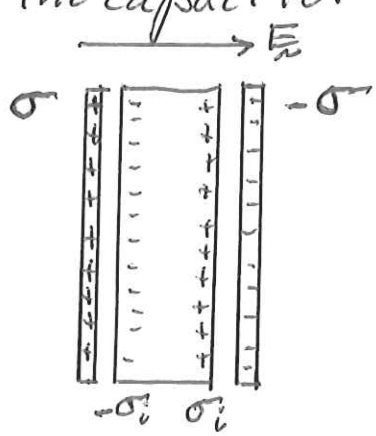
electrons ~~B~~ shift in $-\underset{\sim}{E_m}$ direction
plus changes shift along $\underset{\sim}{E_m}$

$\Rightarrow$ the dielectric is polarized by $\underset{\sim}{E_m}$

The response of the dielectric



In the capacitor



σ_i is the induced charge on the surface of the dielectric

$\Rightarrow$ total $\dfrac{\text{charge}}{\text{area}} = \sigma - \sigma_i$

$$E = \frac{\sigma - \sigma_i}{\epsilon_0}, \quad E_0 = \frac{\sigma}{\epsilon_0}$$

$$\frac{E}{E_0} = \frac{Ed}{E_0 d} = \frac{V}{V_0} = \frac{1}{k}$$

$$\frac{E}{E_0} = \frac{\sigma - \sigma_i'}{\epsilon_0} \frac{1}{\sigma/\epsilon_0} = \frac{\sigma - \sigma_i'}{\sigma} = \frac{1}{k}$$

$$\sigma - \sigma_i' = \frac{\sigma}{k} \Longrightarrow \sigma_i' = \sigma\left(1 - \frac{1}{k}\right)$$

Can also define a permittivity of the dielectric

$$E = \frac{\sigma - \sigma_i'}{\epsilon_0} = \frac{\sigma}{k\epsilon_0} \equiv \frac{\sigma}{\epsilon}$$

$$\epsilon = k\epsilon_0 > \epsilon_0 \quad \text{with } C = \frac{\epsilon A}{d}$$

ϵ = permittivity

Dielectric constants of some materials

	k	dielectric strength
air	1.0006	3×10^6 V/m
oil	4	12×10^6 V/m
glass	5	14×10^6 V/m
water (liquid)	80	

The "dielectric strength" is the electric field above which the dielectric breaks down and a current flows.

Energy density with dielectrics

$U = \frac{1}{2} Q V$ $\Rightarrow$ where V is reduced compared with no dielectric

Since $Q = VC$

$$U = \frac{1}{2} C V^2 = \frac{1}{2} \frac{\epsilon A}{d} E^2 d^2 = \frac{1}{2} \epsilon E^2 (Ad)$$

$u = \frac{1}{2} \epsilon E^2$ = energy density with a diedectric

$\Rightarrow$ for a given E, it is greater than in vacuum

Gauss' Law with dielectrics

Gauss' Law is given by

$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \frac{q}{\epsilon_0}$ where is the total charge within the Gaussian surface

$$q = q_f + q_i$$

q_f = free charge

q_i = induced charge

A charge in a uniform dielectric is shielded by the induced charge

$$q_f + q_i = \frac{q_f}{K}$$

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \frac{q_f}{K\epsilon_0} = \frac{q_f}{\epsilon}$$

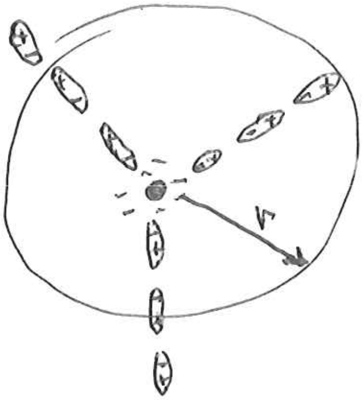
If the dielectric is not uniform, then must include ϵ in the calculation of the electric flux

$$\oint \epsilon \underset{\sim}{E} \cdot d\underset{\sim}{A} = q_f$$

$\underset{\sim}{D} = \epsilon \underset{\sim}{E}$ = electric displacement

$\Rightarrow$ evaluate ϵ at the Gaussian surface

<u>example</u>: Charge Q in a uniform dielectric



The charge Q is shielded by the dielectric

$$\oint \epsilon \underset{\sim}{E} \cdot d\underset{\sim}{A} = Q$$

$$\epsilon_0 K E 4\pi r^2 = Q$$

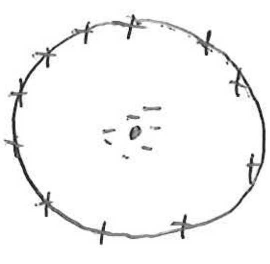
$$E = \frac{Q}{K} \frac{1}{4\pi\epsilon_0 r^2}$$

$\Rightarrow$ reduced E

What if the dielectric is a sphere of radius R with Q at the center?

$\Rightarrow$ for $r < R$ E is unchanged from above

For $r > R$

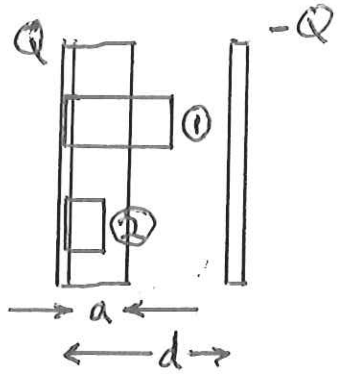


$$\oint \epsilon E \cdot d\underset{\sim}{A} = \epsilon_0 E 4\pi r^2 = Q$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \Rightarrow \text{same as vacuum}$$

$\Rightarrow$ positive induced charge on surface at $r = R$

example: Capacitor with dielectric



The dielectric has a width a that is smaller than the plate separation d.

First use Gauss' Law in ① to find $E = E_1$ in vacuum

$$\oint_{①} \epsilon \, \underset{\sim}{E} \cdot d\underset{\sim}{A} = \epsilon_0 E_1 A = Q$$

$$E_1 = \frac{Q}{\epsilon_0 A}$$

$\Longrightarrow$ dielectric has no effect. Why?

Now use Gauss' Law in ② to find $E = E_2$ inside the dielectric

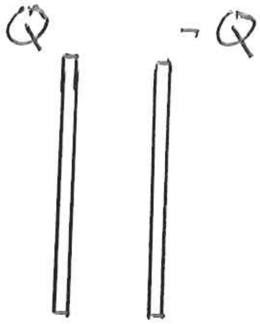
$$\oint_{②} \epsilon \, \underset{\sim}{E} \cdot d\underset{\sim}{A} = \epsilon E_2 A = Q$$

$$E_2 = \frac{Q}{\epsilon A} = \frac{Q}{K \epsilon_0 A}$$

Total potential

$$V = E_2 a + E_1 (d-a) = \frac{Q}{A\epsilon_0}\left[\frac{a}{K} + d - a\right]$$

Forces of capacitor plates



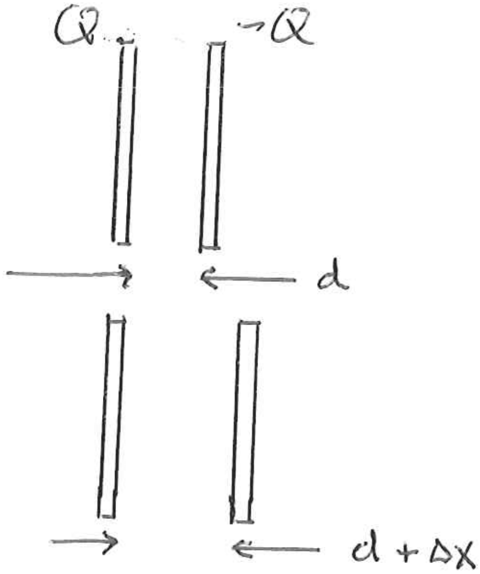
Can calculate the force on the plates (attractive) in two ways:

(i) Direct evaluation

$$F = qE$$

(2) Find the change in energy due to a small virtual displacement of a plate and equate to the work done

⇒ Do (2) first



$$U_i = \frac{1}{2}\epsilon_0 E^2 (Ad)$$

= initial energy

$$U_f = \frac{1}{2}\epsilon_0 E^2 (d + \Delta x) A$$

⇒ note E is unchanged if Q is fixed

$$\text{work done} = F\Delta x$$

$$= W_f - W_i = \frac{1}{2}\epsilon_0 E^2 \Delta x A$$

$$F = \frac{1}{2}\epsilon_0 E^2 A$$

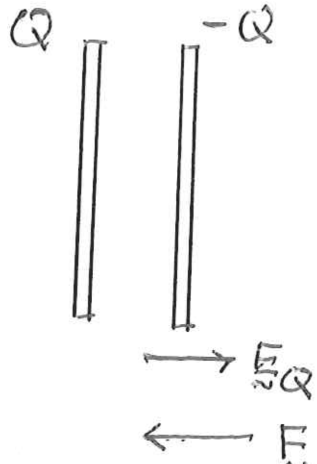
⇒ note the force between the plates is attractive

⇒ rewrite F in terms of E and Q

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$$

$$F = \frac{1}{2}\epsilon_0 E A\left(\frac{Q}{A\epsilon_0}\right) = \frac{1}{2} Q E$$

Why is there a factor of $\frac{1}{2}$?



$\Rightarrow$ a charge can not accelerate itself. E comes from both Q and $-Q$.. What is E in the location of $-Q$ due only to Q ?

$E_Q = \frac{\sigma}{2\epsilon_0}$ $\Rightarrow$ E from a sheet of charge σ

$$F = E_Q(-Q) = \frac{\sigma}{2\epsilon_0} Q \quad \text{(attractive)}$$

$$= \frac{1}{2}\epsilon_0 E^2 A$$

$\Longrightarrow$ same as from energy argument.