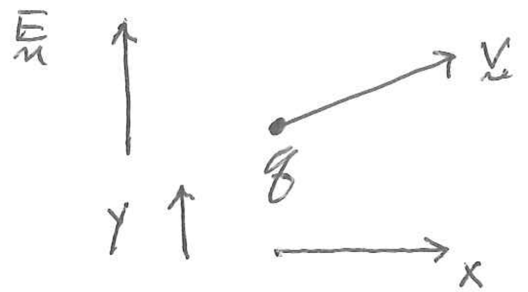
Energy and electric potential

Consider a charge q moving in a uniform electric field. If q is positive, the charge is pushed in the direction of $\underset{\sim}{E}$ with a constant force



$\Rightarrow$ similar to the motion in a gravitational field

$$qE \longleftrightarrow mg$$

Can define a potential function U similar to the gravitational potential mgy.

To show this, write the equation of motion

$$m \frac{d}{dt} \underset{\sim}{v} = q \underset{\sim}{E} \quad \text{with } E \text{ along } y.$$

To turn this into an energy equation, take the dot product with $\underset{\sim}{v}$

$$m\, \underset{\sim}{v} \cdot \frac{d}{dt} \underset{\sim}{v} = q\, \underset{\sim}{E} \cdot \underset{\sim}{v} = q E v_y$$

$$\parallel$$

$$m\left(v_x \frac{d}{dt} v_x + v_y \frac{d}{dt} v_y + v_z \frac{d}{dt} v_z \right)$$

$$v_x \frac{dv_x}{dt} = \frac{d}{dt}\left(\frac{1}{2} v_x^2\right), \; \ldots$$

$$m\underset{\sim}{v}\cdot\frac{d}{dt}\underset{\sim}{v} = \frac{m}{2}\frac{d}{dt}\left(v_x^2+v_y^2+v_z^2\right) = \frac{m}{2}\frac{d}{dt}v^2$$

so

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = qE\frac{dy}{dt} = \frac{d}{dt}\left(qEy\right)$$

or

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 - qEy\right) = 0$$

$\mathcal{U} \equiv -qEy$ = potential energy

$\frac{1}{2}mv^2$ = kinetic energy

$\frac{d}{dt}\left(K+\mathcal{U}\right) = 0 \Rightarrow$ energy is conserved

energy $= K + \mathcal{U}$

Can also define the potential energy per unit charge

$$V \equiv \frac{\mathcal{U}}{q} = -Ey$$

energy $= K + qV$ = const.

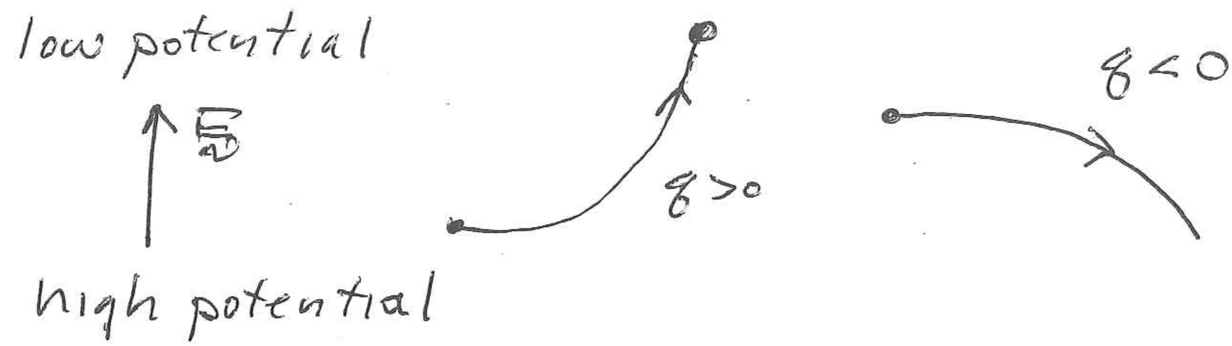
units of V: volt $= \frac{\text{joule}}{C}$

The advantage of using the potential to calculate the motion of charged particles is that the energy is a scalar

$\Rightarrow$ don't worry about vector directions

Particle motion in a potential

The potential goes down in the direction of $\underset{\sim}{E}$



A positive charge moves toward lower potential

A negative charge moves toward higher potential.

example: An electron is accelerated through a potential of 10^3 V. What is its velocity

$$K_i + qV_i = K_f + qV_f$$

Let $K_i = 0$

$$K_f = q(V_i - V_f) = +e\underbrace{(V_f - V_i)}_{\Delta V > 0}$$

$$\frac{1}{2}mv^2 = e\,\Delta V$$

$$v^2 = 2\,\frac{1.6\times10^{-19}\,C}{9.11\times10^{-31}\,Kg}\,10^3\,\frac{\cancel{Joule}^{\,Kg\,m^2/s^2}}{\cancel{C}}$$

$$= 3.5\times10^{14}\,\frac{m^2}{s^2}$$

$$v = 1.9\times10^{7}\,\frac{m}{s}$$

Potential energy of point charges

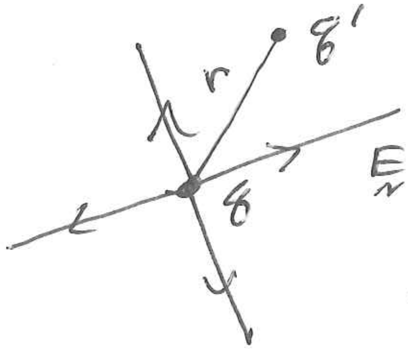
Consider a charge $q > 0$. Move a charge $q' > 0$ towards q. Have to do work on q' to move it toward q since the two charges repell each other.

We can also describe this interaction in terms of potential energy

$\Rightarrow$ a positive charge q produces a positive potential V

Calculate the motion of q' in an electric field produced by the fixed charge q

$$\underset{\sim}{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}$$



Motion of q':

$$m \frac{d}{dt} \underset{\sim}{V} = q' \underset{\sim}{E}$$

Take dot product with $\underset{\sim}{V}$.

$$m \underset{\sim}{v} \cdot \frac{d}{dt} \underset{\sim}{v} = \frac{d}{dt} \frac{1}{2} m v^2 = q' \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \cdot \underset{\sim}{V}$$

$$= \frac{qq'}{4\pi\epsilon_0} \frac{V_r}{r^2} = \frac{qq'}{4\pi\epsilon_0} \frac{1}{r^2} \frac{dr}{dt}$$

$$\frac{d}{dt} \frac{1}{r} = -\frac{1}{r^2} \frac{dr}{dt}$$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -\frac{d}{dt}\left(\frac{qq'}{4\pi\epsilon_0}\frac{1}{r}\right)$$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{qq'}{4\pi\epsilon_0 r}\right) = 0$$

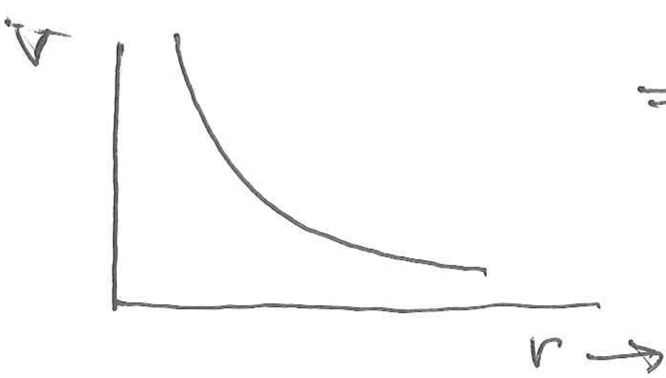
$$U = \frac{qq'}{4\pi\epsilon_0 r}$$ = potential energy of q' as it moves toward q

As with a uniform field, define the potential energy per charge

$$V = \frac{U}{q'} = \frac{q}{4\pi\epsilon_0 r}$$

V is the electric potential that is produced by and surrounds the charge q.

units: volts = joule / coulomb

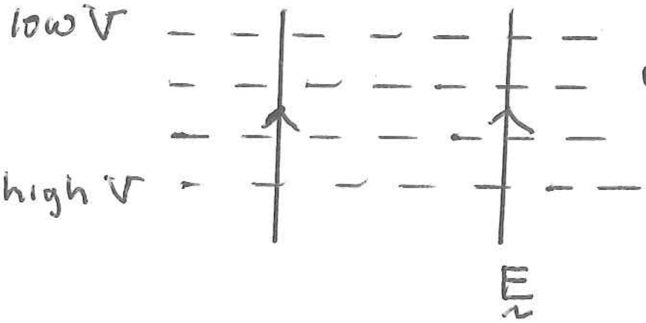


⇒ positive charges $q'>0$ are repelled from the positive potential

⇒ negative charges $q'<0$ are attracted.

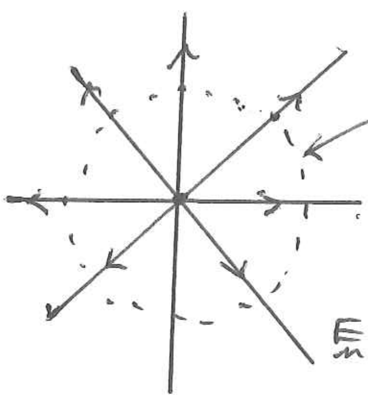
Contours of constant V and $\underset{\sim}{E}$

For a uniform E, $V = -Ey \Rightarrow$ constant values of V are horizontal lines



$\Rightarrow \underset{\sim}{E}$ is $\perp$ to lines of constant V and points down the potential

For a point charge $\Rightarrow \quad E = -\frac{\partial}{\partial y} V$



contours of constant V are shells with radius r.

$$V = \frac{q}{4\pi\epsilon_0 r}$$

$\Rightarrow \underset{\sim}{E}$ is $\perp$ to shells of constant V.

$$\Rightarrow \quad E = -\frac{\partial}{\partial r} V$$

The electric field is the negative derivative of the potential

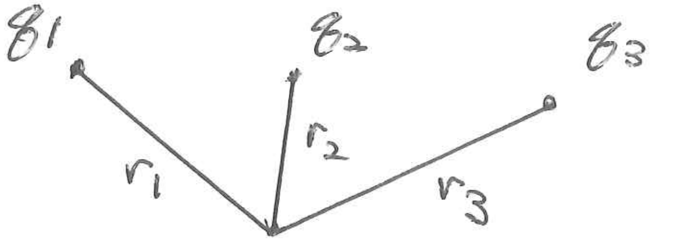
$\Rightarrow$ always true

More generally

$$\underset{\sim}{E} = -\nabla V$$

Potential with multiple charges

What is the potential when several charges are present?



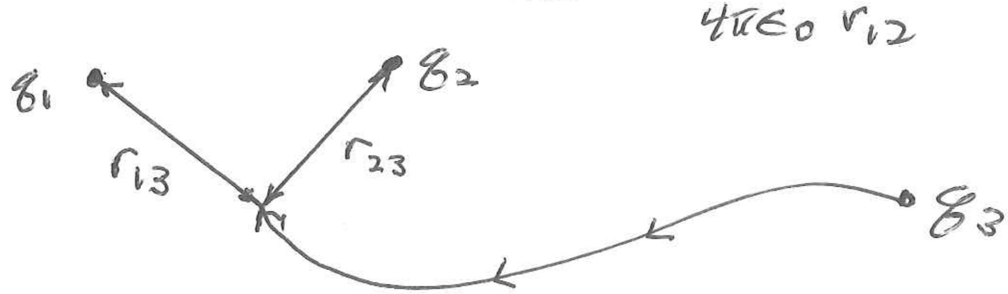
$$V = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{r_i}$$

$\Longrightarrow$ note that the potential is a scalar so can add the potentials without vector addition.

What is the energy required to assemble charge?

To move q_2 from infinity to r_{12} requires energy

$$U_{r_{12}} = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}}$$



To bring q_3 to r_{13}, r_{23} requires

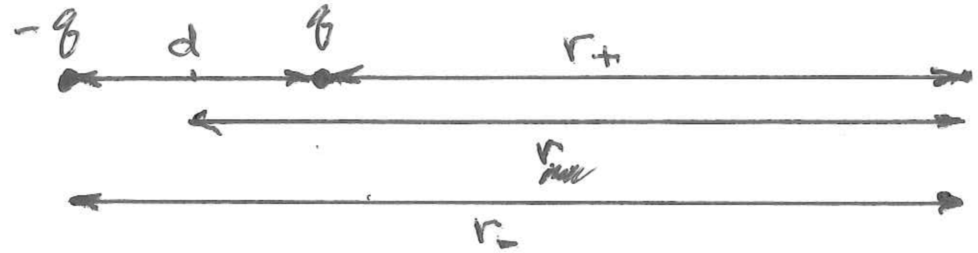
$$u_{13} = \frac{q_1 q_3}{4\pi\epsilon_0 r_{13}} \quad \text{and} \quad u_{23} = \frac{q_2 q_3}{4\pi\epsilon_0 r_{23}}$$

Total energy:

$$u = u_{12} + u_{23} + u_{13} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_3}{r_{13}}\right)$$

Potential from a ~~[illegible]~~ dipole

Positive and negative charge separated by a distance d.



$$r_+ = r - \frac{d}{2}, \quad r_- = r + \frac{d}{2}$$

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

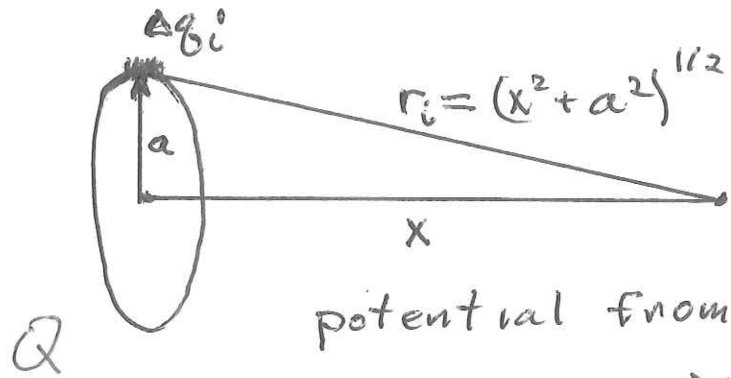
$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r - \frac{d}{2}} - \frac{1}{r + \frac{d}{2}}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{r + \frac{d}{2} - \left(r - \frac{d}{2}\right)}{r^2 - \frac{d^2}{4}}\right)$$

$$= \frac{p}{4\pi\epsilon_0} \frac{1}{r^2 - \frac{d^2}{4}} \quad \Rightarrow \quad r \gg d$$

$$V \simeq \frac{p}{4\pi\epsilon_0 r^2}$$

Potential from a charged ring



potential from Δq_i

$$\Delta V_i = \frac{\Delta q_i}{4\pi\epsilon_0 r_i}$$

$\Rightarrow$ r_i same for all charges around the ring

$$V = \sum_i \Delta V_i = \sum_i \frac{\Delta q_i}{4\pi\epsilon_0 r_i} = \frac{\sum \Delta q_i}{4\pi\epsilon_0 (x^2+a^2)^{1/2}}$$

$$= \frac{Q}{4\pi\epsilon_0 (x^2+a^2)^{1/2}}$$

$\Rightarrow$ like a point charge for $x \gg a$

Techniques for evaluating the potential

(1) Sum over the potential from individual charges

(2) Calculate the potential by integrating $\underset{\sim}{E}$

$\Rightarrow$ previously found

$$\frac{d}{dt} K = q' \underset{\sim}{E} \cdot \underset{\sim}{v}$$

$\Rightarrow$ with K the kinetic energy of charge q'

Integrate to find the change in kinetic energy as an object moves from position $\mathbf{x}_1$ to $\mathbf{x}_2$,

$$\Delta K \equiv K_2 - K_1 = \int_1^2 dK$$

$$\frac{dK}{dt} = q' \, \mathbf{E} \cdot \frac{d\mathbf{x}}{dt}$$

$$dK = q' \, \mathbf{E} \cdot d\mathbf{x}$$

$\Rightarrow$ motion along $\mathbf{E}$ causes K to change

$$K_2 - K_1 = q' \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{E} \cdot d\mathbf{x}$$

Can again write as an energy conservation law

$$K_2 - K_1 + U_2 - U_1 = 0$$

with $$U_2 - U_1 = q' (V_2 - V_1)$$

and $$V_2 - V_1 = - \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{E} \cdot d\mathbf{x}$$

where $U_2 - U_1$ is the change in potential energy of q' as it moves in $\mathbf{E}$.

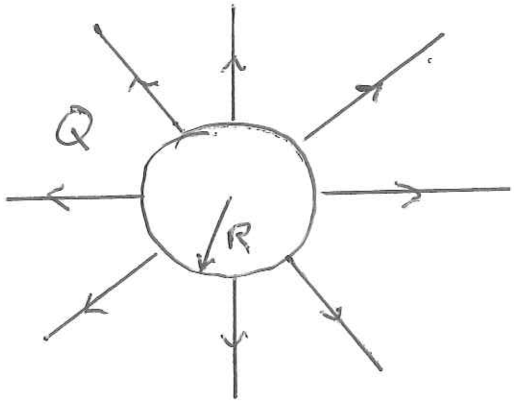
The change in potential energy of q' is its charge times the change in potential.

$\Rightarrow$ Can find the change in potential V between two locations by integrating $\underset{\sim}{E}$.

$\Rightarrow$ Note that only the difference in V is important

$\Rightarrow$ it is the change in V that produces a change in kinetic energy.

Example: Potential around a conducting sphere with charge Q.



$r < R \quad \underset{\sim}{E} = 0$

$r > R \quad \underset{\sim}{E} = \dfrac{Q}{4\pi\epsilon_0 r^2}\,\hat{r}$

Take $V = 0$ at $r = \infty$

$$V(r) = -\int_\infty^r \underset{\sim}{E} \cdot d\underset{\sim}{r}$$

$$= -\int_\infty^r \frac{Q}{4\pi\epsilon_0 r^2}\,dr \quad \text{for } r > R.$$

$$= -\frac{Q}{4\pi\epsilon_0}\int_\infty^r \frac{dr}{r^2} = -\frac{Q}{4\pi\epsilon_0}\left(-\frac{1}{r}\right)\Bigg|_\infty^r$$

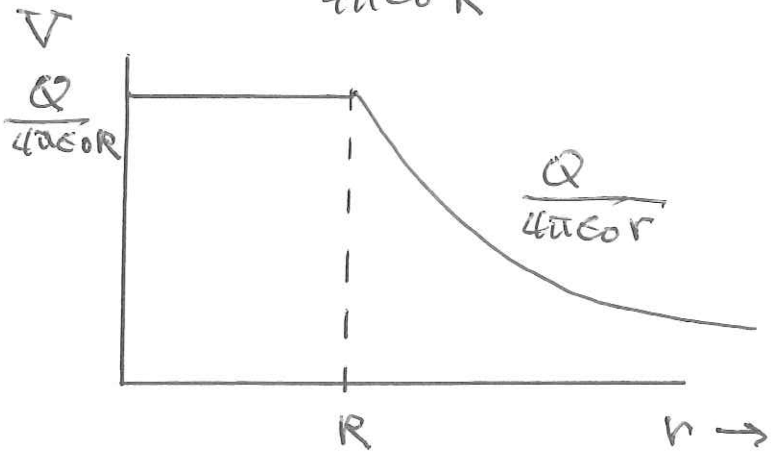
$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{r} \Rightarrow \text{same as point charge}$$

For $r < R$

$$V = -\int_{\infty}^{r} \underset{\sim}{E} \cdot d\underset{\sim}{r} = -\int_{\infty}^{R} E\,dr + \cancel{\int_{R}^{r} E\,dr}$$

since $\underset{\sim}{E} = 0$ for $r < R$

$$= \frac{Q}{4\pi\varepsilon_0 R}$$



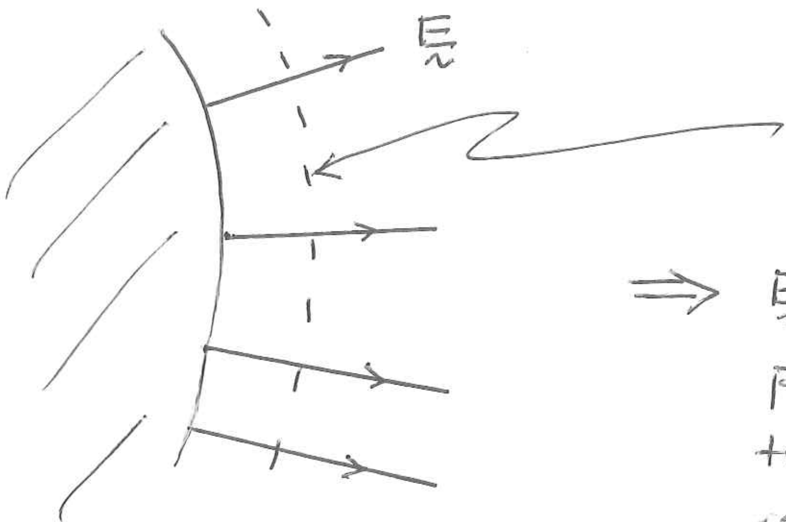
Potentials in and near conductors

The electric field in a conductor is zero. Can calculate the potential difference between any two points in a conductor

$$\Delta V = -\int_{\underset{\sim}{x}_1}^{\underset{\sim}{x}_2} \underset{\sim}{E} \cdot d\underset{\sim}{x} = 0$$

$\Rightarrow$ the potential is constant throughout a conductor.

Since $\underset{\sim}{E}$ is perpendicular to the surface just outside a conductor, the contours of constant potential must be parallel to the conductor



surface of constant potential

$\Rightarrow$ $\underset{\sim}{E}$ points down the potential and is $\perp$ to the surface of constant potential

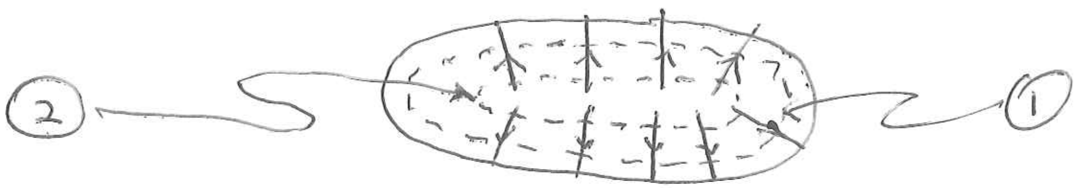
What about an open cavity in a conductor?

$\Rightarrow$ assume no charge inside

Is the potential constant in the cavity?

Just inside the cavity must have a potential surface that maps the conductor

$\Rightarrow$ surface 1



Suppose have another surface 2 inside 1 that has a different potential, greater than 1.

An electric field must point outward from ② to ①. Integrate the electric flux across a Gaussian surface

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \frac{\text{charge enclosed}}{\epsilon_0}$$

$\Rightarrow$ since no charge is inclosed,

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = 0 \Rightarrow \underset{\sim}{E} = 0$$

$\Rightarrow$ The potential in the cavity is constant.

You can shield electric fields out of a region of space by enclosing the region with a conducting material.

Properties of the electric potential

① The electric potential is the energy per charge

$$U = q' V$$

② A positive potential V surrounds a positive charge

A negative potential surrounds a negative charge.

③ The potential from a point charge q is

$$V = \frac{q}{4\pi\epsilon_0 r}$$

④ The change in potential between locations $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ is

$$\Delta V = V_2 - V_1 = -\int_{\underset{\sim}{x}_1}^{\underset{\sim}{x}_2} \underset{\sim}{E} \cdot d\underset{\sim}{x}$$

⑤ Constant potential contours are $\perp$ to the local electric field

$\Longrightarrow$ $\underset{\sim}{E}$ points down the potential

⑥ $\underset{\sim}{E}$ can be calculated from V

$$\left.\begin{aligned} E_x &= -\frac{\partial V}{\partial x} \\ E_y &= -\frac{\partial V}{\partial y} \\ E_z &= -\frac{\partial V}{\partial z} \end{aligned}\right\} \quad \underset{\sim}{E} = -\nabla V$$

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Potential differences are path independent

The potential difference between any two points ~~is~~ depends only on the two points and not on the path used to find ΔV,

$$\Delta V = V_2 - V_1 = -\int_{\underset{\sim}{x}_1}^{\underset{\sim}{x}_2} \underset{\sim}{E} \cdot d\underset{\sim}{x} \, .$$

Use the result the $\underset{\sim}{E} = -\nabla V$

$$V_2 - V_1 = +\int_{\underset{\sim}{x}_1}^{\underset{\sim}{x}_2} d\underset{\sim}{x} \cdot \nabla V = \int_{\underset{\sim}{x}_1}^{\underset{\sim}{x}_2} dV$$

$\Rightarrow$ jump in V is independent of path

Electron volt

An "electron volt" is the energy associated with moving an electron through a potential drop of one Volt.

$$1\,eV = 1.6 \times 10^{-19}\, C \;\; \frac{1 \text{ Joule}}{C}$$

$$= 1.6 \times 10^{-19} \text{ Joules}$$

$\Rightarrow$ an "eV" is a commonly used unit of energy