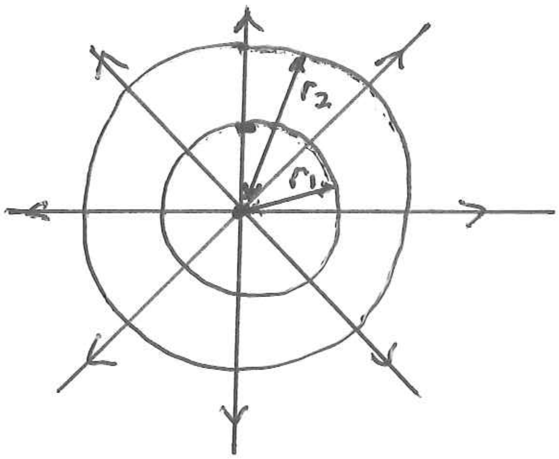
# Electric Field Flux and Gauss's Law

Consider the electric field from a single point charge $q$. The electric field points radially outward every where




The number of field lines crossing the two radii $r_1$ and $r_2$ is the same since the lines extend to $\infty$ if there is no other charge present.

$$E = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}$$

Consider a spherical shell of radius $r$.

$\Rightarrow$ shell area $= A = 4\pi r^2$

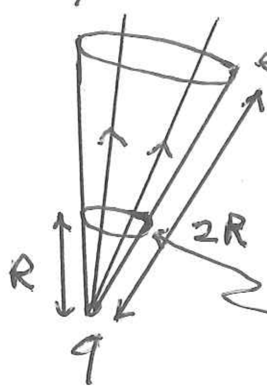
$\Rightarrow$ $EA = \dfrac{q}{\epsilon_0}$ $\Rightarrow$ independent of $r$

$\Rightarrow$ ~~only~~ EA only depends on the size of the charge

$\Rightarrow$ This expresses a conservation law for electric field lines

$\Rightarrow$ The electric field starts and ends on charges so if some number of lines crosses a radius $r$ then the same number will cross through $2r$ if there are no other charges

The same conservation of lines holds for any fraction of the surface area




$4\Delta A$, $\quad E = \dfrac{q}{4\pi\epsilon_0}\dfrac{1}{4R^2}$

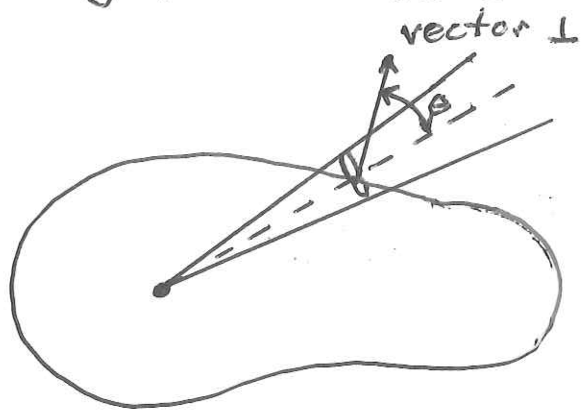
$\Delta A$, $\quad E = \dfrac{q}{4\pi\epsilon_0}\dfrac{1}{R^2}$.

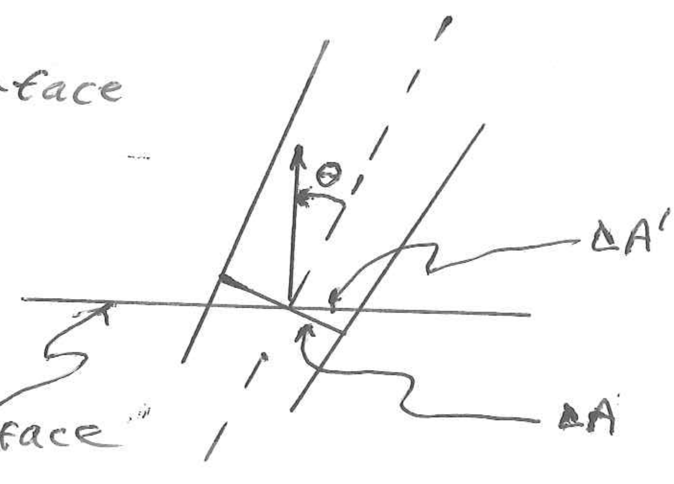
$E\Delta A$ is independent of radius

~~Thus~~ Summing all the small areas

$$\sum_{\text{all areas}} E\Delta A = \frac{q}{\epsilon_0}$$

Now surround the charge with a closed, irregular surface




$\Delta A = \Delta A' \cos\theta$

note $\Delta A \leq \Delta A'$

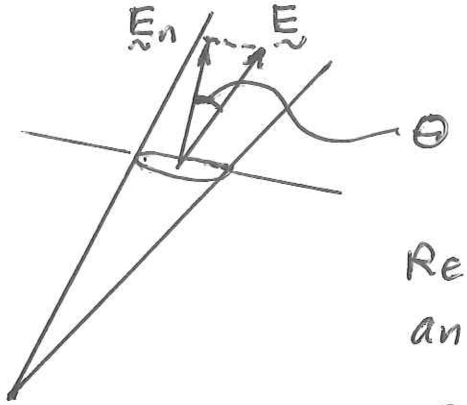
$\Delta A'$ = area of the surface

$\Delta A$ = area ⊥ to the radius from $q$.

$$\Longrightarrow \quad E\Delta A = E\Delta A' \cos\theta$$

$$\Longrightarrow \quad \sum E\Delta A = \sum E\Delta A' \cos\theta = \frac{q}{\epsilon_0}$$

$E \cos\theta = E_n$ = electric field $\perp$ to the surface $\Delta A'$




$$\sum E_n \Delta A' = \frac{q}{\epsilon_0}$$

Rewrite the discrete sum as an integral over the surface

$$\oint E_n \, dA' = \frac{q}{\epsilon_0}$$

$\oint \Rightarrow$ means an integral over a closed surface

Take any closed surface around $q$ then ~~any~~ the integral of the electric field $\perp$ to the surface times the area element is equal to the charge enclosed by the closed surface.

$\Rightarrow$ a statement that the # of electric field lines crossing the surface is conserved

$\Rightarrow$ The total "electric flux" is preserved.

Define $E_n \, dA' \equiv \underset{\sim}{E} \cdot d\underset{\sim}{A}'$

where the vector $d\underset{\sim}{A}'$ points outward and $\perp$ to the local surface $dA'$ and has a magnitude $dA'$.

$$\boxed{\oint \underset{\sim}{E} \cdot d\underset{\sim}{A}' = \frac{q}{\epsilon_0}}$$ Gauss's Law

$\Rightarrow$ closed surface and $d\underset{\sim}{A}'$ is outward and $\perp$ to the surface.

Can extend this to multiple changes, e.g. $q_1, q_2$

$$\oint \underset{\sim}{E}_1 \cdot d\underset{\sim}{A} = \frac{q_1}{\epsilon_0}$$

$$\oint \underset{\sim}{E}_2 \cdot d\underset{\sim}{A} = \frac{q_2}{\epsilon_0}$$

$$\Rightarrow \oint \underbrace{(\underset{\sim}{E}_1 + \underset{\sim}{E}_2)}_{\underset{\sim}{E}} \cdot d\underset{\sim}{A} = \frac{q_1 + q_2}{\epsilon_0}$$

$$\Rightarrow \oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \frac{q}{\epsilon_0}$$

$\Rightarrow$ $q$ is the total change enclosed by the surface

$\Rightarrow$ Gauss's Law is fully equivalent o Coulomb's law

$\Rightarrow$ It is always valid but only useful when $\underset{\sim}{E} \cdot d\underset{\sim}{A}$ is constant on a surface.

~~example: charged conducting sphere~~

Electric fields in conductors

The change in a conductor (electrons) moves in response to a electric field. The charge moves to shield the electric field out of the conductor. The change only stops moving when $\underset{\sim}{E} = 0$ inside the conductor.

$\Rightarrow$ for a time independent system $\underset{\sim}{E} = 0$ inside a conductor

$\Rightarrow$ our present focus is on "electrostatics" in which everything is time independent

$\Rightarrow$ since $\underset{\sim}{E} = 0$ inside a conductor, there can be no net charge in a conductor $\Rightarrow$ from Gauss's Law

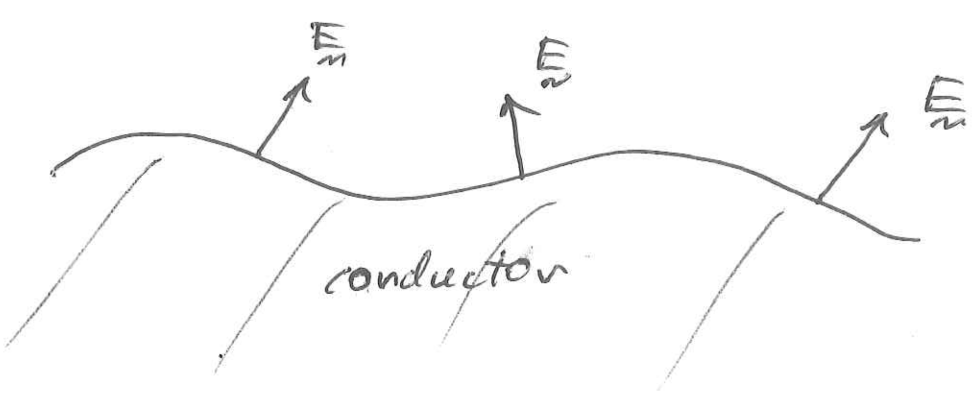
$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = 0$ inside any closed surface in a conductor so $q$ inside that surface is zero.

$\Rightarrow$ All net charge on a conductor must lie on the surface

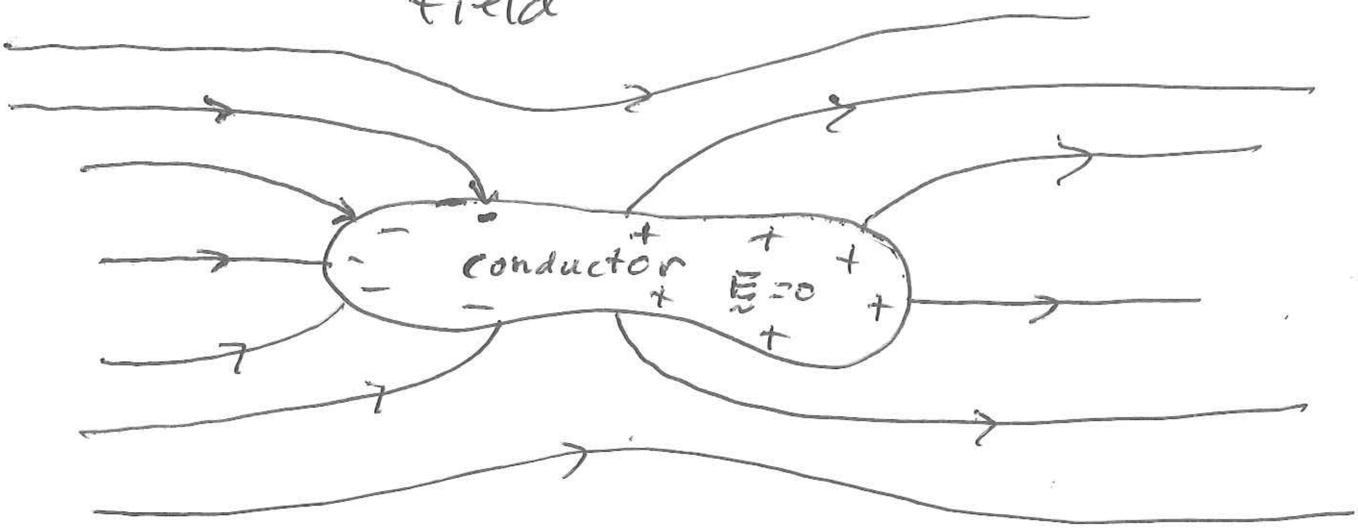
At the surface of a conductor $\underset{\sim}{E}$ is $\perp$ to the surface

$\Rightarrow$ any $E_{\parallel}$ at the surface would cause charge to flow along the surface

$\Rightarrow$ we will show later in the class that if $E_{\parallel}$ is zero at the surface, it is also zero just outside of the conducting surface

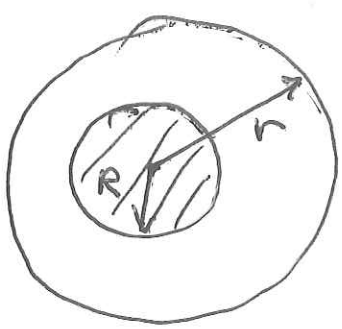


<u>example</u> conductor in an external electric field




$\Rightarrow$ electrons move in the conductor to shield out the electric field

<u>example</u>: Charged conducting sphere (charge $Q$, radius $R$)




$\Rightarrow$ $\underset{\sim}{E} = 0$ inside the conductor

$\Rightarrow$ charge lies on the surface

$\Rightarrow$ wants to spread out

$\Rightarrow$ by symmetry the electric field is radial

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \int E\, dA = E \int dA = E\, 4\pi r^2 = \frac{Q}{\epsilon_0}$$

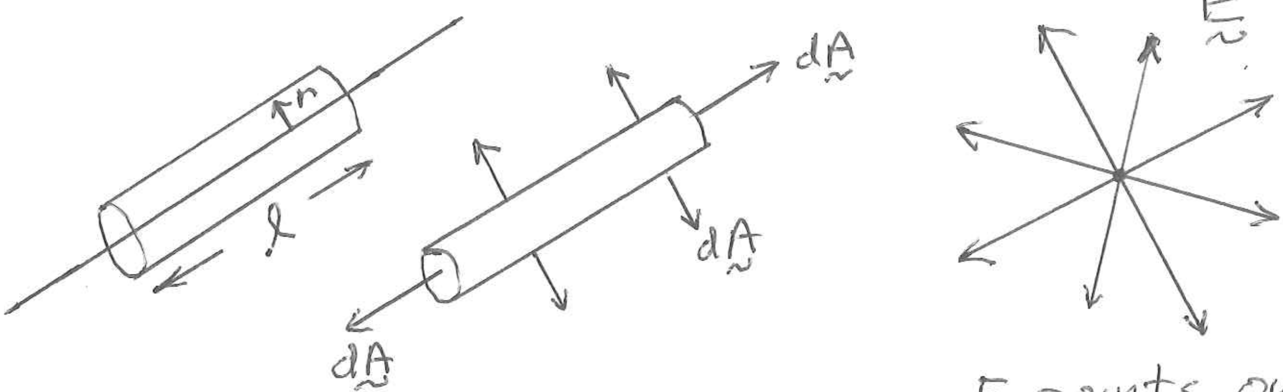
$$E = \frac{Q}{4\pi \epsilon_0 r^2}$$

$\Rightarrow$ same as if the charge were at the center of the sphere

<u>example</u> : $\underset{\sim}{E}$ from line charge

$\lambda$ = charge / length

Consider a cylindrical closed surface




$\underset{\sim}{E}$ points outward from the line charge

$\underset{\sim}{E}\cdot d\underset{\sim}{A} \neq 0$ at sides of the cylinder

$\underset{\sim}{E}\cdot d\underset{\sim}{A} = 0$ at the ends

Gauss's Law :
$$\oint \underset{\sim}{E}\cdot d\underset{\sim}{A} = \overbrace{\cancel{\int_{\text{ends}} \underset{\sim}{E}\cdot d\underset{\sim}{A}}}^{0} + \int_{\text{sides}} \underset{\sim}{E}\cdot d\underset{\sim}{A}$$

$$= E\,2\pi r \ell$$

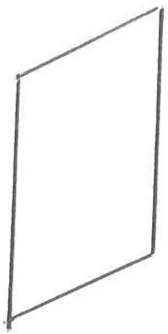
charge enclosed = $\lambda \ell$

$$E\,2\pi r \ell = \lambda \ell \frac{1}{\epsilon_0}$$

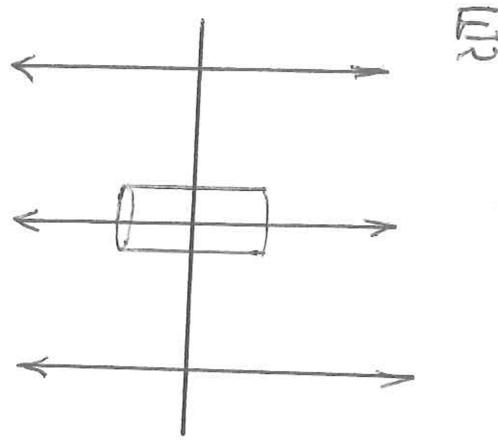
$$E = \frac{\lambda}{2\pi r \epsilon_0} \Rightarrow \text{as before}$$

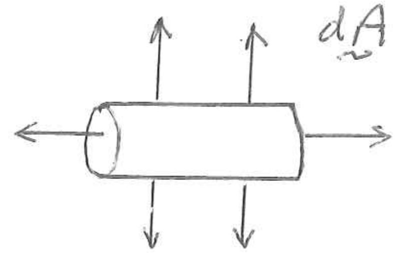
<u>example</u>: charged sheet

$\sigma$ = charge / area



infinite charged sheet




choose closed surface

$\mathbf{E} \cdot d\mathbf{A} = 0$ on side of surface

$\mathbf{E} \cdot d\mathbf{A} \neq 0$ on ends

$$\oint \mathbf{E} \cdot d\mathbf{A} = \int_{\text{ends}} \mathbf{E} \cdot d\mathbf{A} + \cancelto{0}{\int_{\text{side}} \mathbf{E} \cdot d\mathbf{A}}$$

$$= 2EA = \frac{q}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

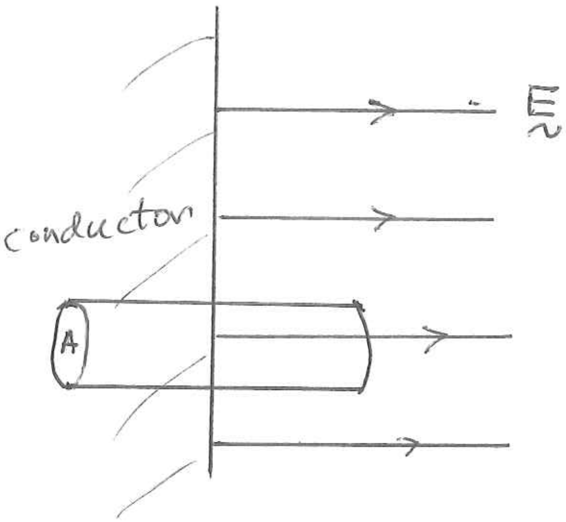
A = area of end

from two ends

$$E = \frac{\sigma}{2\epsilon_0}$$

$\Rightarrow$ note that $\mathbf{E}$ does not fall off with distance from the sheet example : conducting surface

$\sigma$ = charge/area




$$\oint \underset{\sim}{E}\cdot d\underset{\sim}{A} = \overset{0}{\cancel{\int_{\text{side}} \underset{\sim}{E}\cdot d\underset{\sim}{A}}} + \overset{0}{\cancel{\int_{\substack{\text{end}\\\text{conductor}}} \underset{\sim}{E}\cdot dA}}$$

$$+ \int_{\substack{\text{end}\\\text{vacuum}}} \underset{\sim}{E}\cdot dA$$

$$= EA = \frac{q}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$E = \frac{\sigma}{\epsilon_0}$$

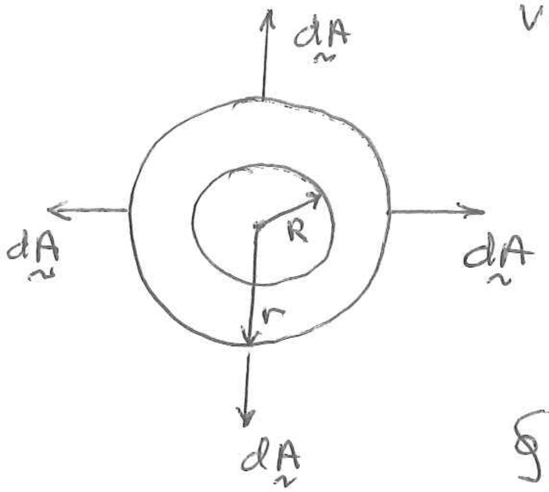
This result is generally true very close to the surface of any conducting surface

$\Rightarrow$ any surface seems flat if you are close enough

However, this relation does not help in solving for E since $\sigma$ will in general not be known for an irregular surface.

example: sphere with uniform charge density

$$\rho = \frac{\text{charge}}{\text{volume}} = \frac{Q}{\frac{4}{3}\pi R^3}$$




$Q$ = total charge

$R$ = charge radius

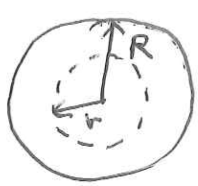
$r > R$

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = 4\pi r^2 E = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

$\Rightarrow$ behaves like all the charge is at the center of the sphere

$r < R$




$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = 4\pi r^2 E = \frac{q}{\epsilon_0}$$

$q$ is only the charge enclosed within the close surface at radius $r < R$

total charge enclosed $= q = \rho \frac{4}{3}\pi r^3$

$$= Q\left(\frac{r}{R}\right)^3$$

$$4\pi r^2 E = \frac{Q}{\epsilon_0}\left(\frac{r}{R}\right)^3$$

$$E = \frac{Q}{4\pi\epsilon_0} \frac{r}{R^3}$$