Maxwell's Equations and Electromagnetic Waves

We have found that a magnetic field that varies in time produces an induced electric field that fills space.

However, the equations for electric and magnetic fields that we have explored thus far do not reveal that a time varying electric field produces a magnetic field.

$\Rightarrow$ ~~Such~~ We expect a symmetry in the equations such that if $\frac{\partial \underline{B}}{\partial t}$ produces $\underline{E}$, then $\frac{\partial \underline{E}}{\partial t}$ produces $\underline{B}$

$\Rightarrow$ This asymmetry ~~suches~~ suggests that there is something missing from our equations.

$\Rightarrow$ We now write the complete set of equations describing $\underline{E}$ and $\underline{B}$ — Maxwell's equations — and discuss the additional term that describes how a $\partial \underline{E}/\partial t$ produces $\underline{B}$.

Maxwell's equations take two characteristic forms: those that involve integrals over closed surfaces and ~~that~~ those that involve integrals over an area with a bounding curve.

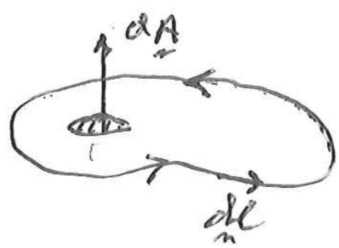
$$\left.\begin{aligned}\oint \underset{\sim}{E}\cdot d\underset{\sim}{A} &= \frac{q}{\epsilon_0}\\ \oint \underset{\sim}{B}\cdot d\underset{\sim}{A} &= 0\end{aligned}\right\}$$ integrals over closed surfaces

The latter means that magnetic fields do not start or stop — no magnetic monopoles

$\Rightarrow$ just dipole magnetic fields

$$\oint \underset{\sim}{E}\cdot d\underset{\sim}{l} = -\frac{d}{dt}\Phi = -\frac{d}{dt}\oint \underset{\sim}{B}\cdot d\underset{\sim}{A}$$

$$\oint \underset{\sim}{B}\cdot d\underset{\sim}{l} = \mu_0 I + \mu_0\epsilon_0 \frac{d}{dt}\oint \underset{\sim}{E}\cdot d\underset{\sim}{A}$$



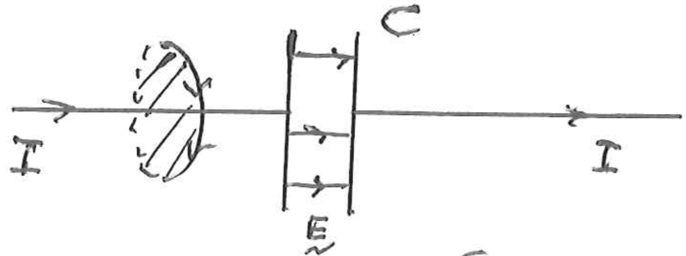
Integrals over an area with a bounding curve

$\epsilon_0 \frac{d}{dt}\oint \underset{\sim}{E}\cdot d\underset{\sim}{A}$ is the displacement current

$\Rightarrow$ a time varying $\underset{\sim}{E}$ produces a $\underset{\sim}{B}$.

Induced magnetic fields

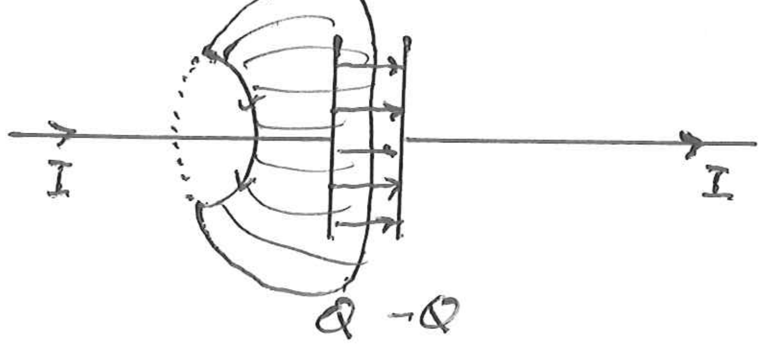
To show that this new term is required, we consider a parallel plate capacitor



$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 \int_A d\mathbf{A}\cdot\mathbf{J} = \mu_0 I$$

$\Rightarrow$ total current I threads through A.

However we can distort the area A as long as the bounding curve is unchanged



No current cuts through the surface so Ampere's Law says

$$\oint \mathbf{B}\cdot d\mathbf{l} = 0$$

$\Rightarrow$ This is not possible since $\mathbf{B}$ has not changed where $\mathbf{B}\cdot d\mathbf{l}$ is calculated.

$\Rightarrow$ now show that the displacement current resolves this contradiction.

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d}{dt} \oint \vec{E}\cdot d\vec{A}$$

In the capacitor

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$$

$$\oint \vec{E}\cdot d\vec{A} = \oint E\,dA \quad (d\vec{A} \text{ points to the right})$$

$$= EA = \frac{Q}{\epsilon_0}$$

so

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d}{dt} \frac{Q}{\epsilon_0}$$

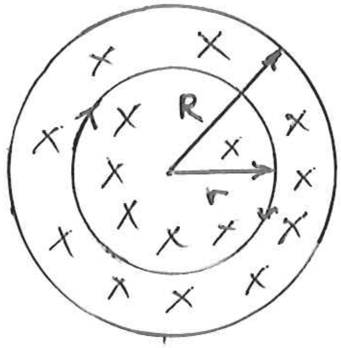
$\Rightarrow$ but $I = \dfrac{dQ}{dt}$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I \quad \Rightarrow \text{same as before}$$

$\Rightarrow$ within the capacitor there is a displacement current associated with the time varying $\vec{E}$. This produces a magnetic within the capacitor just like the current in the wire produces $\vec{B}$.

<u>Magnetic field inside a capacitor</u>

Consider a capacitor with cylindrical symmetry. Draw a bounding curve inside the capacitor $\Rightarrow$ evaluate $\vec{B}$ on this bounding curve



$A = \pi R^2$ = area of capacitor

$\underset{\sim}{E}$ points inward in direction of I

Bounding curve of radius r clockwise to find $B(r)$

$$E = \frac{Q}{A \epsilon_0}$$

$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = \oint B\, dl = B\, 2\pi r = \mu_0 \epsilon_0 \frac{d}{dt} \oint \underset{\sim}{E} \cdot d\underset{\sim}{A}$$

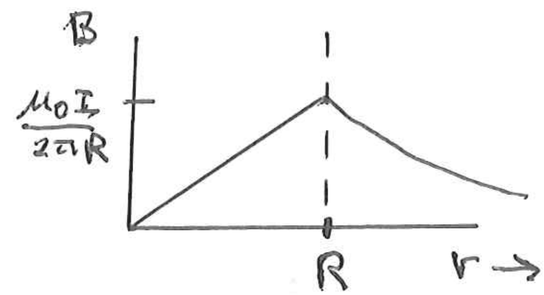
$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = E \int dA = E \pi r^2$$

πr^2 = area within bounding curve

$$B\, 2\pi r = \mu_0 \epsilon_0 \frac{d}{dt} \left(\pi r^2 \frac{Q}{\pi R^2 \epsilon_0} \right)$$

$$= \mu_0 \frac{r^2}{R^2} I$$

$$B = \frac{\mu_0 r}{R^2 2\pi} I$$



For $r > R$

$$B = \frac{\mu_0 I}{2\pi r}$$

$\Rightarrow$ a time changing electric field produces a magnetic field with $\underset{\sim}{E}$ and $\underset{\sim}{B}$ $\perp$ to each other.

The displacement current completes Maxwell's eqns.

changing $\underset{\sim}{B} \Rightarrow \underset{\sim}{E}$

changing $\underset{\sim}{E} \Rightarrow \underset{\sim}{B}$

$\Rightarrow$ electromagnetic waves (light)

Electromagnetic waves

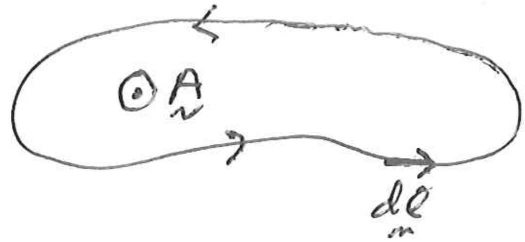
Maxwell's equations tell us that changing $\underset{\sim}{E}$ and $\underset{\sim}{B}$ can self-generate each other to produce waves that can propagate in space even in the absense of conductors.

Showing that such waves follow from the integral form of Maxwell's Eqns is messy and not very illuminating. Instead, let's consider some basic characteristics of the equations and make some dimensional arguments. The equations in free space are

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{l} = -\frac{d}{dt} \oint \underset{\sim}{B} \cdot d\underset{\sim}{A}$$

$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = \mu_0 \epsilon_0 \frac{d}{dt} \oint \underset{\sim}{E} \cdot d\underset{\sim}{A}$$

where the integrals involve bounding curves



The top equation relates $\underset{\sim}{E}$ along $d\underset{\sim}{l}$ to $\underset{\sim}{B}$ perpendicular to $d\underset{\sim}{l}$ along $d\underset{\sim}{A}$. Similarly the second equation ~~and~~ relates $\underset{\sim}{B}$ along $d\underset{\sim}{l}$ to $\underset{\sim}{E}$ perpendicular to $d\underset{\sim}{l}$ ~~and~~ along $d\underset{\sim}{A}$.

This suggests that in electromagnetic waves $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are perpendicular to each other. Now consider the scale length λ and time scale τ. The two equations yield

$$E\lambda \sim \frac{1}{\tau} B\lambda^2$$

$$B\lambda \sim \mu_0 \epsilon_0 \frac{1}{\tau} E\lambda^2$$

Recall that $\mu_0 \epsilon_0 = \frac{1}{c^2}$ with $c = 3\times10^8$ m/s.
Thus eliminating E using top equation reduces to a single equation

$$B\lambda \sim \frac{1}{c^2}\frac{1}{\tau}\left(\frac{B\lambda}{\tau}\right)\lambda^2$$

B the cancels leaving

$$1 \sim \frac{1}{c^2}\frac{\lambda^2}{\tau^2} \quad \text{or} \quad \frac{\lambda}{\tau} = c$$

λ/τ is the wave velocity or c.
Thus, c is the velocity of light in free space. Knowing λ/τ, we can find the relation between E and B in these waves

$$E \sim \frac{\lambda}{\tau} B \sim cB$$

$\Rightarrow$ λ is the wavelength of the wave

$\Rightarrow$ τ is the wave period

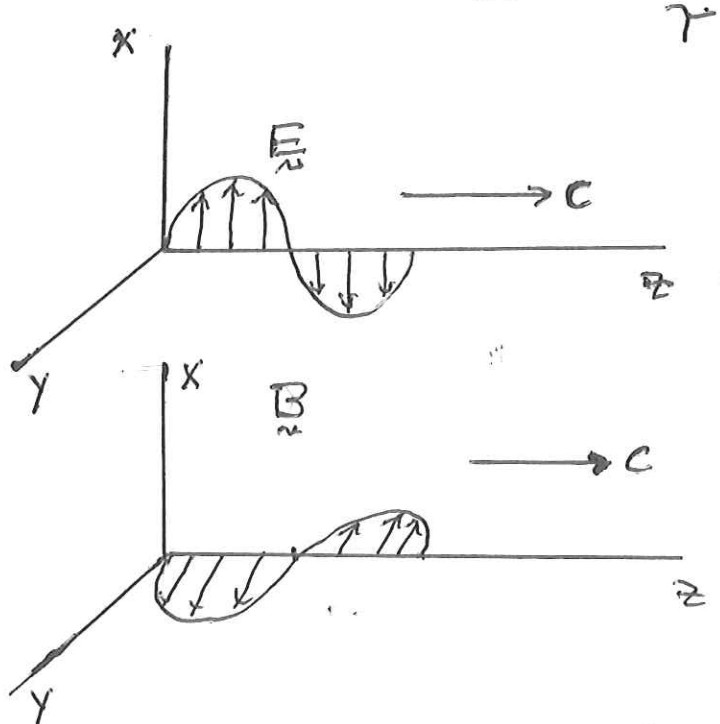
These characteristics follow from more rigorous solutions to Maxwell's Eqns

$\Rightarrow$ $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are $\perp$ to each other

$\Rightarrow$ The wave velocity is c

It is convenient to describe the waves in terms of

$$\omega = \frac{2\pi}{T}, \quad k = \frac{2\pi}{\lambda}$$



$$\underset{\sim}{E} = E_0 \cos(kz - \omega t)\,\hat{i}$$

$$\underset{\sim}{B} = B_0 \cos(kz - \omega t)\,\hat{j}$$

with $E_0 = B_0 c$.

The waves propagate along z with velocity c.

Electromagnetic waves in nature span an enormous range of scales

$\Rightarrow$ from long wavelength radio waves ~ 10's of meters

$\Rightarrow$ X-rays with λ as small as 10^{-12} m

$\Rightarrow$ visible light is in the range of 400–700 nm

Wave energy

Electromagnetic waves carry energy in the form of electric and magnetic energy

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{B^2}{2\mu_0} = \text{wave energy density} = \frac{\text{energy}}{\text{volume}}$$

but $E = Bc$ so

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}\left(\frac{E^2}{c^2}\right)$$

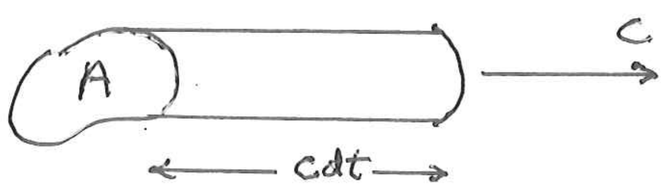
$$= \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{\mu_0\epsilon_0}{\mu_0}E^2 = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\epsilon_0 E^2$$

$$= \epsilon_0 E^2$$

$\Rightarrow$ the energy in em waves is split equally between electric and magnetic energy

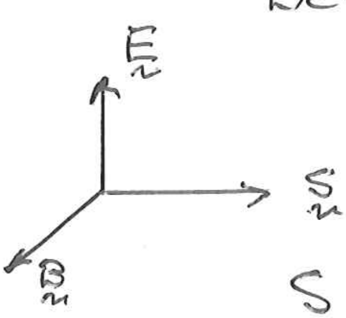
Wave energy flow

Consider the wave energy in an area A moving ~~thus~~ with a velocity c. In a time dt the total energy crossing a surface is



$$dW = \epsilon_0 E^2 A c\, dt$$

Define S as the energy per unit area crossing a surface



$$S = \frac{dW}{dt}\frac{1}{A} = \epsilon_0 E^2 c = \epsilon_0 E(Bc)c$$

$$= \epsilon_0 (EB)c^2 = \frac{1}{\mu_0}EB$$

$$\underset{\sim}{S} \equiv \frac{1}{\mu_0}\underset{\sim}{E}\times\underset{\sim}{B}$$ is the Poynting flux.

$\Rightarrow$ gives the direction of energy propagati

wave momentum

Electromagnetic waves also carry momentum. The momentum per unit volume is given by

$$\frac{dp}{dV} = \frac{EB}{\mu_0 c^2} = \frac{S}{c^2}$$

Light shining on and being absorbed on a surface produces a pressure, which is the force per unit area

$$P_{rad} = \frac{\text{rate of momentum transfer}}{\text{area}}$$

$$= \frac{\left(\frac{S}{c^2}\right) c\Delta t}{\Delta t} = \frac{S}{c}$$

Averaging of energy, Poynting flux or pressure

When calculating the time averaged energy in em waves, always reduce the result by a factor of 1/2 compared to the peak value. For example, for the wave energy density

$$u = \epsilon_0 E^2 = \epsilon_0 \cos^2(kz-\omega t)\, E_0^2$$

$$\langle u \rangle_t = \epsilon_0 E_0^2 \langle \cos^2(kz-\omega t)\rangle_t = \frac{1}{2}\epsilon_0 E_0^2$$

Average over the wave period τ,

$$\langle\,\rangle_t = \frac{1}{\tau}\int_0^\tau dt\ \frac{1-\cos[2(kz-\omega t)]}{2}$$

$$= \frac{1}{2\tau}\int_0^\tau dt = \frac{1}{2}$$