# NON-BASIC RIGID PACKETS FOR DISCRETE $L$-PARAMETERS

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Abstract. This article introduces the theory of non-basic rigid inner forms over $p$-adic local fields, extending the basic theory developed by Kaletha. Motivated by the recent work of Bertoloni Meli–Oi on the $B(G)$-parametrization of the local Langlands conjectures, our main application is to extend the basic rigid refined local Langlands conjectures for a discrete $L$-parameter $\phi$ of a quasi-split connected reductive group $G$. The packets of our extended construction are Weyl orbits of representations of inner forms of twisted Levi subgroups $N$ of $G$ for which $\phi$ factors through a member of the canonical $\widehat{G}$-conjugacy class of embeddings ${}^LN_{\pm} \to {}^LG$ constructed by Kaletha.

## 1. Introduction

### 1.1. Motivation.

1.1.1. *The refined local Langlands conjecture.* Let $\phi$ be a discrete $L$-parameter for a connected reductive group $G$ over a nonarchimedean local field $F$. The goal of this paper is prove that the rigid refined local Langlands conjecture implies a new, extended correspondence for $\phi$ concerning all of the twisted Levi subgroups $N$ of $G$ such that $\phi$ factors through an $L$-embedding ${}^LN \to {}^LG$. Twisted Levi subgroups of $G$ are rich sources of supercuspidal representations, for example the construction of [Yu01] using elliptic maximal tori—the extended correspondence given in this paper is a step toward deepening the relationship between the local Langlands correspondence and the representation theory of $p$-adic groups.

The local Langlands conjecture predicts a finite-to-one map from irreducible (smooth) representations of $G(F)$ on $\mathbb{C}$-vector spaces to $\widehat{G}$-conjugacy classes of $L$-parameters $\phi\colon W_F \times \mathrm{SL}_2 \to {}^LG := \widehat{G} \rtimes W_F$, where $\widehat{G}$ is a Langlands dual group of $G$ and $W_F$ is the Weil group of $F$. This map is expected to satisfy numerous desiderata (see e.g. [KT22, §6.1] for the full details).

A refined local Langlands correspondence is a parametrization of the fibers of the above map using data associated to $\phi$ (we will see examples of this "data" shortly). When $G$ is quasi-split, the conjectural parametrization is that, for a fixed parameter $\phi$, the fiber over $[\phi]$, denoted by $\Pi_{\phi}(G)$, is in bijection, in a canonical way up to a choice of Whittaker datum for $G$, with irreducible representations of the finite group $\pi_0(Z_{\widehat{G}}(\phi)/Z(\widehat{G})^{\Gamma})$. In the non-quasi-split case, Adams, Barbasch, and Vogan discovered, originally for $F = \mathbb{R}$ (cf. [ABV92]), that one should group together representations of inner forms of a fixed quasi-split group $G$ (which all have the same set of $L$-parameters) and parametrize this so-called "compound $L$-packet." Following this philosophy, in the rest of the introduction we always assume $G$ to be quasi-split.

An important technicality in the study of compound $L$-packets is the question of when to identify representations of two inner twists $(G_1, \psi_1, \pi_1)$, $(G_2, \psi_2, \pi_2)$, where $\psi_i\colon G_{\overline{F}} \to G_{i,\overline{F}}$ is a choice of twisting isomorphism and $\pi_i$ is a representation of $G_i(F)$. As Vogan observed in [Vog93], equivalence of inner twists (in other words, being cohomologous in $H^1(F, G_{\mathrm{ad}})$) does not work,

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since there are automorphisms of inner twists which identify non-isomorphic representations. To overcome this obstacle, Vogan chooses a 1-cocycle $z_\psi$ valued in $G$ which lifts $\bar{z}_\psi \in Z^1(F, G_{\mathrm{ad}})$, the cocycle associated to the inner twisting $\psi$, and works with triples $(G', z_\psi, \pi)$ instead, insisting that isomorphisms of such triples preserve the cocycle up to a coboundary.

More generally, it has proved useful to take the cocycle $z_\psi$ with coefficients in $G$ to be a cocycle not for the absolute Galois group $\Gamma$ of $F$ but, more generally, for a certain type of group $\mathscr{E}$ called by Langlands and Rapoport a "Galois gerbe" [LR87, §2]. Such a group $\mathscr{E}$ is, by definition, an extension

$$1 \to A(\overline{F}) \to \mathscr{E} \to \Gamma \to 1$$

of $\Gamma$ by a (commutative) group scheme $A$, called the "band", and the group $\mathscr{E}$ acts on $G(\overline{F})$ through the inflation of the Galois action. There are three standard choices for $\mathscr{E}$, yielding either pure inner forms ($\mathscr{E} = \Gamma$), extended pure inner forms ($\mathscr{E} = \mathscr{E}_{\mathrm{Kott}}$), or rigid inner forms ($\mathscr{E} = \mathscr{E}_{\mathrm{Kal}}$); see [KT22, §6.3] for a discussion of all three of these approaches. The third of these will be the focus in this paper, but first we discuss extended pure inner forms.

1.1.2. *The $B(G)$ local Langlands correspondence.* In the case where $\mathscr{E} = \mathscr{E}_{\mathrm{Kott}}$ is the "Kottwitz gerbe", sometimes also called the "Dieudonné gerbe", the band is the pro-torus $\mathbb{D}$ with character group $\mathbb{Q}$, namely

$$\mathbb{D} := \varprojlim_n \mathbb{G}_m$$

where the limit is taken over the nonnegative integers with transition map for $n \mid m$ the $(m/n)$th-power map. Define $B(G)$ to be the classes in $H^1(\mathscr{E}_{\mathrm{Kott}}, G)$ whose restriction to $\mathbb{D}$ is algebraic and define the set of basic classes $B(G)_{\mathrm{bas}} \subset B(G)$ to be those classes whose image in $H^1(\mathscr{E}_{\mathrm{Kott}}, G_{\mathrm{ad}})$ is contained in $H^1(F, G_{\mathrm{ad}})$. Given a cocycle $b \in Z^1_{\mathrm{alg}}(\mathscr{E}_{\mathrm{Kott}}, G)$, we can form, using $b$, a certain inner twist $G_b$ of the centralizer $Z_G(f_b)$ of the restriction $f_b \colon \mathbb{D}_{\overline{F}} \to G_{\overline{F}}$ of $b$ to the band, choosing $b$ within its class so that $f_b$ is defined over $F$. If $b$ is basic, or equivalently, if $f_b$ factors through $Z(G)$, then $G_b$ is an inner twist of $G$.

Kottwitz has conjectured [Kal14, Conjecture 4.1] that there is a bijection

$$\bigsqcup_{b \in B(G)_{\mathrm{bas}}} \Pi_\phi(G_b) \xrightarrow{\iota_{\mathfrak{w}}} \mathrm{Irr}(Z_{\widehat{G}}(\phi)/[Z_{\widehat{G}_{\mathrm{der}}}(\phi)]^\circ), \tag{1}$$

where $\mathfrak{w}$ is a Whittaker datum for $G$. When the center of $G$ is connected, so that the map $B(G)_{\mathrm{bas}} \to H^1(F, G_{\mathrm{ad}})$ is surjective, the bijection parameterizes $\Pi_\phi(G')$ for any inner twist $G'$ of $G$. In fact, there are several such parameterizations, since different elements $b \in B(G)_{\mathrm{bas}}$ can yield the same inner form $G_b$.

The sets $B(G)$ and $B(G)_{\mathrm{bas}}$ also appear in the geometry of the space $\mathrm{Bun}_G$, the moduli stack of $G$-bundles on the Fargues–Fontaine curve, which underlies Fargues and Scholze's geometrization of the local Langlands correspondence [FS24]. Namely, the topological space $|\mathrm{Bun}_G|$ is homeomorphic to $B(G)$ and contains $B(G)_{\mathrm{bas}}$ as the semistable locus. In light of this description and the role of $\mathrm{Bun}_G$ in the local Langlands correspondence, it is natural to seek to extend the conjectural bijection (1) to all of $B(G)$.

This extension was carried out by Bertoloni Meli and Oi in [BMO23, Theorem 1.1]. Given a basic local Langlands correspondence (1) for $G$ and its Levi subgroups, they define for every

$L$-parameter $\phi$ and every $b \in B(G)$ an $L$-packet $\Pi_\phi(G_b)$ and construct a bijection

$$\bigsqcup_{b \in B(G)} \Pi_\phi(G_b) \xrightarrow{\iota_{\mathfrak{w}}} \mathrm{Irr}(S_\phi) \qquad (S_\phi := Z_{\widehat{G}}(\phi)). \tag{2}$$

The extended parameterization has many advantages. For example, as explained in the introduction to [BMO23], irreducible representations of $S_\phi$ correspond to coherent sheaves on the classifying stack $[*/S_\phi]$, which naturally embeds into $Z^1(W_F, \widehat{G})/\widehat{G}$, the stack of $L$-parameters of $G$. As a result one expects the extended parameterization to provide a rough framework for relating the construction of Fargues and Scholze to the refined local Langlands conjectures. The Bertoloni Meli–Oi framework also allows for a generalization, [BMO23, Theorem 5.13], of the endoscopic character identities involving Levi subgroups of endoscopic groups.

A key construction in [BMO23] is to extract from an irreducible representation of $S_\phi$ a dominant weight of the maximal torus $(S_\phi \cap \widehat{M})^\circ$ of $S_\phi$, where $M$ is a minimal Levi subgroup of $G$ through which $\phi$ factors, and to then, using linear-algebraic duality, obtain a dominant coweight $\nu$ of $A_{M'}$—the maximal split torus in the center of $M'$, a conjugate of $M$. One thereby also obtains, by taking the centralizer of $\nu$, a Levi subgroup $M' \subseteq N \subseteq G$. From here, Bertoloni Meli and Oi use the representation theory of disconnected reductive groups (as developed in [AHR20]) to construct an irreducible representation of $S_{\phi'} \cap \widehat{N}$ for some appropriate conjugate $\phi'$ of $\phi$ which descends to its quotient by the subgroup $Z_{\widehat{N}_{\mathrm{der}}}(\phi')^\circ$, thus producing a basic enhancement for $N$. The set $B(G)$ is equipped with a *Newton map* $\nu\colon B(G) \to X_*(A_T)_{\mathbb{Q}}$, where $T$ is a minimal Levi subgroup of $G$, and the element $b \in B(G)$ that is associated, by the Bertoloni Meli–Oi bijection, to an element of $\mathrm{Irr}(S_\phi)$ recovers the Levi subgroup $N$ from above via the formula $N = Z_G(\nu(b))$. Moreover, the coweight $\nu(b)$ for $A_{M'} \subseteq A_T$ is exactly the one obtained from the highest weight of $(S_\phi \cap \widehat{M})^\circ$ corresponding to $\rho$.

1.1.3. *Twisted Levi subgroups.* When $\phi$ is a discrete $L$-parameter, as we assume throughout the paper, the Bertoloni Meli–Oi parametrization reduces to the original basic conjecture (1): by definition, $\phi$ does not factor through the $L$-embedding associated to any proper Levi subgroup of $G$. For the study of discrete parameters, it is much more convenient to work with *twisted* Levi subgroups of $G$, those $F$-rational subgroups that become isomorphic to a Levi subgroup of $G_{\overline{F}}$ over $\overline{F}$.

If $M \subseteq G$ is a twisted Levi subgroup then there is in general no canonical embedding ${}^L M \to {}^L G$. To circumvent this difficulty, Kaletha constructed in [Kal21a] a double cover $M(F)_\pm \to M(F)$ and an $L$-group ${}^L M_\pm$ which does admit a canonical $\widehat{G}$-conjugacy class of embeddings ${}^L M_\pm \to {}^L G$. For brevity we will refer to any such $L$-embedding as "standard". Although $\phi$ cannot factor through the $L$-embedding for a Levi subgroup, it can, and often does, factor through the $L$-embedding of a nontrivial twisted Levi subgroup. For example, when $p$ does not divide the order of the Weyl group of $G$, every semisimple discrete $L$-parameter factors through a standard $L$-embedding for an elliptic maximal torus. These factorizations are key to Kaletha's construction of torally wild supercuspidal $L$-packets [Kal21b].

These developments, and the analogy with [BMO23], lead us to ask:

**Question 1.1.** Is there a version of the refined local Langlands conjectures which constructs, for every twisted Levi subgroup $N$ of $G$ such that $\phi$ factors through a standard $L$-embedding ${}^L N_\pm \to {}^L G$, a packet of representations of inner forms of $N$?

One goal of this paper is to propose an answer to Question 1.1 and give evidence that this answer is reasonable. Our answer consists, in brief, of redeveloping the $B(G)$ local Langlands conjectures after replacing the Kottwitz gerbe $\mathscr{E}_{\mathrm{Kott}}$ by the Kaletha gerbe $\mathscr{E}_{\mathrm{Kal}}$. In spite of the simple slogan, however, there are many new features of the rigid setting absent from the $B(G)$ setting.

1.1.4. *Non-basic classes for $\mathscr{E}_{\mathrm{Kal}}$.* For $\mathscr{E}_{\mathrm{Kal}}$, the band is the group

$$u := \varprojlim_{E,n} \frac{\mathrm{Res}_{E/F}(\mu_n)}{\mu_n},$$

where the projective limit is taken over finite Galois extensions $E/F$ and positive integers $n$ and where the transition map for $(K/E, n \mid m)$ is the $(m/n)$th-power of the $K/E$ norm map. We can again define $H^1_{\mathrm{alg}}(\mathscr{E}_{\mathrm{Kal}}, G)$ as the classes whose restriction to the band is algebraic, and define the basic subset $H^1_{\mathrm{bas}}(\mathscr{E}_{\mathrm{Kal}}, G)$ as we defined $B(G)_{\mathrm{bas}}$, replacing $\mathscr{E}_{\mathrm{Kott}}$ with $\mathscr{E}_{\mathrm{Kal}}$. In this case, the map $H^1_{\mathrm{bas}}(\mathscr{E}_{\mathrm{Kal}}, G) \to H^1(F, G_{\mathrm{ad}})$ is always surjective. In the same way that $b \in B(G)$ gave rise to a reductive $F$-group $G_b$, a class $[x] \in H^1_{\mathrm{alg}}(\mathscr{E}_{\mathrm{Kal}}, G)$ gives rise to a reductive $F$-group $G_{[x]}$: choosing a representative $x$ of $[x]$ whose restriction to the band $f_x\colon u_{\overline{F}} \to G_{\overline{F}}$ is defined over $F$, we form the centralizer $Z_G(f_x)$, then use $x$ (which is automatically valued in $Z_G(f_x)$) to construct the inner twist $G_{[x]}$ of this centralizer.

The rigid analogue of the $B(G)_{\mathrm{bas}}$ local Langlands conjecture (1), articulated in [Kal16b, (1.3)], is a conjectural bijection

$$\bigsqcup_{[x] \in H^1_{\mathrm{bas}}(\mathscr{E}_{\mathrm{Kal}}, G)} \Pi_\phi(G_{[x]}) \xrightarrow{\iota_{\mathfrak{w}}} \mathrm{Irr}(\pi_0(S_\phi^+)). \tag{3}$$

Here $\mathfrak{w}$ is again a Whittaker datum for $G$ and $S_\phi^+$ is the preimage of $S_\phi$ in the group

$$\widehat{\overline{G}} := \varprojlim_{Z \to Z(G)} \widehat{G/Z} \simeq \widehat{G}_{\mathrm{sc}} \times \varprojlim_{n} Z(\widehat{G})^\circ,$$

in which the first projective limit is taken over the poset of finite central subgroups of $G$ and the second is taken over positive integers $n$ with $n$th-power transition maps. To extend the basic rigid bijection (3), we study arbitrary algebraic cocycles $x \in H^1_{\mathrm{alg}}(\mathscr{E}, G)$. The structure of these objects is much richer than for $B(G)$.

To begin with, there are strange classes $[x] \in H^1_{\mathrm{alg}}(\mathscr{E}, G)$ for which $G_{[x]}$ has smaller rank than $G$, or is even a finite $F$-group. For example, in Lemma A.2, we construct an algebraic cocycle $x$ of $\mathscr{E}$ for $\mathrm{PGL}_2$ such that the image of $f_x$ is $\mu_2^2$, meaning that $Z_{G_{\overline{F}}}(f_x) = \mu_2^2$. The source of these examples is the potential failure of $f_x$ to factor through a maximal torus. To avoid this pathology, we restrict our attention to the *regular classes* in $H^1_{\mathrm{alg}}(\mathscr{E}_{\mathrm{Kal}}, G)$, defined to be those lying in the image of $H^1_{\mathrm{alg}}(\mathscr{E}_{\mathrm{Kal}}, T)$ for some maximal torus $T$. In particular, if $[x]$ is regular then the map $f_x$ factors through a torus and we can use this torus to analyze the structure of $x$ in terms of standard objects from the theory of reductive groups.

Fortunately, even after we restrict to the regular classes $H^1_{\mathrm{reg}}(\mathscr{E}_{\mathrm{Kal}}, G)$, there is a rich supply of groups of the form $Z_G(f_x)$, which we call *rigid Newton centralizers*. It is difficult to combinatorially describe this class of groups, but, as we show in Theorem 3.18, it includes every twisted Levi subgroup of $G$ that contains an elliptic maximal torus. This observation gives strong evidence that $\mathscr{E}_{\mathrm{Kal}}$ is equipped to answer Question 1.1: unlike $\mathscr{E}_{\mathrm{Kott}}$, it can produce subgroups of $G$ whose $L$-embeddings are amenable to the study of discrete parameters.

On the other hand, not every rigid Newton centralizer is a twisted Levi subgroup (Example 3.21), or is even connected. It would be extremely interesting to fit this more general class of groups into the local Langlands conjectures, and in § 4.3.5 and the introduction to § 4 we offer a few thoughts as to how this might go. Nonetheless, for simplicity, and because twisted Levi subgroups are already a rich set of tools with which to probe $L$-parameters, in this paper (at least for the applications to the local Langlands correspondence) we restrict further to those classes $[x] \in H^1_{\text{reg}}(\mathscr{E}, G)$ such that $Z_{G_{\overline{F}}}(f_x)$ is a Levi subgroup of $G_{\overline{F}}$, which we call *Levi-regular* and denote by $H^1_{\text{L-reg}}(\mathscr{E}, G)$.

A key feature of the set $H^1_{\text{bas}}(\mathscr{E}_{\text{Kal}}, G)$ is the Tate-Nakayama duality isomorphism ([Kal16b, Theorem 4.11])

$$H^1_{\text{bas}}(\mathscr{E}_{\text{Kal}}, G) \to X^*(\pi_0(Z(\widehat{G})^{\Gamma,+})).$$

Conjecturally, this isomorphism identifies which rigid inner twist of $G$ carries the representation corresponding to a fixed $\rho \in \text{Irr}(\pi_0(S^+_\phi))$, by passing the restriction of $\rho$ to $\pi_0(Z(\widehat{G})^{\Gamma,+})$ through the isomorphism. A prerequisite for any kind of reasonable "non-basic" rigid correspondence is therefore the extension of this duality map to all of $H^1_{\text{L-reg}}(\mathscr{E}_{\text{Kal}}, G)$. Actually, we can even construct this map for all regular classes, not just the Levi-regular ones, but in this introduction we present the more restricted formulation for expository convenience.

**Theorem 1.2.** *(Definition 3.14) The Tate-Nakayama duality isomorphism for $H^1_{\text{bas}}(\mathscr{E}_{\text{Kal}}, G)$ can be extended to an injective map, the "rigid Kottwitz map"*

$$\kappa \colon H^1_{\text{L-reg}}(\mathscr{E}_{\text{Kal}}, G) \longrightarrow \bigsqcup_{[N]} \frac{X^*(\pi_0(Z(\widehat{G})^{\Gamma,+}_{(N)}))}{W(G,N)(F)},$$

*where the union is taken over conjugacy classes $[N]$ of twisted Levi subgroups $N$ that arise from classes in $H^1_{\text{L-reg}}(\mathscr{E}_{\text{Kal}}, G)$.*

Here $Z(\widehat{G})^{\Gamma,+}_{(N)}$ denotes the preimage of $Z(\widehat{G})^{\Gamma}$ in $\widehat{\overline{N}}$. In Proposition 3.16 we explicitly describe the image of the rigid Kottwitz map,

1.1.5. *The main theorem.* For a discrete $L$-parameter $\phi$, one then hopes to use the basic rigid refined correspondence to construct $L$-packets for rigid inner twists of the twisted Levi subgroups $M$ of $G$ such that $\phi$ factors through a standard $L$-embedding for $M$. Parametrizing these packets requires expanding $\text{Irr}(\pi_0(S^+_\phi))$ to some larger set which captures rigid inner twists of all such twisted Levi subgroups. We will refer to such group-theoretic data as "enhancements."

On the Galois side, the main accomplishment of this paper is to propose such an enhancement and use it to extend the basic rigid conjecture. The precise definition of our enhancement is somewhat technical; to avoid losing ourselves in the details, we will first give a rough shape of the enhancement and explain the resulting rigid local Langlands conjectures.

Our set of enhancements is a certain *twisted extended quotient*, a general construction defined, for instance, in [AMS18, §1]. Constructing a twisted extended quotient $(X /\!/ \mathsf{G})_\natural$ requires three pieces of data: a set $X$, a finite group $\mathsf{G}$ acting on $X$, and a certain family of cocycles $\{\natural_x \in Z^2(\mathsf{G}_x, \mathbb{C}^\times)\}_{x \in X}$, where $\mathsf{G}_x$ is the stabilizer of $x$ in $\mathsf{G}$. For our enhancement one takes a certain set $X^+_\phi(\widehat{G})$, acted on by the group $\pi_0(S_\phi)$, together with a certain family of cocycles that we will comment on later. All in all, then, the set of enhancements of $\phi$ has the shape

$$[(X^+_\phi(\widehat{G}) /\!/ \pi_0(S_\phi))_\natural],$$

where the brackets denote that we are considering this set up to $\widehat{G}$-conjugacy. With the enhancements as black box, we can state the main Theorem.

**Theorem 1.3.** *(Theorem 4.41) Let $G$ be a quasi-split connected reductive group and $\phi\colon W_F\times \mathrm{SL}_2 \to {}^LG$ a discrete L-parameter. Assume that the basic local Langlands correspondence (Conjecture 4.10) holds for $G$ and all quasi-split twisted Levi subgroups of $G$ that contain an elliptic maximal torus of $G$. Then there is a bijection*

$$\bigsqcup_{[N]} \iota_{\mathfrak{w}_{N,\pm}}\colon \bigsqcup_{[N]} [\![\Pi^+_{\phi_{N,\pm}}]\!]_{W(G,N)(F)} \longrightarrow [(X^+_\phi(\widehat{G}) /\!\!/ \pi_0(S_\phi))_\natural], \tag{4}$$

*where the disjoint union is taken over all conjugacy classes of twisted Levi subgroups $N$ such that $\phi$ factors, via a standard L-embedding for $N$, through an L-parameter $\phi_{N,\pm}\colon W_F\times \mathrm{SL}_2 \to {}^LN_\pm$ for the double cover $N(F)_\pm$.*

A few words are in order about the objects appearing in the Theorem. As usual, $\mathfrak{w}_N$ is a choice of Whittaker datum for $N$.

- The parameter $\phi_{N,\pm}$ is uniquely determined by $\phi$ and $N$ up to $\widehat{G}$-conjugacy, though not $\widehat{N}$-conjugacy.
- The map $\iota_{\mathfrak{w}_{N,\pm}}$ is a version of the basic rigid refined local Langlands correspondence for $N(F)_\pm$, deduced canonically from the correspondence for $N$ (Theorem 4.15).
- $\Pi_{\phi_{N,\pm}}$ is the $L$-packet for $\phi_{N,\pm}$ of representations of rigid inner twists of the double cover $N(F)_\pm$.
- The superscript "+" is a technical restriction on the members $(N'(F)_\pm, z_\psi, \pi_\pm)$ of the $L$-packet $\Pi_{\phi_{N,\pm}}$ which ensures that the class $[z_\psi]$ is such that $Z_G(z_\psi|_u) = N$ (up to conjugacy).
- $W(G,N) := N_G(N)/N$ is the relative Weyl group, and $W(G,N)(F)$ acts on the set $\Pi(N)^{\mathrm{rig}}$ of isomorphism classes of rigid inner twists of $N$.
- $[\![\Pi_{\phi_{N,\pm}}]\!]_{W(G,N)(F)}$ denotes the image of the set $\Pi_{\phi_{N,\pm}}$ in the quotient $\Pi(N)^{\mathrm{rig}}/W(G,N)(F)$.
- A choice of $\Gamma$-stable pinning (see §4.3.2 for details) determines an action of $W(G,N)(F)$ on the set of $L$-parameters for $N(F)_\pm$, and if two $L$-parameters are in the same $W(G,N)(F)$-orbit then their $L$-packets have the same $W(G,N)(F)$-orbit.

Moreover, our conjectural correspondence is compatible with the rigid Kottwitz map. More precisely, there is a canonical map (explained in Remark 4.42)

$$\bigsqcup_{[N]} [\![\Pi^+_{\phi_{N,\pm}}]\!]_{W(G,N)(F)} \to H^1_{\text{L-reg}}(\mathscr{E}_{\mathrm{Kal}}, G).$$

If an element $[\dot\pi]$ on the left-hand side of (4) has image $[x]$ under this map then we may view $[\dot\pi]$ concretely as the $W(G,N)(F)$-orbit of a fixed representation $\dot\pi = (N', z, \pi_\pm) \in \Pi_{\phi_{N,\pm}}$ such that $z\in Z^1_{\mathrm{bas}}(\mathscr{E}_{\mathrm{Kal}}, N)$ has image $[x]$ in $H^1(\mathscr{E}_{\mathrm{Kal}}, G)$. On the other side of the correspondence, there is a canonical map

$$[(X^+_\phi(\widehat{G}) /\!\!/ \pi_0(S_\phi))_\natural] \to \bigsqcup_{[N]} X^*(\pi_0(Z(\widehat{G})^{\Gamma,+}_{(N)}))/W(G,N)(F),$$

to the disjoint union appearing as the codomain of the rigid Kottwitz map.

**Theorem 1.4.** *(Theorem 4.46) In the setting of the previous Theorem, there is a commutative diagram as follows, in which the top map is a bijection:*

$$\begin{array}{ccc} \bigsqcup_{[N]} [\![\Pi^+_{\phi_{N,\pm}}]\!]_{W(G,N)(F)} & \xrightarrow{\sqcup \iota_{\mathfrak{w}_N,\pm}} & [(X^+_\phi(\widehat{G}) /\!/ \pi_0(S_\phi))_\natural], \\ \downarrow & & \downarrow \\ H^1_{\text{L-reg}}(\mathscr{E}, G) & \xrightarrow{\kappa} & \bigsqcup_{[N]} X^*(\pi_0(Z(\widehat{G})^{\Gamma,+}_{(N)}))/W(G,N)(F). \end{array}$$

1.1.6. *Rigid enhancements of discrete L-parameters.* To build intuition for our definition of rigid enhancement, assume temporarily that $\phi$ normalizes a unique maximal torus $\mathscr{T}$ of $\widehat{G}$ and further that $S_\phi \subseteq \mathscr{T}$ (this is the case e.g. for many regular supercuspidal parameters, see Lemma 4.51). In this case, the $\phi$-twisted Galois action produces an elliptic maximal torus $T$ of $G$ which is dual to $\mathscr{T}$ and thus identifies $S_\phi$ with $\widehat{T}^\Gamma$ and $S^+_\phi$ with the preimage of $\widehat{T}^\Gamma$ in the group $\varprojlim_{Z\to Z(G)} \widehat{T/Z}$, where the limit is over all finite central subgroups $Z$ of $G$. Tate-Nakayama duality for the Kaletha gerbe ([Kal16b, Theorem 4.8]) then identifies $X^*(\pi_0(S^+_\phi))$ with the cohomology set $\varprojlim_{Z\to Z(G)} H^1(\mathscr{E}_{\text{Kal}}, Z \to T)$, where the $Z$'s are as above and the notation "$Z \to T$" means that each $x|_u$ must factor through $Z$. This perspective makes it clear what one should do: Use the preimage of $H^1_{\text{L-reg}}(\mathscr{E}_{\text{Kal}}, G)$ in $H^1(\mathscr{E}_{\text{Kal}}, T)$ instead of just $\varprojlim_{Z\to Z(G)} H^1(\mathscr{E}_{\text{Kal}}, Z \to T)$ (which is the preimage of $H^1_{\text{bas}}(\mathscr{E}_{\text{Kal}}, G)$). Applying Tate-Nakayama duality to $H^1(\mathscr{E}_{\text{Kal}}, T)$ but now in the other direction yields $X^*(\pi_0(\widehat{T}^{\Gamma,+}))$, where $\widehat{T}^{\Gamma,+}$ denotes the preimage of $\widehat{T}^\Gamma$ in $\varprojlim_{t\mapsto t^n} \widehat{T}$, which can be (non-canonically) identified with the preimage of $S_\phi$ in the infinite cover $\varprojlim_{t\mapsto t^n} \mathscr{T}$, denoted by $(S_\phi)^+_{(\mathscr{T})}$—this last object has the advantage of being independent of any choices. We thus obtain a rough candidate for the more general objects on the Galois side: characters of $\pi_0((S_\phi)^+_{(\mathscr{T})})$ which come from (via Tate-Nakayama duality) a class in the preimage of $H^1_{\text{L-reg}}(\mathscr{E}_{\text{Kal}}, G)$.

Another way to view characters of $\pi_0((S_\phi)^+_{(\mathscr{T})})$ is by using the central extension

$$0 \to \pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T})}) \to \pi_0((S_\phi)^+_{(\mathscr{T})}) \to \pi_0(S_\phi) \to 1,$$

where $Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T})}$ denotes the preimage of $Z(\widehat{G})^{\Gamma,\circ}$ in the infinite cover of $\mathscr{T}$ given by the power maps described above. The subgroup $\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T})})$ appearing as the kernel turns out to be important: A choice of identification of $\widehat{T}$ with $\mathscr{T}$ gives an injection (cf. Proposition 4.21)

$$X^*(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T})})) \hookrightarrow \text{Hom}(\mu_{\overline{F}}, T_{\overline{F}}), \tag{5}$$

and, as is the case with characters of the torus $(S_\phi \cap \widehat{M})^\circ$ in [BMO23], the twisted Levi subgroup of $G$ corresponding to a point $\pi \in X^*(\pi_0((S_\phi)^+_{(\mathscr{T})}))$ in our parametrizing data is determined by the combinatorics of the torsion cocharacter given by the image of $\pi|_{\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T})})}$ via (5) (it is not in general given by the centralizer of the torsion cocharacter—see §2.3.3 for a full explanation). From this point of view, Clifford theory (applied to the above central extension) tells us that characters of $\pi_0((S_\phi)^+_{(\mathscr{T})})$ are equivalent to pairs $(\chi, \rho)$, where $\chi$ is a character of $\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T})})$ and $\rho$ is a simple module over the twisted group algebra $\mathbb{C}[\pi_0(S_\phi)_\chi, \theta_\chi]$ for the stabilizer of $\chi$ in $\pi_0(S_\phi)$

and the cocycle $\theta_\chi$ determined by a choice of section of $\varprojlim \mathscr{T} \to \mathscr{T}$ and the character $\chi$. As explained above, the map (5) applied to $\chi$ gives a quasi-split twisted Levi subgroup $N$ (unique up to $F$-isomorphism) such that $\phi$ factors through ${}^L N_\pm$ and the pair $(\chi, \rho)$ yields (via the local Langlands correspondence) a representation of some (basic) rigid inner form of $N(F)_\pm$. The fact that we consider $\phi$ up to its $\widehat{G}$-conjugacy class $[\phi]$ means that one must consider such pairs $(\chi, \rho)$ up to $\widehat{G}$-conjugacy (in particular, a pair $(\chi, \rho)$ for $\phi$ will go to an analogous pair $(\chi', \rho')$ for $\mathrm{Ad}(g^\vee) \circ \phi$).

The Clifford-theoretic approach to the above example gives a blueprint for how to define the analogous enhancements for an arbitrary discrete parameter $\phi$. There are two pieces of data $([\chi], \rho)$: a "highest weight" $\chi$ which is a character of $\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})})$, where $\mathscr{M}$ is a Levi subgroup (not necessarily a torus) of $\widehat{G}$ which is the image of $\widehat{M}$ under standard $L$-embedding $\eta \colon {}^L M_\pm \to {}^L G$ (for $M$ a twisted Levi subgroup of $G$) such that $\phi$ factors through $\eta$ and $\mathscr{M}$ is minimal with these properties, and $Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})}$ denotes the preimage of $Z(\widehat{G})^{\Gamma,\circ}$ in the cover

$$\mathscr{M}_{\mathrm{sc}} \times \varprojlim_{m \mapsto m^n} Z(\mathscr{M})^\circ \to \mathscr{M}.$$

We declare two such "weights" (for potentially distinct $\mathscr{M}, \mathscr{M}'$) to be equivalent if they coincide on some suitably-defined (cf. Lemma 4.29 and the ensuing discussion) common quotient $Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})} \to Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{N})} \leftarrow Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M}')}$—the equivalence class is denoted by $[\chi]$. The other piece of data $\rho$ is the "non-abelian data" and is a simple module over the twisted group algebra $\mathbb{C}[\pi_0(S_\phi)_{[\chi]}, \natural_{[\chi]}]$—the cocycle $\natural_{[\chi]}$ is more subtle to define than in the above easier case (chiefly because $S_\phi$ is no longer contained in $\mathscr{T}$, cf. Example 4.3.4); in particular, it is obtained by combining $\chi$ with a choice of a section of the map $\widehat{\widehat{N}} := \widehat{N}_{\mathrm{sc}} \times \varprojlim_{x \mapsto x^n} Z(\widehat{N})^\circ \to \widehat{N}/Z(\widehat{G})^{\Gamma,\circ}$, where $N = N_{[\chi]}$ is a quasi-split twisted Levi subgroup of $G$ obtained from $\chi$ via (5) (as in the preceding easier example).

In summary, the "non-basic enhancements" on the Galois side for a fixed representative $\phi$ of $[\phi]$ are pairs $([\chi], \rho)$ as above, considered up to $\pi_0(S_\phi)$-conjugacy; in fact, one can show (cf. §4.2) that the family of cocycles $\{\natural_{[\chi]}\}_{[\chi]}$ along with the action of $\pi_0(S_\phi)$ on the aforementioned set of highest weights allows one to quotient the set of all pairs $([\chi], \rho)$ by the $\pi_0(S_\phi)$-action to form the *twisted extended quotient* $(X^+_\phi(\widehat{G}) /\!/ \pi_0(S_\phi))_\natural$ (as defined, e.g., in [AMS18, §1]), a representation-theoretic object which generalizes Clifford theory, where $X^+_\phi(\widehat{G})$ is the set of equivalences classes of highest weights. As in the previous example, we must take the limit over $\widehat{G}$-conjugacy (any $g^\vee \in \widehat{G}$ induces a canonical bijection between the two twisted extended quotients), and the resulting set is denoted by $[(X^+_\phi(\widehat{G}) /\!/ \pi_0(S_\phi))_\natural]$.

1.1.7. *Future directions.* The story is far from finished. The next task is to carry out a similar analysis for arbitrary nondiscrete $L$-parameters, which will have a different relationship with the non-basic correspondence of [BMO23]. We intend to return to return to this matter in a sequel paper.

In a different direction, we hope that the constructions of this paper can be extended to arbitrary regular cohomology classes, or even more optimistically, the algebraic classes, rather than the Levi-regular classes. §4.3.5 and the introduction to § 4 reflect on this problem.

Although our definition of rigid enhancements has the correct formal properties, we do not know how to interpret the enhancement as a more classical object of geometric representation theory, such as a sheaf on a moduli stack of $L$-parameters. Identifying such an interpretation would, we hope, point the way to an extension of the geometrization program in the rigid setting.

Finally, even though this paper assumes that the characteristic of $F$ is zero, everything written below should hold for an arbitrary nonarchimedean local field $F$, replacing certain group-cohomological arguments with their Čech-cohomological analogues and using the theory of the Kaletha gerbe over local function fields developed in [Dil23]. We chose to avoid this level of generality for expository clarity.

1.2. **Overview.** We begin in §2 by defining the cohomology set $H^1_{\mathrm{reg}}(\mathcal{E}_{\mathrm{Kal}}, G)$ and studying some of its basic properties with special focus on the $G$-conjugacy classes of homomorphisms $u \to G$ obtained by restricting cocycles of $\mathcal{E}_{\mathrm{Kal}}$ to $u$. In particular, we provide a coordinatization of these homomorphisms using alcoves and also give a rough combinatorial description of their centralizers using intersections of $\Gamma$-orbits of facets. The other main construction related to $H^1_{\mathrm{reg}}(\mathcal{E}_{\mathrm{Kal}}, G)$ is the so-called "rigid Kottwitz map," an injective map from this cohomology set to a linear algebraic object $\bar{\mathfrak{Y}}_{+,\mathrm{tor}}(G)$ which extends the Tate-Nakayama isomorphism on $H^1_{\mathrm{bas}}(\mathcal{E}_{\mathrm{Kal}}, G)$ given in [Kal16b]; this construction is the main focus of §3. The main strategy for defining this map is combining its analogue in the basic case with techniques from non-abelian cohomology.

In §4 this article turns its focus to the local Langlands correspondence, beginning by recalling some basic notions, mainly related to double covers and their associated $L$-embeddings. The heart of the paper is §4.2 which starts in §4.2.1 with a generalization of the (basic) rigid refined local Langlands to double covers (and their inner forms). After this in §§4.2.2, 4.2.3, 4.2.4 we define the non-basic rigid enhancements that appear on the Galois side of the extended rigid local Langlands correspondence. The main technical difficulties are defining an equivalence relation on the 'highest weights" mentioned above and defining the family of cocycles $\{\natural_{[\chi]}\}$ for the set of highest weights. In §4.3 we use these enhancements to state and prove the extended rigid refined local Langlands correspondence, using Clifford theory for central extensions of finite groups and the basic rigid local Langlands correspondence for (double covers of) twisted Levi subgroups of $G$. To give some examples, in §4.4 we discuss the relationship between the aforementioned extended correspondence and Kaletha's explicit constructions in [Kal19] and [Kal21b]. Finally, Appendix A is a short section dedicated to a variety of examples of phenomena that can occur in the above setting in order to demonstrate the necessity of making certain technical assumptions.

1.3. **Notation.** In the following, $F$ is a nonarchimedean local field of characteristic zero with fixed algebraic closure $\overline{F}$ with absolute Galois group $\Gamma$ and Weil group $W_F$. When we write $H^1(F, G)$ for a finite type $F$-group scheme $G$ we always mean $H^1_{\mathrm{fppf}}(F, G)$; for us, $G$ will always be either reductive or commutative, and is thus always smooth, so we could just as well write $H^1_{\text{ét}}(F, G)$ or $H^1(\Gamma, G(\overline{F}))$. For a group $\Gamma'$ acting on a (potentially non-abelian) group $\mathsf{G}$, a 1-cocycle $x \in Z^1(\Gamma', \mathsf{G})$ and $g \in \mathsf{G}$, set $x * dg := [\sigma \mapsto gx(\sigma)({}^{\sigma}g^{-1})]$, and call it the *twist of $x$ by the coboundary of $g$*.

For an affine algebraic group $G$ over $F$, we denote by $Z(G)$ the center of $G$, by $G^{\circ}$ its identity component, by $\pi_0(G)$ the quotient $G/G^{\circ}$, by $G_{\mathrm{der}}$ the derived subgroup of $G$; if $G$ is semisimple, we denote by $G_{\mathrm{sc}}$ its simply-connected cover and for general connected reductive $G$ we denote by $G_{\mathrm{sc}}$ the group $(G_{\mathrm{der}})_{\mathrm{sc}}$. For $H \subseteq G$ a subgroup scheme we denote by $Z_G(H)$, $N_G(H)$ the (scheme-theoretic) centralizer and normalizer of $H$, respectively, and for an $F$-rational homomorphism $G \xrightarrow{f} G'$ we denote by $Z_G(f)$ the centralizer of the scheme-theoretic image of $f$. Whenever we say "a maximal torus of $G$" we mean an $F$-rational maximal torus of $G$; an elliptic maximal torus of $G$ is a maximal torus of $G$ such that $T/Z(G)$ is anisotropic. For a maximal torus of $T$ of $G$, we denote by $T_{\mathrm{sc}}$ the preimage of $T$ in $G_{\mathrm{sc}}$.

For a finite Galois subextension $E/F$ we denote by $\Gamma_{E/F}$ the Galois group of $E/F$. We denote by $\widehat{G}$ a dual group of connected reductive $G$ and will frequently conflate reductive groups over $\mathbb{C}$ with their $\mathbb{C}$-points; for example, we write $\mathrm{SL}_2$ rather than $\mathrm{SL}_2(\mathbb{C})$ (in the context of the Galois side of the Langlands correspondence). The Weil Deligne group $W_F \times \mathrm{SL}_2$ of $F$ is denoted by $W'_F$, and an $L$-parameter for $G$ (or for $\widehat{G}$) is a homomorphism $W'_F \to {}^L G$ such that $W_F$ has semisimple projection to $\widehat{G}$ and is a morphism of $W_F$-extensions.

Finally, we list here some non-standard notation that is used heavily in this paper for the reader's convenience (in particular, all of these notations will be repeated later): For $H \subseteq G$ a subgroup scheme, $K_{H,G}$ denotes $\{g \in G(\overline{F}) \mid {}^{\sigma}g^{-1} \cdot g \in H(\overline{F}),\ \forall \sigma \in \Gamma\}$. Let $G$ be a connected reductive group—if $\mathscr{M}$ is a Levi subgroup of $\widehat{G}$, we denote by $\widetilde{\mathscr{M}}$ the cover

$$\mathscr{M}_{\mathrm{sc}} \times \varprojlim_{m \mapsto m^n} Z(\mathscr{M})^{\circ} \to \mathscr{M}$$

given by the usual map on the left direct factor and projection to the first coordinate on the right direct factor; if $V \subseteq \widehat{G}$ is an arbitrary subgroup we denote by $V^+$ its preimage in $\widetilde{\widehat{G}}$ and if $V \subseteq \mathscr{M}$ we denote by $V^+_{(\mathscr{M})}$ its preimage in $\widetilde{\mathscr{M}}$. Now fix a twisted Levi subgroup $M \subset G$; it is easy to show (see the discussion immediately after Notation 4.6 for details) that there is a canonical embedding of $Z(\widehat{G})$ in $\widehat{M}$—denote by $\widetilde{\widehat{M}}$ the group $\widehat{M}_{\mathrm{sc}} \times \varprojlim_{m \mapsto m^n} Z(\widehat{M})^{\circ} \to \widehat{M}$ (which can also be identified with $\varprojlim_A \widehat{M/A}$, where the limit is over any cofinal system of finite central subgroups $A$ of $M$) and for any subgroup $V \subseteq Z(\widehat{G})$ set $V^+_{(M)}$ to be the preimage of $V$ in $\widetilde{\widehat{M}}$.

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## 2. Non-basic $\mathscr{E}$-cohomology

2.1. **Preliminaries.** Fix cofinal systems $\{n_k\}_{k \geq 0}$ of natural numbers (this means that $n_k \mid n_{k+1}$ for all $k$) and nested finite Galois extensions $\{E_k/F\}_{k \geq 0}$; consider the profinite group scheme

$$u := \varprojlim_k \frac{\mathrm{Res}_{E_k/F}(\mu_{n_k})}{\mu_{n_k}},$$

where the transition maps are defined by the morphism of $\Gamma$-modules

$$\mathbb{Z}/n_k\mathbb{Z}[\Gamma_{E_k/F}]_0 \to \mathbb{Z}/n_{k+1}\mathbb{Z}[\Gamma_{E_{k+1}/F}]_0, \quad \sum c_{\gamma}[\gamma] \mapsto \sum_{\gamma} \sum_{\sigma \mapsto \gamma} (n_{k+1}/n_k) c_{\gamma}[\sigma]$$

where the subscript "0" denotes the kernel of the augmentation map. At the level of group schemes the above maps are given by the $E_{k+1}/E_k$-norm followed by the $n_{k+1}/n_k$-power map. Set $u_k := \mathrm{Res}_{E_k/F}(\mu_{n_k})/\mu_{n_k}$ and denote the map $u_l \to u_k$ by $p_{l,k}$ and $u \to u_k$ by $p_k$.

For $Z$ a finite multiplicative group scheme over $F$ which is $n_k$-torsion and split by $E_k/F$ (for $k \gg 0$), we have the following characterization of $u$:

**Lemma 2.1.** *There is a canonical isomorphism*

$$\mathrm{Hom}(u_k, Z) \xrightarrow{\sim} \mathrm{Hom}(\mu_{n_k,\overline{F}}, Z_{\overline{F}})^{N_{E_k/F}},$$

*where the superscript "$N_{E_k/F}$" denotes all elements killed by the $\Gamma_{E_k/F}$-norm. At the level of character modules, this isomorphism is given by*

$$[X^*(Z) \xrightarrow{f} X^*(u_k)] \mapsto [x \mapsto f(x) = \sum_{\gamma} c_\gamma[\gamma] \mapsto c_e].$$

In particular, we can deduce:

**Corollary 2.2.** *For $k \gg 0$ depending on a fixed $Z$ as above, there is a canonical isomorphism*

$$\mathrm{Hom}_F(u_k, Z) \xrightarrow{\sim} \mathrm{Hom}(\mu_{n_k,\overline{F}}, Z_{\overline{F}}).$$

*In particular, there is some $k \gg 0$ such that the natural map*

$$\mathrm{Hom}(u_k, Z) \to \mathrm{Hom}(u, Z)$$

*is an isomorphism.*

*Proof.* We prove the second statement. Any map $u \xrightarrow{f} Z$ factors through some $u_l \xrightarrow{f_l} Z$, which corresponds to some map $\mu_{n_l,\overline{F}} \xrightarrow{g_l} Z_{\overline{F}}$. Finding $k \le l$ (without loss of generality, since we can always enlarge $l$) such that the first part of the Corollary holds for $k$ and $Z$ is $n_k$-torsion, we see that $\mu_{n_l,\overline{F}} \to Z_{\overline{F}}$ factors through the natural map $\mu_{n_l,\overline{F}} \xrightarrow{n_l/n_k} \mu_{n_k,\overline{F}} \xrightarrow{g_k} Z_{\overline{F}}$. It follows that $f_l = f_k \circ p_{l,k}$ (with $f_k \in \mathrm{Hom}(u_k, Z)$) and therefore $f = f_k \circ p_k$. □

The cohomology of $u$ has a nice description using local class field theory, as proved in [Kal16b, Theorem 3.1]:

**Proposition 2.3.** *We have $H^1(F, u) = 0$ and a canonical isomorphism $H^2(F, u) \xrightarrow{\sim} \widehat{\mathbb{Z}}$.*

This leads us to define:

**Definition 2.4.** For any cocycle $\xi \in Z^2(\Gamma, u(\overline{F}))$ with image $-1 \in H^2(F, u)$, we define the *Kaletha gerbe $\mathscr{E}$* to be the gerbe $\mathscr{E}_\xi \to \mathrm{Sch}/\mathrm{Spec}(F)$.

For ease of calculations, since we are in mixed characteristic we may view $\mathscr{E}$ as a group extension

$$0 \to u(\overline{F}) \to \mathscr{E} \to \Gamma \to 1.$$

The gerbe $\mathscr{E} \to \mathrm{Sch}/\mathrm{Spec}(F)$ inherits the fpqc topology from the base, and it thus makes sense, for a finite type affine algebraic group $G$ over $F$, to consider fpqc $G_{\mathscr{E}}$-torsors on $\mathscr{E}$, which we will simply refer to as *$G$-torsors on $\mathscr{E}$*; the set (the reader can verify that this is indeed a set) of such torsors is denoted by $Z^1(\mathscr{E}, G)$ and isomorphism classes by $H^1(\mathscr{E}, G)$. Under the dictionary between $\mathscr{E}$ and the associated group extension (cf. [KT22, §7.1]), these correspond to 1-cocycles $z \in H^1(\mathscr{E}, G(\overline{F}))$ such that $z|_{u(\overline{F})}$ is an algebraic homomorphism.

Although the choice of $\xi$ representing $-1 \in H^2(F, u)$ is far from canonical, for any two choices $\xi, \xi'$ there is an isomorphism (also non-canonical) of $u$-gerbes $\mathscr{E}_\xi \to \mathscr{E}_{\xi'}$ such that the induced bijection $H^1(\mathscr{E}_{\xi'}, G) \to H^1(\mathscr{E}_{\xi'}, G)$ is canonical, due to the vanishing of $H^1(F, u)$. For this reason, it is harmless to fix a representative $\xi$ once and for all and set $\mathscr{E} = \mathscr{E}_\xi$.

2.2. **New cohomology sets.** We now take $G$ to be a connected reductive group over $F$. Restricting a representative $x$ of a class $[x] \in H^1(\mathscr{E}, G)$ to $u$ yields a homomorphism $u_{\overline{F}} \xrightarrow{f_x} G_{\overline{F}}$ (from now on, when we say "homomorphism" we mean an algebraic morphism), and picking a different representative $x'$ which differs from $x$ by a $g \in G(\mathscr{E}) = G(\overline{F})$-coboundary yields $f_{x'} = \mathrm{Ad}(g) \circ f_x$. The fact that $x$ is a 1-cocycle implies that the $G(\overline{F})$-conjugacy class of $f_x$ is defined over $F$. Recall the following fundamental definition introduced in [Kal16b]:

**Definition 2.5.** We say that a cohomology class $[x] \in H^1(\mathscr{E}, G)$ is *basic* if there is a representative $x$ of $[x]$ such that $f_x$ factors through $Z(G)$ (it is then automatically defined over $F$, and the same properties hold for any representative of $[x]$). We denote the set of such classes by $H^1_{\mathrm{bas}}(\mathscr{E}, G)$.

There are three ways to generalize the above Definition which we will focus on in this article.

**Definition 2.6.** We define the set of *regular classes* in $H^1(\mathscr{E}, G)$, denoted by $H^1_{\mathrm{reg}}(\mathscr{E}, G)$, to be the image of

$$\bigsqcup_{S \subset G} H^1(\mathscr{E}, S) \to H^1(\mathscr{E}, G),$$

where the union is over all $F$-rational maximal tori of $G$.

**Remark 2.7.** It is not true that every algebraic cocycle is regular: in Lemma A.2, for the group $G = \mathrm{PGL}_2$ we construct an algebraic $x \in Z^1(\mathscr{E}, G)$ such that $Z_G(f_x)$ is the symmetric group on four letters. We exclude such cocycles because they are radically different from the regular ones. However, it would be extremely interesting to find a role for these cocycles in the local Langlands correspondence.

For $[x] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$ with representative $x \in Z^1(\mathscr{E}, S)$ for some $F$-rational maximal torus $S$, the map $f_x$ is automatically an $F$-rational map from $u$ to $S$. If we choose any other representative $x' \in [x]$ such that $f_{x'}$ defined over $F$, then $f_{x'}$ also factors through an $F$-rational maximal torus of $G$, since $Z_G(f_{x'})$ is a (possibly disconnected) reductive group such that $Z_G(f_{x'})^\circ$ contains an $F$-rational maximal torus $T$ of $G$. Indeed, we know that $Z_G(f_x) \hookrightarrow G$ is a closed subgroup which is $G(\overline{F})$-conjugate to $Z_G(f_{x'})$, and hence the latter is also reductive and $Z_G(f_{x'})^\circ$ has the same rank as $Z_G(f_x)^\circ$, which is just the rank of $G$. The result follows (since $u$ is commutative, the map $f_x$ always factors through $Z_G(f_x)$). Another elementary fact is:

**Lemma 2.8.** *Given $[x] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$ with representative $x$ and $Z_G(f_x)$ such that $u \xrightarrow{f_x} T$ for some $F$-rational maximal $T$, the cocycle $x$ takes values in $Z_G(f_x)(\overline{F})$.*

*Proof.* Using that $f_x = x|_u$ is defined over $F$, for $u \in u(\overline{F})$ and $w \in \mathscr{E}$ we have

$$x(w)x(u) = x(w)[{}^w x(w^{-1}uw)] = x(w)[{}^w x(w^{-1})][{}^{w^{-1}w} x(uw)] = x(e)x(u)[{}^u x(w)] = x(u)x(w).$$

□

**Warning 2.9.** It will not be true in general that, in the setting of Lemma 2.8, the cocycle $x$ takes values in $Z_G(f_x)^\circ(\overline{F})$.

Shapiro's Lemma tells us that an $F$-rational morphism $u \xrightarrow{f} T$ corresponds to the $\Gamma$-orbit of a morphism $\mu_{n_k, \overline{F}} \xrightarrow{f_e} T_{\overline{F}}$ (where we write "$f_e$" to denote the fact that we are using the morphism coming from the identity coordinate of $\prod_{\gamma\in\Gamma_{E_k/F}}(\mu_{n_k,\overline{F}})_\gamma \xrightarrow{f} T_{\overline{F}})$ for some $k \gg 0$. It is clear that in this case we have

$$Z_G(f_x)_{\overline{F}} = \bigcap_{\gamma\in\Gamma_{E_k/F}} Z_{G_{\overline{F}}}({}^\gamma f_e) = \bigcap_{\gamma\in\Gamma_{E_k/F}} {}^\gamma Z_{G_{\overline{F}}}(f_e).$$

It is then immediate that the (absolute) root system of $Z_G(f_x)^\circ$ (with respect to the maximal torus $T$) is given by the following $\Gamma$-stable subsystem of $R(G_{\overline{F}}, T_{\overline{F}})$:

$$R(Z_G(f_x)^\circ_{\overline{F}}, T_{\overline{F}}) = \bigcap_{\gamma\in\Gamma_{E_k/F}} {}^\gamma R(Z_{G_{\overline{F}}}(f_e)^\circ, T_{\overline{F}}).$$

Although the root system $R(Z_{G_{\overline{F}}}(f_e)^\circ, T_{\overline{F}})$ has a classical combinatorial description in terms of extended bases (discussed later), the above intersection construction is not directly amenable to such methods, since $T$ may not be contained in an $F$-rational Borel subgroup of $G$.

**Definition 2.10.** We call a subgroup of the form $H = Z_G(f_x)$ for $[x] \in H^1_{\text{reg}}(\mathscr{E}, G)$ and $f_x$ an $F$-rational morphism a *rigid Newton centralizer (in $G$)*. We will study (but not completely classify) which subgroups arise in this manner in §3.4.

A second strengthening is:

**Definition 2.11.** We say that a class $[x] \in H^1_{\text{reg}}(\mathscr{E}, G)$ is *cyclic* if it contains a representative $x$ such that $\nu_{x,e}$ (defined as the restriction of $f_x$ to the identity coordinate $\mu_{n_k,\overline{F},e}$ as explained immediately above) is such that the inclusion $Z_{G_{\overline{F}}}(f_{x,e}) \to G_{\overline{F}}$ is defined over $F$. Denote the corresponding subset of $H^1_{\text{reg}}(\mathscr{E}, G)$ by $H^1_{\text{cyc}}(\mathscr{E}, G)$. The reason for the name "cyclic" is that the the subgroups $Z_G(f_x)$ are the centralizers of cyclic subgroups of $G(\overline{F})$ (consisting of semisimple elements).

Observe that if $f_x$ is as in Definition 2.11 then the natural inclusion $Z_G(f_x)_{\overline{F}} \hookrightarrow Z_{G_{\overline{F}}}(f_{x,e})$ is an equality; in particular, $Z_G(f_x)^\circ$ has a reasonable combinatorial description in terms of centralizers of ($\overline{F}$-rational) torsion elements, which is the main motivation for considering such objects. Definition 2.11 behaves well across cohomology classes:

**Lemma 2.12.** *For $[x] \in H^1_{\text{cyc}}(\mathscr{E}, G)$ with representative $x$ such that $f_x$ defined over $F$, the inclusion $Z_{G_{\overline{F}}}(\nu_{x,e}) \to G_{\overline{F}}$ is always defined over $F$.*

*Proof.* One always has the inclusion

$$Z_G(f_x)_{\overline{F}} \hookrightarrow Z_{G_{\overline{F}}}(f_{x,e}), \tag{6}$$

where by assumption the inclusion $Z_G(f_x)_{\overline{F}} \hookrightarrow G_{\overline{F}}$ is defined over $F$. It is easy to see that replacing $f_x$ by $f_y = \text{Ad}(g) \circ f_x$ replaces $f_{x,e}$ by $\nu_{y,e} = \text{Ad}(g) \circ f_{x,e}$, and by assumption the inclusion $Z_G(f_y)_{\overline{F}} \hookrightarrow Z_{G_{\overline{F}}}(f_{y,e})$ is an isomorphism for some $y$ with $[y] = [x]$. It follows that the inclusion (6) becomes an equality after conjugating by $g^{-1}$, and is thus itself an equality, whence the desired result. □

**Corollary 2.13.** *A class $[x] \in H^1_{\text{cyc}}(\mathscr{E}, G)$ always has a representative $x$ valued in some $F$-rational maximal torus $T$ such that $f_x$ satisfies the cyclic property.*

The point of the above Corollary is that, for a cyclic class $[x]$, there is always a representative witnessing both the regular and cyclic properties of $[x]$. Another potential strengthening which we will exclusively work with for applications to the local Langlands correspondence in §4 is:

**Definition 2.14.** We say that a class $[x] \in H^1(\mathscr{E}, G)$ is *Levi-regular* if it contains a representative $x$ such that $Z_G(f_x)$ is a twisted Levi subgroup of $G$ (recall that this means that it is $\overline{F}$-rationally a Levi subgroup). Denote the corresponding subset of $H^1_{\text{reg}}(\mathscr{E}, G)$ by $H^1_{\text{L-reg}}(\mathscr{E}, G)$.

One can show that cyclic does not imply Levi-regular in general, as in Example 3.21 (although this implication can sometimes hold, depending on the root system), and it is also the case that $L$-regular does not in general apply cyclic. For example, if $Z_{G_{\overline{F}}}(f_{x,e}) = L \subset G_{\overline{F}}$ is a Levi subgroup which does not descend to $F$, in which case $Z_G(f_x) = \bigcap_\gamma {}^\gamma L$ is a twisted Levi subgroup (the intersection of Levi subgroups containing a common maximal torus is always a Levi subgroup) that is evidently smaller than $Z_{G_{\overline{F}}}(f_{x,e})$. We again have a Lemma of a familiar flavor:

**Lemma 2.15.** *For $[x] \in H^1_{\text{L-reg}}(\mathscr{E}, G)$ with representative $x$ such that $f_x$ defined over $F$, the group $Z_G(f_x) \hookrightarrow G$ is a twisted Levi subgroup. In other words, any representative 1-cocycle with $F$-rational homomorphism to $T$ witnesses the Levi-regular property of $[x]$.*

*Proof.* This is clear. □

We have already stated this fact in some proofs, but it is worth stating explicitly:

**Lemma 2.16.** *For $[x] \in H^1_{\text{reg}}(\mathscr{E}, G)$ with representative $x$, the subgroup $Z_G(f_x)$ is a (possibly disconnected) reductive group of the same rank as $G$.*

We now examine how the group $H := Z_G(f_x)$ varies across all representatives $x$ in a fixed class $[x] \in H^1_{\text{reg}}(\mathscr{E}, G)$. Denote by $K_{H,G}$ the set of $g \in G_{\text{ad}}(\overline{F})$ such that $\text{Ad}(g) \circ f_x$ is defined over $F$, which is the same as all $g \in G(\overline{F})$ such that $({}^\sigma g^{-1})g \in H(\overline{F})$ for all $\sigma \in \Gamma$. We record some basic properties of $K_{H,G}$ which will be used repeatedly later:

**Lemma 2.17.** *For a class $[x] \in H^1_{\text{reg}}(\mathscr{E}, G)$, fix a representative $x$ with $u \xrightarrow{f_x} T \to G$ an $F$-rational morphism to an $F$-rational maximal torus $T$ of $G$ and set $H := Z_G(f_x)$. Then:*

1. *The set of all $H'$ which arise as $Z_G(f_{x'})$ for $x' \in [x]$ is given by the set $K_{H,G} \cdot H$ (acting by conjugation);*
2. *If $H' \in K_{H,G} \cdot H$ then $K_{H',G} \cdot H' = K_{H,G} \cdot H$;*
3. *We have the equality $K_{N_H(T),G} \cdot H(\overline{F}) = K_{N_H(T),G} \cdot H^\circ(\overline{F}) = K_{H,G}$;*
4. *For any $x' \in [x]$ and $F$-rational maximal $T'$ through which $f_{x'}$ factors, the tori $T'$ and $T$ are $K_{N_H(T),G}$ (even $K_{N_{H^\circ}(T),G}$)-conjugate.*

*Proof.* The first two points are obvious. For the third, if $k \in K_{H,G}$ then ${}^kT$ is a maximal torus of ${}^kH$; the latter is an $F$-rational subgroup of $G$ and thus we may find $z' = {}^kz \in {}^kH^\circ(\overline{F})$ (where $z \in H^\circ(\overline{F})$) such that ${}^{z'k}T$ is defined over $F$, which holds if and only if $kz \in K_{N_H(T),G}$, and thus $k = kz \cdot z^{-1}$, as desired. The fourth statement follows easily from a similar argument to the third. □

In this vein, we will need the following notions to allow for the disconnectedness of $H$. In order to study $K_{H^\circ,G} \subseteq K_{H,G}$, we define an equivalence relation on $K_{H,G}$ by declaring $g_1 \sim g_2$ if $g_2 g_1^{-1} \in K_{{}^{g_1}H^\circ,G}$, which has the alternative description:

**Lemma 2.18.** *For $g_1, g_2 \in K_{H,G}$ we have $g_1 \sim g_2$ if and only if $K_{{}^{g_1}H^\circ,G} \cdot g_1 = K_{{}^{g_2}H^\circ,G} \cdot g_2$ (note that both of these two sets are contained in $K_{H,G}$). In particular, we can write*

$$K_{H,G} = \bigsqcup_{i \in I',\circ} K_{{}^{g_1}H^\circ,G} \cdot g_i'$$

*for some finite set $\{g'_i\}_{i\in I',\circ}$ of equivalence class representatives.*

*Proof.* Everything except for the finiteness claim is an elementary computation. For finiteness we observe that lifting $\bar{g} \in (G/H)(F)$ to $\tilde{g} \in (G/H^\circ)(\overline{F})$ and then taking the differential $d$ gives an element of $Z^1(F, \pi_0(H))$. There is a composition

$$K_{H,G} \to (G/H^0)(\overline{F})^{(F)} \to (G/H)(F) \xrightarrow{d} H^1(F, \pi_0(H)), \tag{7}$$

where $(G/H^0)(\overline{F})^{(F)}$ denotes the preimage of $(G/H)(F)$ in $(G/H^0)(\overline{F})$. Then for $x \in K_{H,G}$ the equivalence class defined above containing $x$ is the fiber in $K_{H,G}$ of (7) through $x$. The result follows from the finiteness of $H^1(F, \pi_0(H))$. □

Using the description of $\sim$ from the proof of Lemma 2.18 (in terms of the fibers of the map (7)), it makes sense to restrict the equivalence relation $\sim$ to the subset $K_{N_H(T),G} \subset K_{H,G}$ (where $T$ is an $F$-rational maximal torus of $G$), and we denote the corresponding set of equivalence class representatives by $\{g_i\}_{i\in I^\circ}$ (by part (4) of Lemma 2.17, these equivalence classes and representatives do not depend on the choice of $T$).

It will also be useful for Section 3 to understand the uniqueness of a representative $x \in Z^1(\mathscr{E}, T) \subset Z^1(\mathscr{E}, G)$ witnessing the regularity of a class $[x] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$. For any $x' = x * dg$, since $x(\sigma) \in Z_G(f_x)^\circ(\overline{F})$ for all $\sigma \in \mathscr{E}$, we have $f_{x'} = \mathrm{Ad}(g) \circ f_x$ and $x'(\sigma) = gx(\sigma)\, {}^\sigma g^{-1} = gx(\sigma)g^{-1}(g\, {}^\sigma g^{-1})$, which lies in $Z_G(f_{x'})^\circ(\overline{F}) = \mathrm{Ad}(g)[Z_G(f_x)](\overline{F})$ if and only if $dg(\sigma) = g\, {}^\sigma g^{-1} \in Z_G(f_{x'})^\circ(\overline{F})$, which is equivalent to ${}^\sigma g^{-1} g \in Z_G(f_x)^\circ$ (that is, $g \in K_{Z_G(f_x)^\circ,G}$).

Moreover, once we know that $x'$ lies in $Z^1(\mathscr{E}, Z_G(f_{x'})^\circ)$ (and thus $Z^1_{\mathrm{bas}}(\mathscr{E}, Z_G(f_{x'})^\circ)$), it follows from [Kal16b, Corollary 3.7] that there is some elliptic maximal torus $S \subset Z_G(f'_x)^\circ$ such that $x'$ is cohomologous to some $y \in Z^1(\mathscr{E}, S)$ via $h \in Z_G(f_{x'})^\circ(\overline{F})$. Since $h \in Z_G(f_{x'})^\circ(\overline{F})$ and $Z_G(f_{x'})^\circ = \mathrm{Ad}(g)[Z_G(f_{x'})^\circ)]$, we have $hg \in K_{Z_G(f_x)^\circ,G}$. We see in particular that any other $y \in Z^1(\mathscr{E}, S)$ representing $[x]$ is a $K_{Z_G(f_x)^\circ,G}$-translate of the original $x$ and conversely that any $K_{Z_G(f_x)^\circ,G}$-conjugate of $Z_G(f_x)$ contains a maximal torus $S$ through which a cocycle representing the class $[x]$ factors.

The following definition echoes the basic case:

**Definition 2.19.** For a fixed finite $F$-rational subgroup $A$ of $G$ contained in an $F$-rational (maximal) torus $T$, we define $H^1(\mathscr{E}, A \to G)$ as all classes $[x] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$ which are in the image of some $H^1(\mathscr{E}, A \to S)$ for $S$ an $F$-rational maximal torus of $G$ containing $A$. Equivalently, these are class in $H^1_{\mathrm{reg}}(\mathscr{E}, G)$ which have a representative $x$ with $f_x$ factoring $F$-rationally through $A$ and $x \in Z^1(\mathscr{E}, Z_G(f_x)^\circ)$.

2.3. **The Newton map.** Before going into the details, we explain the rough idea of what will happen in this subsection; consider a class $[x] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$ which for simplicity we assume for the moment is such that $Z_G(f_x)$ is connected. Choose some representative $x$ with $f_x$ an $F$-rational morphism factoring through some $F$-rational maximal torus $T$ of $G$.

As we saw above, restricting to the identity coordinate $f_{x,e}$ determines $f_x$, so the restriction of $[x]$ to the band is determined by the "torsion cocharacter" $f_{x,e} \in \mathrm{Hom}(\mu_{\overline{F}}, T_{\overline{F}})$, a straightforward combinatorial piece of data. As is the case with cocharacters, one can describe torsion cocharacters up to $G(\overline{F})$-conjugacy in a very explicit way by choosing a Borel subgroup containing $T_{\overline{F}}$.

We make the obvious observation that $f_x$ being determined by $f_{x,e}$ is only true when $f_x$ is $F$-rational, so we necessarily must restrict ourselves to this case. Also, changing the cocycle representative for $[x]$ replaces $f_x$ by a $G(\overline{F})$-conjugate, and so we can combine these observations and deduce that one can only modify $f_x$, and therefore $f_{x,e}$, by a $K_{Z_G(f_x),G}$-conjugate, not an arbitrary $G(\overline{F})$-conjugate. One can view $f_x$ as corresponding to the $\Gamma$-orbit $\{{}^\gamma f_{x,e}\}_{\gamma\in\Gamma}$, and we are only allowed to act on $f_{x,e}$ by elements of $G_{\mathrm{ad}}(\overline{F})$ whose action on $f_{x,e}$ is $\Gamma$-equivariant, and thus sends this orbit to the analogous orbit of ${}^g f_{x,e}$ in a way that that preserves $\Gamma$-coordinates.

These observations mean that finding an explicit combinatorial description of the appropriate codomain for "the image of $[x]$ under restriction to the band" is involved, primarily because the image of a given $[x]$ is defined up to conjugacy by a subset of $G(\overline{F})$ (namely, $K_{Z_G(f_x),G}$) which itself depends in a non-obvious way on $[x]$.

In §2.3.2 we use the combinatorics of torsion cocharacters to interpret the image of $[x]$ under the "restriction to the band" operation. When $[x]$ is cyclic, this produces a point $\bar{y}_x \in X_*(T)_{\mathbb{Q}/\mathbb{Z}}$ in the image of a base alcove determined by some carefully-chosen (non-unique) Borel subgroup $B$ containing $T_{\overline{F}}$ which is independent of the choice of $B$ and whose $K_{Z_G(f_x),G}$-conjugacy class (in particular, we allow the torus $T$ to vary) is canonically-associated to $[x]$, yielding a rough coordinatization of the image of $[x]$. For general $[x]$, one has to consider $Z_G(f_x)$-orbits of alcoves inside a $Z_G(f_{x,e})^\circ$-orbit, which results instead in a tuple $\{\bar{y}_{x,i}\}_i \subseteq X_*(T)_{\mathbb{Q}/\mathbb{Z}}$ of points as above whose $K_{Z_G(f_x),G}$-conjugacy class is again canonically associated to $[x]$.

Another natural question is how to compute the centralizer $Z_G(f_x)$ or $K_{Z_G(f_x),G} \cdot Z_G(f_x)$ (again, recall that for this discussion we assume connectedness) using the torsion cocharacter $f_{x,e}$. We give a crude answer in §2.3.3, using subsets of $X_*(T)_{\mathbb{Q}/\mathbb{Z}}$ obtained from intersecting the $\Gamma$-orbits of facets in (the image of) base alcoves. These intersections of facets are not uniquely determined by the subgroup of $G$ that they produce (whose root system consists of all roots pairing to $0 \in \mathbb{Q}/\mathbb{Z}$ with all $\mathbb{Q}/\mathbb{Z}$-coweights in this subset), which means that one needs to group together all facet intersections (up to conjugacy) which yield the same collection of subgroups $K_{H,G} \cdot H$.

2.3.1. *Torsion cocharacters.* For $[x] \in H^1(\mathscr{E}, G)$, we are interested in the image of $f_x$ (for $x \in [x]$) in the quotient

$$\frac{\mathrm{Hom}_F^{\mathrm{reg}}(u, G)}{G(\overline{F})\text{-conjugacy}},$$

where $\mathrm{Hom}_F^{\mathrm{reg}}(u, G)$ denotes the group of $F$-rational homomorphisms $u \to G$ which factor through an $F$-rational maximal torus and when we write the quotient by $G(\overline{F})$-conjugacy we mean inside the larger quotient set $\frac{G(\overline{F})\cdot \mathrm{Hom}_F^{\mathrm{reg}}(u,G)}{G(\overline{F})\text{-conjugacy}}$ (which is equivalent to $K_{Z_G(f_x),G}$-conjugacy for some fixed representative $x$).

In fact, if we choose a representative $x$ of $[x]$ such that $x \in Z^1(\mathscr{E}, Z_G(f_x)^\circ)$ (which is possible, by definition) then $x$ with this property is unique up to $K_{Z_G(f_x)^\circ,G}$-conjugacy, so in fact if we work with this subset of representatives then one can consider the image of $f_x$ in $\mathrm{Hom}_F^{\mathrm{reg}}(u, G)$ modulo $K_{Z_G(f_x)^\circ,G}$-conjugacy (this turns out to be a more useful notion).

Unlike in the isocrystal case, the choice of $F$-rational maximal torus matters here:

**Definition 2.20.** For an $F$-rational maximal torus $T$, define $H^1(\mathscr{E}, G)_T$ as the image of $H^1(\mathscr{E}, T)$ in $H^1_{\mathrm{reg}}(\mathscr{E}, G)$.

There is a straightforward characterization of $H^1(\mathscr{E}, G)_T$:

**Lemma 2.21.** *A class $[x] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$ lies in $H^1(\mathscr{E}, G)_T$ if and only if it has a representative $x$ such that $f_x$ is an $F$-rational morphism factoring through $T$ and $x \in Z^1(\mathscr{E}, Z_G(f_x)^\circ)$.*

*Proof.* This follows readily from the discussion preceding Definition 2.19. □

In particular, when $Z_G(f_x)$ is not connected, to lie in $H^1(\mathscr{E}, G)_T$ it is not enough for $[x]$ to have a representative $x$ with $f_x$ an $F$-rational morphism factoring through $T$.

Inspired by this lemma, denote by $Z^1(\mathscr{E}, G)_T$ the set of all cocycles $x$ of $\mathscr{E}$ in $G(\mathscr{E}) = G(\overline{F})$ such that $f_x$ factors through $T$ and $x \in Z^1(\mathscr{E}, Z_G(f_x)^\circ)$—we alert the reader that $Z^1(\mathscr{E}, G)_T$ is not all possible representatives of cocycles in $H^1(\mathscr{E}, G)_T$, although the natural map $Z^1(\mathscr{E}, G)_T \to H^1(\mathscr{E}, G)_T$ is surjective. Similarly, denote by $Z^1_{\text{reg}}(\mathscr{E}, G)$ the set of all cocycles $x$ such that $x \in Z^1(\mathscr{E}, Z_G(f_x)^\circ)$ (every class in $H^1_{\text{reg}}(\mathscr{E}, G)$ has a representative in $Z^1_{\text{reg}}(\mathscr{E}, G)$ but not every representative for $[x]$ lies in $Z^1_{\text{reg}}(\mathscr{E}, G)$).

For a fixed $T$ as above and $[x] \in H^1(\mathscr{E}, G)_T$ with representative $x \in Z^1(\mathscr{E}, G)_T$, we have a homomorphism $\mu_{\overline{F}} \xrightarrow{f_{x,e}} T_{\overline{F}}$ in the identity coordinate which uniquely determines $f_x$. Choosing a different $x'$ satisfying the same properties has the effect of replacing $f_x$ by $f_{x'} = \text{Ad}(g) \circ f_x$ such that $f_{x'}$ is still an $F$-rational morphism factoring through $T$. Computing what this means, we see that for any $\gamma \in \Gamma$ there is the equality

$$ {}^g Z_G(f_x) = Z_G(f_{x'}) = {}^\gamma Z_G(f_{x'}) = {}^\gamma({}^g Z_G(f_x)) = {}^{\gamma g} Z_G(f_x), $$

which holds if and only if $g^{-1}\,{}^\gamma g \in Z_G(f_x)$. We can thus replace $g$ by a $Z_G(f_x)^\circ(\overline{F})$-translate in order to assume that $g \in N_G(T)(\overline{F})$. It follows that $f_x$ is unique up to $[N_G(T)/N_{Z_G(f_x)}(T)](F)$-conjugacy (acting via picking a preimage in $N_G(T)(\overline{F})$). We already saw in Section 2.2 that if we drop the requirement that $f_x$ factors through $T$ but still insist of $F$-rationality then the resulting map is unique up to $K_{Z_G(f_x),G}$-conjugacy.

Continue to fix $x \in Z^1(\mathscr{E}, G)_T$ along with $f_x$ and $f_{x,e} \in \text{Hom}(\mu_{\overline{F}}, T_{\overline{F}}) = X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$. We recall some basic root combinatorics: For a Borel subgroup $B$ of $G_{\overline{F}}$ containing $T_{\overline{F}}$ with base $\Delta = \{\alpha_i\}_{1 \leq i \leq r} \subset R(G_{\overline{F}}, T_{\overline{F}})$ and longest root $\alpha_0$, one can define the simplex

$$ \tilde{\Delta} := \{y \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q} \mid (y, \alpha_0) \leq 1, (y, \alpha_i) \geq 0, 1 \leq i \leq r\} $$

(sometimes also denoted by $\tilde{\Delta}_{(B,T)}$ If we want to emphasize the earlier choices). To pass from $y \in \tilde{\Delta}$ to a map $\mu_{\overline{F}} \to T_{\overline{F}}$ one applies the (injective) map $\tilde{\Delta} \hookrightarrow X_*(T)_{\mathbb{Q}} \to X_*(T)_{\mathbb{Q}/\mathbb{Z}}$. For $y \in \tilde{\Delta}$ corresponding to a morphism $\mu_{\overline{F}} \xrightarrow{\nu} T_{\overline{F}}$, the connected reductive group $Z_{G_{\overline{F}}}(\nu)^\circ$ has base $I_y \subset \Delta \cup \{-\alpha_0\}$, where $I_y = \{\alpha \in \Delta \cup \{-\alpha_0\} \mid (y, \alpha) \in \mathbb{Z}\}$. For such a simplex $\tilde{\Delta}$, denote by $\bar{\Delta}$ its image in $X_*(T)_{\mathbb{Q}/\mathbb{Z}}$.

**Lemma 2.22.** *There is some Borel subgroup $B$ of $G_{\overline{F}}$ containing $T_{\overline{F}}$ such that $f_{x,e}$ has a preimage in $X_*(T)_{\mathbb{Q}}$ which lies in $\tilde{\Delta}_{(B,T)}$. Moreover, any two choices of $B$ are $N_{Z_{G_{\overline{F}}}(f_{x,e})^\circ}(T)(\overline{F})$-conjugate (by an element which is necessarily unique up to translation by $T(\overline{F})$).*

*Proof.* An arbitrary lift $\bar{f}_{x,e} \in X_*(T)_{\mathbb{Q}}$ lies in (the closure of) some alcove $A \subset X_*(T)_{\mathbb{Q}}$ (see e.g. [Hum15]). Choosing an arbitrary Borel subgroup $B'$ of $G_{\overline{F}}$ containing $T_{\overline{F}}$ gives $\tilde{\Delta}_{(B',T)}$, which is the closure of another alcove $A'$. Since the extended affine Weyl group $\tilde{W}_a$ acts transitively on alcoves (we are just working with alcoves, so in fact we could work with the affine Weyl group $W_a := R(G_{\overline{F}}, T_{\overline{F}}) \rtimes W(G_{\overline{F}}, T_{\overline{F}})$ instead to obtain a simply transitive action, but that is not needed here), there is some $\tilde{w} \in \tilde{W}_a$ such that ${}^{\tilde{w}}(\bar{f}_{x,e}) \in \overline{A'}$. Writing $\tilde{w} = \lambda \rtimes w$ we have

$$ {}^w(\bar{f}_{x,e}) + \lambda \in \overline{A'}, $$

and, applying $w^{-1}$ to the above equation gives

$$\bar{f}_{x,e} + {}^{w^{-1}}\lambda \in {}^{w^{-1}}(\overline{A'}),$$

which gives the first statement, since ${}^{w^{-1}}(\overline{A'}) = \tilde{\Delta}_{({}^{w^{-1}}B,T)}$ and ${}^{w^{-1}}\lambda \in X_*(T) \subset X_*(T)_{\mathbb{Q}}$.

Consider a fixed lift $\bar{f}_{x,e} \in X_*(T)_{\mathbb{Q}}$ which lies in the closure of two alcoves $A_1$ and $A_2$ corresponding to Borel subgroups $B_1$ and $B_2$ (we call these *base alcoves*) of $G_{\overline{F}}$ containing $T_{\overline{F}}$. The parabolic subgroup $P_G(\bar{f}_{x,e})$ contains $B_1$ and $B_2$, which both contain $U_G(\bar{f}_{x,e})$, the unipotent radical of $P_G(\bar{f}_{x,e})$. It follows that, since the images of $B_1$ and $B_2$ are conjugate in the reductive quotient $P_G(\bar{f}_{x,e})/U_G(\bar{f}_{x,e})$, they are conjugate under the Levi factor $Z_G(\bar{f}_{x,e}) \subset Z_G(f_{x,e})^{\circ}$, as desired. Choosing a different lift $\bar{f}'_{x,e} = \bar{f}_{x,e} + \lambda' \in X_*(T)_{\mathbb{Q}}$, one can take the element $\tilde{w} \in \tilde{W}_a$ as above and set $\tilde{w}' := (\lambda - {}^{w}\lambda') \rtimes w$ and obtain ${}^{\tilde{w}'}(\bar{f}'_{x,e}) \in \overline{A'}$, and so in the end one recovers the same alcove ${}^{w^{-1}}(\overline{A'})$ as with $\bar{f}_{x,e}$ and therefore the same $B$. □

The above Lemma lets us view $f_{x,e}$ as in $\tilde{\Delta}_{(B,T)}$ for some $B$, since restricting the projection $X_*(T)_{\mathbb{Q}} \to X_*(T) \otimes \mathbb{Q}/\mathbb{Z}$ to $\tilde{\Delta}_{(B,T)}$ is injective. Once such a $B$ is fixed, we are free to take any of its $Z_G(f_x)(\overline{F})$ conjugates (if we insist that $f_x$ is unchanged), take any of its $(K_{Z_G(f_x)^{\circ},G}) \cap N_G(T)(\overline{F})$-conjugates (allowing $f_x$ to change to another $F$-rational morphism factoring through $T$), or even any of its $K_{N_{Z_G(f_x)^{\circ}}(T),G}$-conjugates (allowing $f_x$ to be any $F$-rational morphism coming from some $y \in Z^1_{\mathrm{reg}}(\mathscr{E}, G)$ with $[x] = [y]$).

**Remark 2.23.** A difficulty relevant to Lemma 2.22 is that, although any two choices of $B$ as in the Lemma are $Z_{G_{\overline{F}}}(f_{x,e})^{\circ}(\overline{F})$-conjugate, we are only allowed (assuming that the $K_{Z_G(f_x),G}$-conjugacy class of $f_x$ stays fixed) to conjugate by elements in $Z_G(f_x)(\overline{F})$, which may not act transitively on the aforementioned set of Borel subgroups. However, if $[x] \in H^1_{\mathrm{cyc}}(\mathscr{E}, G)$ then this is not a problem, since by definition the relevant centralizers coincide.

**Remark 2.24.** Using Lemma 2.22, one obtains from any $[x]$ with representative $x \in Z^1(\mathscr{E}, G)_T$ a subset $I_x \subseteq \Delta_{(B,T)} \cup \{-\alpha_0\}$ for any $(B,T)$ as in the aforementioned Lemma. Unlike in the isocrystal case, the group $Z_G(f_x)^{\circ}$ does not uniquely determine $I_x$ (in the sense that different $I_x$ and $I_{x'}$ can yield the same $Z_G(f_x)^{\circ}$), since one needs to take an intersection over the Galois translates of $Z_{G_{\overline{F}}}(f_{e,x})$ to form $Z_G(f_x)$ (see §2.3.3 for another description of these connected centralizers). This is true, however, if we restrict ourselves to $[x] \in H^1_{\mathrm{cyc}}(\mathscr{E}, G)$.

2.3.2. *Defining the Newton map.* We now define the first version of the "rigid Newton map," which will focus on some fixed $F$-rational maximal torus $T$ of $G$.

For $[x] \in H^1(\mathscr{E}, G)_T$ with fixed representative $x \in Z^1(\mathscr{E}, G)_T$ we have the homomorphism $f_{x,e}$ and from Lemma 2.22 a canonical $N_{Z_{G_{\overline{F}}}(f_{x,e})^{\circ}}(T)(\overline{F})$-orbit of Borel subgroups $B$ of $G_{\overline{F}}$ containing $T_{\overline{F}}$ such that $\bar{f}_{x,e} \in \tilde{\Delta}_{(B,T)}$ for $\bar{f}_{x,e} \in X_*(T)_{\mathbb{Q}}$ some preimage of $f_{x,e}$.

Denote this $N_{Z_{G_{\overline{F}}}(f_{x,e})^{\circ}}(T)(\overline{F})$-orbit of Borel subgroups by $\mathscr{O}_{B,x}$, which decomposes as a finite union $\mathscr{O}_{B,x} = \bigsqcup \mathscr{O}_{B,i,x}$ of $N_{Z_G(f_x)^{\circ}}(T)(\overline{F})$-orbits. For each smaller orbit $\mathscr{O}_{B,i,x}$ we can associate a point $y_{i,x} \in \tilde{\Delta}_{(B_{i,x},T)}$ for any choice of $B_{i,x} \in \mathscr{O}_{B,i,x}$.

Take another $x' \in Z^1(\mathscr{E}, G)_T$ which is cohomologous to $x$ (see Lemma 2.17) by some element $g \in K_{Z_G(f_x)^{\circ},G} = K_{N_{Z_G(f_x)^{\circ}}(T),G} \cdot Z_G(f_x)^{\circ}(\overline{F})$. In particular, ${}^{g}f_{x,e} = f_{x',e}$, which by assumption factors through $T$. We may thus replace $g$ by a right $Z_G(f_x)^{\circ}(\overline{F})$-translate $\tilde{g} = \tilde{g}(x')$ and twist $x$ by

$d\tilde{g}$ to obtain $\tilde{x}$, where now $\tilde{g} \in [K_{Z_G(f_x)^\circ,G}] \cap N_G(T)(\overline{F})$; the key observation here is that, although we have changed the cocycle, we still have the equality $f_{x'} = f_{\tilde{x}}$. Moreover, the $\tilde{g}$ we found in the previous step (using $g$) is unique up to right-translation by $N_{Z_G(f_x)^\circ}(T)(\overline{F})$.

It follows that conjugation by $\tilde{g}$ gives a bijection $\mathscr{O}_{B,x} \to \mathscr{O}_{B,\tilde{x}} = \mathscr{O}_{B,x'}$ which induces bijections from each $\mathscr{O}_{B,i,x}$ to some $\mathscr{O}_{B,j(i),\tilde{x}} = \mathscr{O}_{B,j(i),x'}$ (and also a bijection between the orbit decompositions) depending on the choice of $\tilde{g}$ (and thus also the original $g$ and the representative $x'$) up to changing all individual bijections $\mathscr{O}_{B,i,x} \to \mathscr{O}_{B,j(i),x'}$ simultaneously via pre-composing by conjugation by some $h \in N_{Z_G(f_x)^\circ}(T)(\overline{F})$. It thus makes sense to define a "universal" orbit decomposition $\mathscr{O}_{B,[x]} = \bigsqcup \mathscr{O}_{B,i,[x]}$ by taking the inverse limit of orbit decompositions over conjugation by elements $\tilde{g}(x')$ (as obtained in the previous paragraph) for all $x'$ in $(K_{Z_G(f_x)^\circ,G} \cdot x) \cap Z^1(\mathscr{E}, G)_T$, which is exactly the set of representatives $x'$ for $[x]$ which lie in $Z^1(\mathscr{E}, G)_T$.

**Definition 2.25.** We define the *$T$-Newton point of* $[x] \in H^1(\mathscr{E}, G)_T$ to be the tuple

$$\nu_{G,T}([x]) := (\varprojlim_{x' \in Z^1(\mathscr{E},G)_T,\ x' \in [x]} \bar{y}_{i,x'})_{\mathscr{O}_{B,i,[x]}}, \tag{8}$$

where for a fixed orbit $\mathscr{O}_{B,i,[x]}$ and $x'$, the limit is taken over conjugation by the elements $\tilde{g}(x') \in K_{Z_G(f_x)^\circ,G} \cap N_G(T)(\overline{F})$ constructed above, so that $\bar{y}_{i,x'} \in \bar{\Delta}_{(B_{i,x'},T)}$ is the image in $X_*(T)_{\mathbb{Q}/\mathbb{Z}}$ of the point $y_{i,x'} \in \tilde{\Delta}_{(B_{i,x'},T)}$ corresponding to $f_{x',e}$ for any $B_{i,x'} \in \mathscr{O}_{B,i,x'}$—the point $\bar{y}_{i,x'}$ does not depend the choice of $B_{i,x'} \in \mathscr{O}_{B,i,x'}$.

It is straightforward to verify that the above construction does not depend on any choices. Moreover, when $[x] \in H^1_{\text{cyc}}(\mathscr{E}, G)_T$ (defined in the obvious way) then by construction $\mathscr{O}_{B,x}$ is a single $Z_G(f_x)^\circ(\overline{F})$-orbit, so the value of (8) is a 1-tuple. The above definition makes clear that we should view each $\bar{y}_{i,x'}$ as in the intersection of the (image of the) simplices

$$\bar{y}_{i,x'} \in \bigcap_{g \in N_{Z_G(f_{x'})^\circ}(T)(\overline{F})} \bar{\Delta}_{({}^g B_{i,x},T)}.$$

One can also define a version of the Newton map which is independent of $T$. First, we need for each class $[x] \in H^1_{\text{reg}}(\mathscr{E}, G)$ to pick a representative $x \in Z^1(\mathscr{E}, G)_T$ for some $F$-rational maximal torus $T$. Recall that $x$ with these properties is uniquely determined up to $K_{Z_G(f_x)^\circ,G}$-conjugacy.

We have as above the $N_{Z_{G_{\overline{F}}}(f_{x,e})}(T)(\overline{F})$-orbit of Borel subgroups $\mathscr{O}^{(T)}_{B,x}$ from Lemma 2.22 with decomposition $\mathscr{O}^{(T)}_{B,x} = \bigsqcup \mathscr{O}^{(T)}_{B,i,x}$ into $N_{Z_G(f_x)^\circ}(T)(\overline{F})$-orbits.

Instead of conjugating these orbits across $K_{Z_G(f_x)^\circ,G} \cap N_G(T)(\overline{F})$ as with the $T$-Newton map, we conjugate across all of $K_{N_{Z_G(f_x)^\circ}(T),G}$, which as in the $T$-case, gives a universal orbit decomposition $\mathscr{O}_{B,[x]} = \bigsqcup \mathscr{O}_{B,i,[x]}$. The reason we conjugate by $K_{N_{Z_G(f_x)^\circ}(T),G}$ rather than all of $K_{Z_G(f_x)^\circ,G}$ is to ensure that the image of $T$ is again an $F$-rational torus of $G$.

**Definition 2.26.** For a given $[x] \in H^1_{\text{reg}}(\mathscr{E}, G)$ with representative $x \in Z^1(\mathscr{E}, G)_T$, we define

$$\nu_G([x]) := (\varprojlim_{g \in K_{N_{Z_G(f_x)^\circ}(T),G}} {}^g y_{i,x})_{\mathscr{O}_{B,i,[x]}}, \tag{9}$$

where for a fixed orbit $\mathscr{O}_{B,i,[x]}$ and choice of $B_{i,x} \in \mathscr{O}^{(T)}_{B,i,[x]}$ we have the corresponding point $\bar{y}_{i,x} \in \bar{\Delta}_{(B_{i,x},T)}$ and the limit is taken over conjugation by elements in $K_{N_{Z_G(f_x)^\circ}(T),G}$, so that ${}^g(\bar{y}_{i,x}) \in \bar{\Delta}_{({}^g B_{i,x},{}^g T)}$ (each point ${}^g(\bar{y}_{i,x}) \in X_*({}^g T)_{\mathbb{Q}/\mathbb{Z}}$ does not depend on the choice of $B_{i,x} \in \mathscr{O}^{(T)}_{B,i,[x]}$).

Since all possible choices of $x$ as above are, by construction, related by $K_{Z_G(f_x)^\circ,G}$-conjugacy (which is the same as $K_{N_{Z_G(f_x)^\circ}(T),G} \cdot Z_G(f_x)^\circ(\overline{F})$-conjugacy, and conjugation by $Z_G(f_x)(\overline{F})$ does nothing to $\bar{y}_{i,x}$) and, for a fixed $x$ (which, for our purposes, only matters insofar as the output $f_x$), all possible choices of $T$ are related by $[K_{N_{Z_G(f_x)^\circ}(T),G}] \cap Z_G(f_x)^\circ(\overline{F})$-conjugacy, the map (9) is independent of all choices. As with Definition 2.25, the map is valued in 1-tuples when restricted to $H^1_{\mathrm{cyc}}(\mathscr{E}, G)$ (again, by the conjugacy part of Lemma 2.22).

2.3.3. *A crude combinatorial description.* We will describe a combinatorial procedure for computing the connected centralizer of $f_x$ for $[x] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$. First, fix a representative $x$ of $[x] \in H^1(\mathscr{E}, G)_T$ such that $x \in Z^1(\mathscr{E}, G)_T$.

The identity coordinate $f_{x,e} \in X_*(T)_{\mathbb{Q}/\mathbb{Z}}$ lies in the image $\bar{\Delta}_{(B,T)}$ of some base alcove $\tilde{\Delta}_{(B,T)}$ (as in Lemma 2.22); moreover, it lies in the facet $\mathscr{F}_{x,T} \subseteq X_*(T)_{\mathbb{Q}/\mathbb{Z}}$ defined by the intersection of the images of all such base alcoves $\tilde{\Delta}_{(B,T)}$, which is canonically associated to $f_{x,e}$ and $T$.

It is then clear that $Z_G(f_x)^\circ$ is determined by the intersection of facets $\bigcap_{\gamma\in\Gamma} {}^{\gamma}(\mathscr{F}_{x,T})$, in the sense that the root system of $Z_G(f_x)^\circ$ is given by

$$\{\alpha \in R(G_{\overline{F}}, T_{\overline{F}}) \mid \langle z, \alpha\rangle \in \mathbb{Z} \mid \forall z \in \bigcap_{\gamma\in\Gamma} {}^{\gamma}(\mathscr{F}_{x,T})\}. \tag{10}$$

Of course, the representative $x$ with the above properties is only unique to $K_{Z_G(f_x)^\circ,G}$-conjugation (which we can take to be $K_{Z_G(f_x)^\circ,G} \cap N_G(T)(\overline{F})$-conjugation without changing the corresponding morphism $u \xrightarrow{f_{x'}} T$). The effect of this change is to replace $\mathscr{F}_{x,T}$ with ${}^{g}\mathscr{F}_{x,T}$ and $\bigcap_{\gamma\in\Gamma} {}^{\gamma}(\mathscr{F}_{x,T})$ with ${}^{g}[\bigcap_{\gamma\in\Gamma} {}^{\gamma}(\mathscr{F}_{x,T})]$—the fact that $g \in K_{Z_G(f_x)^\circ,G} \subseteq K_{Z_G({}^{\sigma}f_{x,e})^\circ,G}$ means that $\gamma({}^{g}(\mathscr{F}_{x,T})) = {}^{g}(\gamma(\mathscr{F}_{x,T}))$ for any $\gamma \in \Gamma$. It follows that the $(K_{Z_G(f_x)^\circ,G}) \cap N_G(T)(\overline{F})$-conjugacy class of $\bigcap_{\gamma\in\Gamma} {}^{\gamma}(\mathscr{F}_{x,T})$ (and of $\mathscr{F}_{x,T}$) is canonically associated to $[x]$ and $T$.

For a different $T'$ with $[x] \in H^1(\mathscr{E}, G)_{T'}$ and representative $x' \in Z^1(\mathscr{E}, G)_{T'}$ we can choose $g \in K_{Z_G(f_x)^\circ,G}$ whose twist of $x$ yields $x'$ and (after possibly changing $x'$ but keeping $f_{x'}$ the same) conjugates $T$ to $T'$, and it is unique with this property up to right-translation by $(K_{Z_G(f_x)^\circ,G}) \cap N_G(T)(\overline{F})$. It follows that $\mathrm{Ad}(g)$ induces a canonical bijection between the $(K_{Z_G(f_x)^\circ,G}) \cap N_G(T)(\overline{F})$-conjugacy class of $\bigcap_{\gamma\in\Gamma} {}^{\gamma}(\mathscr{F}_{x,T})$ and the $(K_{Z_G(f_{x'})^\circ,G}) \cap N_G(T')(\overline{F})$-conjugacy class of $\bigcap_{\gamma\in\Gamma} {}^{\gamma}(\mathscr{F}_{x',T'})$.

Define a *rigid subset* $S_{\mathscr{F}}$ to be a subset of $\tilde{X}_*(T)_{\mathbb{Q}/\mathbb{Z}}$ that arises as an intersection over the $\Gamma$-orbit of any facet $\mathscr{F}$ contained in the image of a base alcove (if we want to emphasize the torus $T$ we will denote these sets by $S_{\mathscr{F}_T}$) with associated connected centralizer denoted by $G_{S_{\mathscr{F}}}$, defined as in (10). The preceding discussion defines an obvious equivalence relation (namely, $K_{G_{S_{\mathscr{F}}},G}$-conjugacy) on the collection of all rigid subsets of all possible $X_*(T)_{\mathbb{Q}/\mathbb{Z}}$, as we range over all $F$-rational maximal tori $T$ of $G$.

In other words, for a fixed $T$ as above, one has the (very large) union

$$\bigsqcup_{T\subseteq G} \bigcup_{\mathscr{F}_T} \{S_{\mathscr{F}_T}\} \subseteq \mathscr{P}(\bigsqcup_{T\subseteq G} X_*(T)_{\mathbb{Q}/\mathbb{Z}}),$$

where the outer union is over all $F$-rational maximal tori of $G$ and the inner union is over rigid subsets $S_{\mathscr{F}_T} \subseteq X_*(T)_{\mathbb{Q}/\mathbb{Z}}$. We may then take the quotient of this union by the equivalence relation defined in the previous paragraph, denoting the resulting set by $X_G$, and the image of any rigid subset $S_{\mathscr{F}_T}$ (for some $T$) in $X_G$ by $[S_{\mathscr{F}_T}]$.

For a rigid Newton centralizer $H$ in $G$, one can associate a family of (equivalence classes of) rigid subsets

$$S_H = S_{H^\circ} := \{[S_{\mathscr{F}_T}] \mid G_{S_{\mathscr{F}_T}} \in Z_{H^\circ,G} \cdot H^\circ\} \subseteq X_G$$

By construction, one has the decomposition

$$X_G = \bigsqcup_{K_{H^\circ,G}\cdot H^\circ} S_{H^\circ}, \tag{11}$$

where the union is over a set of representatives of each class $K_{H^\circ,G} \cdot H^\circ$.

Now fix an arbitrary class $[x] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$ along with a choice of representative $x \in Z^1(\mathscr{E}, G)_T$ for some $F$-rational maximal torus $T$ and set $H = Z_G(f_x)$; recall that $x$ such that $x \in Z^1_{\mathrm{reg}}(\mathscr{E}, G)$ (forgetting about the $T$-factorization part) is unique with this property is unique up to $K_{H^\circ,G}$-conjugacy. From $x$ and $T$ we obtain $f_{x,e} \in X_*(T)_{\mathbb{Q}/\mathbb{Z}}$ which, as explained above, yields a facet $\mathscr{F}_T \subseteq X_*(T)_{\mathbb{Q}/\mathbb{Z}}$ and therefore a rigid set $S_{\mathscr{F}_T}$ whose image $[S_{[x]}]$ in $X_G$ does not depend on $x$ or $T$ with the aforementioned properties. By construction, the collection of subgroups $K_{H^\circ,G} \cdot H^\circ$ associated to the class $[x]$ is completely determined by the piece of the decomposition (11) the class $[S_{[x]}]$ lies in.

We thus obtain an induced "Newton decomposition"

$$H^1_{\mathrm{reg}}(\mathscr{E}, G) = \bigsqcup_{K_{H^\circ,G}\cdot H^\circ} H^1(\mathscr{E}, G)_{H^\circ}, \tag{12}$$

where the union is over possible classes $K_{H^\circ,G} \cdot H^\circ$ for all rigid Newton centralizers $H$ (in particular, for a given $H$ there are multiple such classes inside $K_{H,G} \cdot H$) and $H^1(\mathscr{E}, G)_{H^\circ}$ is the preimage of $[S_{H^\circ}]$ under the map constructed above.

## 3. The rigid Kottwitz map

The goal of this section is to give an analogue of the Kottwitz map (cf. [Kot14, §11])

$$B(G) \to \pi_1(G)_\Gamma \xrightarrow{\sim} X^*(Z(\widehat{G})^\Gamma)$$

in the rigid setting developed above. More precisely, we construct (in Definition 3.14) a map

$$H^1_{\mathrm{reg}}(\mathscr{E}, G) \to \bar{\mathfrak{Y}}_{+,\mathrm{tor}}(G), \tag{13}$$

where $\bar{\mathfrak{Y}}_{+,\mathrm{tor}}(G)$ is a linear algebraic object analogous to $\pi_1(G)_\Gamma$ (see (29) for the precise definition). In §4.3.3 we re-interpret the right-hand side of (13) in terms of $\widehat{G}$ in an analogous way to the identification $\pi_1(G)_\Gamma \xrightarrow{\sim} X^*(Z(\widehat{G})^\Gamma)$ used in the Kottwitz map.

We begin by observing that one can extend the definition of regular classes to arbitrary (i.e., potentially disconnected) reductive groups in the obvious way:

**Definition 3.1.** For an arbitrary reductive group $G$, define $H^1_{\mathrm{reg}}(\mathscr{E}, G)$ to be the image of $H^1_{\mathrm{reg}}(\mathscr{E}, G^\circ)$ in $H^1(\mathscr{E}, G)$. Equivalently, $H^1_{\mathrm{reg}}(\mathscr{E}, G) \subset H^1(\mathscr{E}, H)$ is the image of

$$\bigsqcup_{S \subset H} H^1(\mathscr{E}, S) \to H^1(\mathscr{E}, G)$$

where the union is over all $F$-rational maximal tori of $G$.

Using Definition 3.1, we can, for maximal generality, relax some of the usual assumptions of this paper:

**Assumption 3.2.** For the rest of this section we drop the assumption that $G$ is connected (unless explicitly stated otherwise).

3.1. **A linear algebraic functor.** Define the category $\mathsf{R}$ whose objects are pairs $[A \xrightarrow{f} G']$ of a (potentially disconnected) reductive group $G'$ defined over $F$ with finite multiplicative group $A \xrightarrow{f} G'$ mapping via a fixed monomorphism $f$ into $G'$ which factors through an $F$-rational torus. The morphisms $[A_1 \xrightarrow{f} G_1] \to [A_2 \xrightarrow{g} G_2]$ in $\mathsf{R}$ are $F$-rational homomorphisms $G_1 \xrightarrow{h} G_2$ sending $A_1$ into $A_2$ such that $Z_{G_2}(h(A_1)) = Z_{G_2}(A_2)$.

In [Kal16b], one works with the full subcategory of $\mathsf{R}$, denoted by $\mathsf{R}_Z^{\circ}$, whose objects $[Z \xrightarrow{f} G] \in \mathrm{Ob}(\mathsf{R})$ are such that $G$ is connected and $f(Z)$ is central; the assignment

$$[Z \to G] \mapsto H^1(\mathscr{E}, Z \to G)$$

is a functor from $\mathsf{R}_Z$ to $\mathsf{Ab\text{-}gp}$ (for the group structure on $H^1(\mathscr{E}, Z \to G)$ see Theorem 3.4 below), denoted by $H^1(\mathscr{E}, -)$. In loc. cit., one first constructs a linear algebraic functor on $\mathsf{T}_Z$, where $\mathsf{T}_Z$ is the full subcategory of $\mathsf{R}_Z^{\circ}$ where $G = T$ is a torus, given by

$$[Z \to S] \mapsto \bar{Y}_{+,\mathrm{tor}}[Z \to Y] := \varinjlim_k \frac{X_*(S/Z)}{IX_*(S)}[\mathrm{tor}] = \varinjlim_k \frac{X_*(S/Z)^N}{IX_*(S)}, \tag{14}$$

where $I = I_{E_k/F}$ is the augmentation ideal for $\Gamma_{E_k/F}$ and we have chosen $k \gg 0$ such that $E_k$ splits $S$ and $Z$ is $n_k$-torsion and the superscript "$N$" denotes the elements killed by the $N_{E_k/F}$-norm map. This evidently gives another functor $\mathsf{R}_Z^{\circ} \to \mathsf{Ab\text{-}gp}$. One then has:

**Theorem 3.3.** *(*[Kal16b, Theorem 4.8]*) There is a canonical isomorphism of functors on $\mathsf{T}_Z$:*

$$H^1(\mathscr{E}, -) \xrightarrow{\sim} \bar{Y}_{+,\mathrm{tor}}.$$

The isomorphism of Theorem 3.3 will be called the *rigid Tate-Nakayama isomorphism for tori*. The functor $\bar{Y}_{+,\mathrm{tor}}$ extends to all of $\mathsf{R}_Z^{\circ}$ via the formula

$$\bar{Y}_{+,\mathrm{tor}}[Z \to G] = \varinjlim_{S \subset G} \varinjlim_k \frac{[X_*(S/Z)/X_*(S_{\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{\mathrm{sc}})]},$$

where the outer limit is over all maximal $F$-rational tori $S$ of $G$ with $S_{\mathrm{sc}}$ denoting the preimage of $S$ in $G_{\mathrm{sc}}$, the transition maps are given by $G(\overline{F})$-conjugation, and the inner limit is the same as in (14) (one verifies that this conjugation gives a canonical $\Gamma$-equivariant map between the relevant groups in the limit). This gives a well-defined functor $\mathsf{R}_Z^{\circ} \to \mathsf{Ab\text{-}gp}$, as shown in [Kal16b, §4.1].

We then have:

**Theorem 3.4.** *(*[Kal16b, Theorem 4.11]*) There is a canonical bijection of functors on $\mathsf{R}_Z^{\circ}$:*

$$H^1(\mathscr{E}, -) \xrightarrow{\iota_-} \bar{Y}_{+,\mathrm{tor}}.$$

*extending the isomorphism from Theorem 3.4.*

The first new step is extending $\bar{Y}_{+,\mathrm{tor}}$ from $\mathsf{R}_Z^{\circ}$ to all of $\mathsf{R}$. For ease of notation, we will, for $[A \xrightarrow{f} G] \in \mathrm{Ob}(\mathsf{R})$, identify $A$ with $f(A) \subset G$ in order to think of it as an $F$-rational multiplicative subgroup of $G$ and also set $H := Z_G(A)$ and $W(G,H) := N_G(H)/H$.

**Proposition 3.5.** *Fix $S$ an $F$-rational maximal torus of $G$ containing $A$. We have a functor $\bar{Y}_{+,tor}\colon \mathsf{R} \to \mathsf{Sets}^*$ given on objects by*

$$[A \xrightarrow{f} G] \mapsto \varinjlim_{g \in K_{N_{H^\circ}(S),G}} [\varinjlim_{k} \frac{[X_*({}^gS/{}^gA)/X_*({}^gS_{\mathrm{sc}})]^N}{I[X_*({}^gS)/X_*({}^gS_{\mathrm{sc}})]}/W(G, {}^gH^\circ)(F)]. \tag{15}$$

The construction in Proposition 3.5 extends $\bar{Y}_{+,\mathrm{tor}}$ on $\mathsf{R}_Z$, since in that case $H = G$ and hence $W(G, H^\circ)(F) = \{*\}$ and the outer limit of (15) is over all $F$-rational maximal tori in $G$.

We first need to check that the quotient

$$\frac{[X_*({}^gS/{}^gA)/X_*({}^gS_{\mathrm{sc}})]^N}{I[X_*({}^gS)/X_*({}^gS_{\mathrm{sc}})]}/W(G, {}^gH^\circ)(F)$$

appearing in (15) (for a fixed $k \gg 0$ and $g \in K_{N_{H^\circ}(S),G}$) is well-defined. This follows from (continuing the same notation as above):

**Lemma 3.6.** *For any $g \in K_{N_H(S),G}$ with $S' := {}^gS$, $A' := {}^gA$, and $H' := Z_G(A')$, the induced isomorphism*

$$\frac{X_*(S/A)}{X_*(S_{\mathrm{sc}})} \xrightarrow{\mathrm{Ad}(g)^\sharp} \frac{X_*(S'/A')}{X_*(S'_{\mathrm{sc}})}$$

*is independent of the choice of $g \in K_{N_H(S),G}$ conjugating $[A \to S]$ to $[A' \to S']$ up to post-composing by the lift in $G(\overline{F})$ of an element of $W(G, H')(F)$ which normalizes $S'$. In particular, each $\mathrm{Ad}(g)^\sharp$ is $\Gamma$-equivariant. Moreover, the same result holds if we replace all occurrences of $H$ with $H^\circ$ (and $H'$ with $H^{',\circ}$).*

*Proof.* We will prove the result for $H$—the identical arguments work if one replaces every occurrence of "$H$" in the argument with "$H^\circ$". Since $K_{N_H(S),G}$ is preserved under left-translation by $K_{N_{H'}(S'),G}$, we may assume without loss of generality that $H = H'$ and $g \in N_G(H)(\overline{F})$. Replacing $g$ again by a left $H(\overline{F})$-translate lets us assume further that $S' = S$ and we can therefore view $g$ as an element $w_g \in W(N_G(H), S)(\overline{F})$. Note that if $g \in H(\overline{F})$, then

$$\frac{X_*(S/A)}{X_*(S_{\mathrm{sc}})} \xrightarrow{\mathrm{Ad}(g)^\sharp} \frac{X_*(S/A)}{X_*(S_{\mathrm{sc}})}$$

is the identity, using the arguments of [Kal16b, Lemma 4.2] and the fact that (by construction) $A \subset Z(H)$. It follows that, for the purposes of the stated claim, we may further view $w_g$ as in the quotient $[N_{N_G(H)}(T)](\overline{F})/[N_H(T)](\overline{F})$. We obtain the desired result by observing that the subset $N_G(H)(\overline{F}) \cap K_{N_H(S),G} \subset K_{H,G}$ has image in $(G/H)(F)$ which is contained in $[N_G(H)/H](F) = W(G, H)(F)$.

The $\Gamma$-equivariance of $\mathrm{Ad}(g)^\sharp$ for $g \in K_{N_H(S),G}$ follows from the fact that $g^{-1}\,{}^\sigma g \in H(\overline{F})$ for any $\sigma \in \Gamma$, and we argued above that $N_H(S)(\overline{F})$ acts trivially on $\frac{X_*(S/A)}{X_*(S_{\mathrm{sc}})}$ by conjugation. □

It follows that for any $g \in K_{G,N_{H^\circ}(S)}$ with $S', A', H'$ as in Lemma 3.6 the map $\mathrm{Ad}(g)$ induces a bijection

$$\frac{[X_*(S/A)/X_*(S_{\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{\mathrm{sc}})]} \to \frac{[X_*(S'/A')/X_*(S'_{\mathrm{sc}})]^N}{I[X_*(S')/X_*(S'_{\mathrm{sc}})]}$$

which sends $W(G, H^\circ)$-orbits to $W(G, H^{',\circ})$-orbits, and after passing to these orbits it only depends on $S'$, $A'$, and $H'$, not on $g$. We immediately obtain:

**Corollary 3.7.** *The value $\bar{Y}_{+,\mathrm{tor}}[A \to G]$ defined in* (15) *does not depend on the choice of $S$ containing $A$.*

It remains to show that the assignment (15) is a functor on $\mathsf{R}$:

*Proof.* (of Proposition 3.5) Fix a morphism $[A_1 \xrightarrow{f} G_1] \xrightarrow{\phi} [A_2 \xrightarrow{g} G_2]$ in $\mathsf{R}$; the goal is to show that there is a canonical way to assign a morphism in $\mathsf{Sets}^*$ between $\bar{Y}_{+,\mathrm{tor}}[A_1 \to G_1]$ and $\bar{Y}_{+,\mathrm{tor}}[A_2 \to G_2]$; set $H_j = Z_{G_j}(A_j)$ for $j = 1, 2$.

The assumption that $\phi(H_1) \subseteq Z_{G_2}(\phi(A_1)) = H_2$ implies that $\phi$ sends $K_{H_1^\circ, G_1}$ into $K_{H_2^\circ, G_2}$, since if $x \in G_1(\overline{F})$ with $g\,{}^{\sigma}g^{-1} \in H_1^\circ(\overline{F})$ for all $\sigma \in \Gamma$ then we also have $\phi(g)\,{}^{\sigma}\phi(g)^{-1} = \phi(g\,{}^{\sigma}g^{-1}) \in \phi(H_1^\circ(\overline{F})) \subseteq H_2^\circ(\overline{F})$ (where we are using that $\phi$ is defined over $F$).

The map $G_1 \xrightarrow{\phi} G_2$ lifts uniquely to a map $G_{1,\mathrm{sc}} \xrightarrow{\phi_{\mathrm{sc}}} G_{2,\mathrm{sc}}$. Denote the corresponding choices of tori as in the statement of the Proposition by $S_1$ and $S_2$. By assumption, there is some by $h \in K_{N_{H_2^\circ}(S_2), G}$ such that ${}^h\phi(S_1) \subseteq S_2$ and so ${}^h\phi$ induces a map $S_{1,\mathrm{sc}} \to S_{2,\mathrm{sc}}$; we thus have a $\Gamma$-equivariant (using Lemma 3.6) homomorphism

$$\frac{X_*(S_1/A_1)}{X_*(S_{1,\mathrm{sc}})} \xrightarrow{{}^h\phi_{S_1,S_2}} \frac{X_*(S_2/A_2)}{X_*(S_{2,\mathrm{sc}})}, \tag{16}$$

which therefore induces a map

$$\varinjlim_k \frac{[X_*(S_1/A_1)/X_*(S_{1,\mathrm{sc}})]^N}{I[X_*(S_1)/X_*(S_{1\mathrm{sc}})]} \xrightarrow{{}^h\phi'_{S_1,S_2}} \varinjlim_k \frac{[X_*(S_2/A_2)/X_*(S_{2,\mathrm{sc}})]^N}{I[X_*(S_2)/X_*(S_{2,\mathrm{sc}})]}. \tag{17}$$

Now fix $g \in K_{H_1^\circ, G}$ which maps into $[N_{G_1}(H_1^\circ)/H_1^\circ](F)$ and normalizes $S_1$. Then $\phi(g) \in K_{H_2^\circ, G}$ and we may replace $\phi(g)$ by a right-translate $\phi(g)h' \in K_{N_{H_2^\circ}(S_2), G}$ with $h' \in H_2(\overline{F})$ so that the map (17) induces a map (independent of the choice of $h'$)

$$[\varinjlim_k \frac{[X_*(S_1/A_1)/X_*(S_{1,\mathrm{sc}})]^N}{I[X_*(S_1)/X_*(S_{1,\mathrm{sc}})]}]/W(G, H_1^\circ)(F) \to \tag{18}$$

$$\varinjlim_{K_{N_{H_2^\circ}(S_2),G}} [\varinjlim_k \frac{[X_*({}^gS_2/{}^gA_2)/X_*({}^gS_{2,\mathrm{sc}})]^N}{I[X_*({}^gS_2)/X_*({}^gS_{2,\mathrm{sc}})]}]/W(G, {}^gH_2^\circ)(F);$$

note that it is harmless to pass the Weyl-orbits across the internal direct limit and we have done so in the above equation. The map 18 does not depend on the choice of $h$ made above (by Lemma 3.6).

Applying $\varinjlim_{K_{N_{H_1^\circ}(S),G}}(-)$ to the left-hand side of (18) and then interchanging the colimits gives the desired well-defined map which is independent of any choices (again by Lemma 3.6). □

3.2. **Defining the Kottwitz map.** Continue with the notation of the previous subsection. The goal of this subsection is to extend the Tate-Nakayama isomorphism of functors on $\mathsf{R}_Z^\circ$ from Theorem 3.4 to a morphism from $H^1(\mathscr{E}, -)$ to $\bar{Y}_{+,\mathrm{tor}}$ (as in Proposition 3.5) as functors on the category $\mathsf{R}$.

**Notation 3.8.** For $M$ a connected reductive subgroup of $G$ and $T$ a maximal torus of $G$ contained in $M$, denote by $T_{M,\mathrm{sc}}$ the preimage of $T$ in $M_{\mathrm{sc}}$. Recall that $T_{\mathrm{sc}}$ denotes $T_{G,\mathrm{sc}}$.

The main technical ingredient needed is:

**Proposition 3.9.** *Let $M$ be a connected, reductive, equal-rank subgroup of $G$ with finite central subgroup $A$ such that $Z_G(A)^\circ = M$. If $[x],[y] \in H^1(\mathscr{E}, A \to M)$ have the same image $[z] \in H^1(\mathscr{E}, A \to G)$ and $Z_G(f_x)^\circ = M$ (and hence also $Z_G(f_y)^\circ = M$) then the elements $\iota_{[A\to H]}([x]), \iota_{[A\to H]}([y])$ in $\bar{Y}_{+,tor}[A \to M] = \varinjlim_S \varinjlim_k \frac{[X_*(S/A)/X_*(S_{M,\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{M,\mathrm{sc}})]}$ have the same image in the set $\bar{Y}_{+,tor}[A \to G]$.*

**Remark 3.10.** One verifies easily that the condition on $[x]$ that $Z_G(f_x)^\circ = M$ is equivalent to $[z] \in H^1(\mathscr{E}, G)_M$ via the Newton decomposition (12).

*Proof.* Choose representatives $x, y \in Z^1(\mathscr{E}, A \to M)$ of $[x],[y]$. For ease of exposition we will view $x$ and $y$ as cocycles of the group $\mathscr{E}$ valued in $M(\overline{F})$ rather than torsors on an abstract gerbe (cf. the discussion at the end of Section 2.1). In this setting, $G_{\mathscr{E}}$ denotes $G(\overline{F})$ as an $\mathscr{E}$-module with the action inflated from the usual $\Gamma$-action.

By assumption there is some $g \in G(\overline{F})$ such that $y(e) = gx(e)\,{}^e g^{-1}$ for all $e \in \mathscr{E}$; note that we have $g \in N_G(M)(\overline{F})$ with image in $[N_G(M)/M](F)$, since

$$ {}^g M = {}^g Z_G(f_x)^\circ = Z_G({}^g f_x)^\circ = Z_G(f_y)^\circ = M, \tag{19} $$

and by assumption $g \in K_{M,G}$, whose intersection with $N_G(M)(\overline{F})$ is the preimage of $W(G,M)(F)$. We may replace $y$ by a cohomologous element in $Z^1(\mathscr{E}, A \to M)$ in order to assume further that $y \in Z^1(\mathscr{E}, A \to S)$, where $S$ is a fixed elliptic maximal torus of $M$ (using [Kal16b, Corollary 3.7]).

The desired result will follow for $[x]$ and $[y]$ if we can show that $\iota_{[A\to M]}[x]$ and $\iota_{[A\to M]}[y]$ have the same image (for some fixed $k \gg 0$) under the map

$$ \bar{Y}_{+,\mathrm{tor}}[A \to M] = \frac{[X_*(S/A)/X_*(S_{M,\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{M,\mathrm{sc}})]} \to \Big(\frac{[X_*(S/A)/X_*(S_{\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{\mathrm{sc}})]}\Big)/W(G,M)(F). \tag{20} $$

Indeed, because $[x],[y] \in H^1(\mathscr{E}, A \to M)$ and $M$ is connected, there is only one equivalence class (as in Lemma 2.18) in $K_{Z_M(S),M}$. In (20) we are, by abuse of notation, using the colimit description of $\bar{Y}_{+,\mathrm{tor}}[A \to M]$ to use the torus $S$ for the images of both classes under $\iota_{[A\to M]}$.

Note that $W(G,M)(F)$ acts on $H^1(\mathscr{E}, A \to M)$ (as a set) via the following formula, for $w \in W(G,M)(F)$ with lift $g_w \in N_G(M)(\overline{F})$ and $[a] \in H^1(\mathscr{E}, A \to M)$:

$$ (g_w \cdot a)(e) = g_w a(e)\,{}^e g_w^{-1}; \tag{21} $$

one checks that the above formula produces an element of $M(\overline{F})$, is a 1-cocycle whose class is independent of the choice of lift $g_w$, and gives a well-defined group action of $W(G,M)(F)$.

Using the $W(G,M)(F)$-action defined above, we claim that the composition

$$ H^1(\mathscr{E}, A \to M) \xrightarrow{\iota_{[A\to M]}} \frac{[X_*(S/A)/X_*(S_{M,\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{M,\mathrm{sc}})]} \to \frac{[X_*(S/A)/X_*(S_{\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{\mathrm{sc}})]} \tag{22} $$

sends $W(G,M)(F)$-orbits to $W(G,M)(F)$-orbits (where $W(G,M)(F)$ acts on the right-hand side of (22) by lifting elements to $N_G(S)(\overline{F})$ and then acting as usual). This claim would give the desired result, since we showed above that $[x]$ is a $W(G,M)(F)$-translate of $[y]$.

First consider the map $\iota_{[A\to M]}$; for a given $w \in W(G,M)(F)$ with arbitrary lift $g_w \in G(\overline{F})$ we have $dg_w \in Z^1(\Gamma, M(\overline{F}))$ whose class in $H^1(F,M)$ is cohomologous to a class in the image of $H^1(F,S)$ (using that $S$ is elliptic)—it follows that we may replace $g_w$ by an $M(\overline{F})$-translate to assume that $dg_w \in Z^1(\Gamma, S(\overline{F}))$ and let $G'$ be the twisted form of $G$ determined by the cocycle $dg_w$. In particular, we have an $F$-rational isomorphism $G \xrightarrow{\mathrm{Ad}(g_w)} G'$ which restricts to an $F$-rational isomorphism $M \xrightarrow{\sim} M' := {}^{dg_w}M$. By construction, the composition $S \to M \xrightarrow{\mathrm{Ad}(g_w)} M'$ is still the embedding of an $F$-rational maximal torus, so we can and do view $S$ as a maximal torus of $M'$ (and hence we also have an embedding $A \to S \to M'$).

Consider the following commutative diagram of bijections:

$$\begin{array}{ccccc} H^1(\mathscr{E}, A \to M) & \xrightarrow{\mathrm{Ad}(g_w)} & H^1(\mathscr{E}, A \to M') & \xrightarrow{\cdot dg_w} & H^1(\mathscr{E}, A \to M) \\ \downarrow \iota_{[A\to M]} & & \downarrow \iota_{[A\to M']} & & \downarrow \iota_{[A\to M]} \\ \frac{[X_*(S/A)/X_*(S_{M,\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{M,\mathrm{sc}})]} & \xrightarrow{\mathrm{Ad}(g_w)} & \frac{[X_*(S/A)/X_*(S_{M,\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{M,\mathrm{sc}})]} & \xrightarrow{+\iota_{[A\to M]}(dg_w)} & \frac{[X_*(S/A)/X_*(S_{M,\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{M,\mathrm{sc}})]}, \end{array} \tag{23}$$

where we are using above that formation of the group $\frac{[X_*(S/A)/X_*(S_{M,\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{M,\mathrm{sc}})]}$ is unaffected by twisting the $\Gamma$-action on $M(\overline{F})$ by $dg_w$ (since $dg_w$ takes values in $S$). The left-hand square commutes because of the functoriality of $\iota$ and the right-hand square commutes because of [Kot86, Lemma 1.4].

For the cocycle $a \in Z^1(\mathscr{E}, A \to M)$, consider the cocycle $a' := e \mapsto g_w^{-1}a(e)\,{}^e g_w \in Z^1(\mathscr{E}, A \to M)$. The left-hand square in (23) tells us that

$$\iota_{[A\to M]}(a') = \mathrm{Ad}(g_w^{-1})[\iota_{[A\to M']}(g_w a' g_w^{-1})],$$

and one computes easily that $g_w a' g_w^{-1} = a \cdot (dg_w)^{-1}$. The right-hand square in (23) then gives $\iota_{[A\to M']}(g_w a' g_w^{-1}) = \iota_{[A\to M]}(a) - \iota_{[A\to M]}(dg_w)$, and so we have

$$\iota_{[A\to M]}(a') = \mathrm{Ad}(g_w^{-1})[\iota_{[A\to M]}(a) - \iota_{[A\to M]}(dg_w)] = \mathrm{Ad}(g_w^{-1})[\iota_{[A\to M]}(a)] + \iota_{[A\to M]}(d(g_w^{-1})). \tag{24}$$

To see where we obtained the identity

$$\mathrm{Ad}(g_w^{-1})[\iota_{[A\to M]}(dg_w)^{-1}] = \iota_{[A\to M]}(d(g_w^{-1})) \tag{25}$$

appearing in the second summand in the rightmost term of (24), let $\xi \in Z^2(\Gamma, u(\overline{F}))$ be the representative of the canonical class corresponding to $\mathscr{E}$ described in [Kal16b, §4.5] and used loc. cit. to construct the Tate-Nakayama isomorphism (for tori, cf. Theorem 3.3), so that by definition of the Tate-Nakayama isomorphism we have $\xi \cup \iota_{[A\to M]}(dg_w) = dg_w$. One then computes that

$$\xi \cup [\mathrm{Ad}(g_w^{-1})\iota_{[A\to M]}(dg_w)] = [e \mapsto {}^e g_w^{-1} dg_w\,{}^e g_w] = (d(g_w^{-1}))^{-1},$$

which is equivalent to saying that $\iota_{[A\to M]}((d(g_w^{-1}))^{-1}) = \mathrm{Ad}(g_w^{-1})\iota_{[A\to M]}(dg_w)$, and so the identity (25) follows by taking the inverse of both sides (since $\iota_{[A\to M]}$ is a group homomorphism, by definition).

The identity (24) makes the fact that (22) preserves $W(G,M)(F)$-orbits clear, since $\iota_{[A\to M]}(dg_w^{-1})$ lies in the kernel of

$$\frac{[X_*(S/A)/X_*(S_{M,\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{M,\mathrm{sc}})]} \to \frac{[X_*(S/A)/X_*(S_{\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{\mathrm{sc}})]},$$

yielding the desired result. □

**Remark 3.11.** Theorem 3.4 gives $H^1(\mathscr{E}, A \to M)$ the canonical structure of an abelian group—using this structure, the action (21) of $W(G,M)(F)$ is not via group automorphisms. For example, setting $[a] = 0$, the class of the cocycle $e \mapsto g_w\,{}^e g_w^{-1}$ in $H^1(\mathscr{E}, A \to M)$ is in general not trivial (and of course the image of 0 via $\iota_{[A\to M]}$ is 0, which is fixed by $w$).

We can now define the rigid Kottwitz map on $H^1_{\mathrm{reg}}(\mathscr{E}, G)$; given $[x] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$ which has representative $x$ (with $F$-rational $f_x$, as usual) such that $[x]$ is in the image of

$$H^1(\mathscr{E}, f_x(u) \to Z_G(f_x)^\circ) \to H^1(\mathscr{E}, G), \tag{26}$$

and this representative is unique up to $K_{Z_G(f_x)^\circ, G}$-translation. Proposition 3.9 applied to $A = f_x(u)$ and $M = Z_G(f_x)^\circ$ implies that any two preimages of $[x]$ via (26) have the same image in $\bar{Y}_{+,\mathrm{tor}}[f_x(u) \to G]$, and it follows from the same Proposition and the construction of the functor $\bar{Y}_{+,\mathrm{tor}}$ that replacing $x$ by a $K_{Z_G(f_x)^\circ, G}$-translate also yields the same element of the orbit space $\bar{Y}_{+,\mathrm{tor}}[f_x(u) \to G]$.

**Definition 3.12.** For $[x] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$ we call the uniquely-determined element of $\bar{Y}_{+,\mathrm{tor}}[f_x(u) \to G]$ obtained above the *finite rigid Kottwitz map* applied to $[x]$. We will give a precise description of the codomain of this map shortly. It will also be clear why where the word "finite" comes when we define the rigid Kottwitz map (which involves an infinite projective limit) in §3.3.

The proof of Proposition 3.9 can be reverse-engineered to show:

**Lemma 3.13.** *Using the same notation and assumptions as in Proposition 3.9, if $[x], [y] \in H^1(\mathscr{E}, A \to M)$ are such that $\iota_{[A \to M]}([x])$ and $\iota_{[A \to M]}([y])$ have the same image in $\bar{Y}_{+,\mathrm{tor}}[A \to G]$ and $\nu_G([x]) = \nu_G([y])$, then their images in $H^1(\mathscr{E}, A \to G)$ are equal.*

*Proof.* Fix representatives $x$ and $y$ so that $f_x$, $f_y$ are both $F$-rational morphisms $u \to Z(M)$ with $Z_G(f_x)^\circ = Z_G(f_y)^\circ = M$. The assumption that $\nu_G([x]) = \nu_G([y])$ implies that $f_x$ is $K_{M,G}$-conjugate to $f_y$, and we may take a $K_{M,G} \cap N_G(M)(\overline{F})$-cohomologous (but not necessarily $M$-cohomologous) class $[y']$ which satisfies $f_{y'} = f_x$; in particular, we observe that

$$\iota_{[A \to G]}(\iota_{[A \to M]}[y']) = \iota_{[A \to G]}([y']) = \iota_{[A \to G]}([y]) = \iota_{[A \to G]}(\iota_{[A \to M]}[y]) = \iota_{[A \to G]}(\iota_{[A \to M]}([x]),$$

so the hypothesis of the statement holds for $y'$ as well, and it therefore suffices to prove the result with $[y]$ replaced by $[y']$ (we will just call it $[y]$). We may thus choose representatives $x, y \in Z^1(\mathscr{E}, A \to M)$ satisfying $f_x = f_y$. It will also be useful to replace $x$ with an $M$-cohomologous representative in order to assume that $x \in Z^1(\mathscr{E}, A \to S)$ for an $F$-rational maximal torus $S$ of $M$.

The cocycles $x, y$ give a twisted forms ${}^xG$, ${}^yG$ of $G_{\mathscr{E}}$ and, via taking the images of $x$ and $y$ in $Z^1(\Gamma, M_{\mathrm{ad}}(\overline{F}))$ (denoted by $\bar{x}, \bar{y}$), twisted forms ${}^{\bar{x}}M, {}^{\bar{y}}M$ of $M$ such that $({}^{\bar{x}}M)_{\mathscr{E}} \times^{M_{\mathscr{E}}} G_{\mathscr{E}} = {}^xG$ (similarly with $y$). For a group scheme $N$ (over $\overline{F}$) on which $\mathscr{E}$ acts by algebraic automorphisms along with an algebraic group monomorphism $A_{\overline{F}} \to Z(N)$ compatible with the $\mathscr{E}$-action on both schemes, denote by $H^1(\mathscr{E}, A \to N)$ the subset of $H^1(\mathscr{E}, N(\overline{F}))$ consisting of classes $[c]$ which have a representative $c$ such that $c|_{u(\overline{F})}$ is a morphism of algebraic groups from $u(\overline{F})$ to $A(\overline{F})$, which is necessarily defined over $F$.

Take the images of $[x], [y] \in H^1(\mathscr{E}, A \to M)$ in $H^1(\mathscr{E}, A \to {}^{\bar{x}}M)$ under the standard twisting map $[c] \mapsto [cx^{-1}]$, denoted by $[x]'$ and $[y]'$ (so that $[x]'$ is, by design, the neutral class) which lie in the subset $H^1(F, {}^{\bar{x}}M) \subset H^1(\mathscr{E}, A \to {}^{\bar{x}}M)$, since their restriction to $u$ is trivial. This twisting bijection reduces the problem to showing that the image of $[y]$ in $H^1(\mathscr{E}, A \to {}^xG)$, or, equivalently, in $H^1(\mathscr{E}, A \to {}^xN_G(M))$, is the neutral class. Set

$$H^1(\mathscr{E}, A \to {}^xN_G(M))^\circ := \mathrm{Im}[H^1(\mathscr{E}, A \to {}^{\bar{x}}M) \to H^1(\mathscr{E}, A \to {}^xN_G(M)]$$

so that, by construction, the image $[y]' \in H^1(\mathscr{E}, A \to {}^{\bar{x}}M)$ in $H^1(\mathscr{E}, A \to {}^xN_G(M))$ lies in $H^1(\mathscr{E}, A \to {}^xN_G(M))^\circ$.

The short exact sequence of (non-abelian) $\mathscr{E}$-modules

$$1 \to {}^{\bar{x}}M \to {}^{x}N_G(M) \to W(G,M) \to 1$$

yields the exact sequence

$$W(G,M)(F) \to H^1(\mathscr{E}, A \to {}^{\bar{x}}M) \to H^1(\mathscr{E}, A \to {}^{x}N_G(M))^\circ \to 1,$$

(in the right-most term of the short exact sequence and the left-most term of the above sequence we have deliberately omitted the twist, since the $\mathscr{E}$-action is inflated from the usual $\Gamma$-action) hence, essentially by construction,

$$H^1(\mathscr{E}, A \to {}^{x}N_G(M))^\circ = H^1(\mathscr{E}, A \to {}^{\bar{x}}M)/W(G,M)(F), \tag{27}$$

where the action is the one defined in (21) introduced in the proof of Proposition 3.9 and hence $\iota_{[A\to M]}$ induces a pointed bijection

$$H^1(\mathscr{E}, A' \to {}^{x}N_G(M))^\circ \xrightarrow{\sim} \bar{Y}_{+,\mathrm{tor}}[A \to {}^{\bar{x}}M]/W(G,M)(F).$$

Moreover, one uses the commutative diagram (cf. (23) from the proof of Proposition 3.9)

$$\begin{array}{ccc} H^1(\mathscr{E}, A \to M) & \xrightarrow{\cdot x^{-1}} & H^1(\mathscr{E}, A \to {}^{\bar{x}}M) \\ \downarrow{\scriptstyle \iota_{[A\to M]}} & & \downarrow{\scriptstyle \iota_{[A'\to {}^{\bar{x}}M]}} \\ \dfrac{[X_*(S/A)/X_*(S_{M,\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{M,\mathrm{sc}})]} & \xrightarrow{+\iota_{[A\to M]}(x^{-1})} & \dfrac{[X_*(S/A)/X_*(S_{M,\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{M,\mathrm{sc}})]} \end{array}$$

along with the commutative diagram

$$\begin{array}{ccccc} H^1(\mathscr{E}, A \to N_G(M))^\circ & \longrightarrow & H^1(\mathscr{E}, A \to {}^{x}N_G(M))^\circ & \xrightarrow{\sim} & \bar{Y}_{+,\mathrm{tor}}[A \to {}^{\bar{x}}M]/W(G,M)(F) \\ \uparrow & & \uparrow & & \uparrow \\ H^1(\mathscr{E}, A \to M) & \longrightarrow & H^1(\mathscr{E}, A \to {}^{\bar{x}}M) & \xrightarrow{\sim} & \bar{Y}_{+,\mathrm{tor}}[A \to {}^{\bar{x}}M], \end{array}$$

where the first maps in both rows are given by twisting by $[x^{-1}]$, to deduce that the image of $[y]'$ is trivial in $\bar{Y}_{+,\mathrm{tor}}[A \to {}^{\bar{x}}M]/W(G,M)(F)$, and is therefore the neutral class in $H^1(\mathscr{E}, A \to {}^{x}N_G(M))$, as desired. □

3.3. **Limit constructions.** If we restrict the category $\mathsf{R}$ to pairs $[A \to G]$ where $G$ is fixed and $A$ is a finite $F$-rational subgroup contained in a torus, then there is a morphism $[A' \to G] \to [A \to G]$ if and only if $A' \subseteq A$ and $Z_G(A) = Z_G(A')$; it's clear that this defines a subcategory $\mathsf{R}_{(G)}$ which decomposes as a disjoint union of categories

$$\mathsf{R}_{(G)} = \bigsqcup_{[H]} \mathsf{R}^H_{(G)},$$

where $\mathsf{R}^H_{(G)}$ is the subcategory of all $[A \to G]$ with $Z_G(A) = H$ and $[H] = K_{H,G} \cdot H$.

Recall from Definition 3.12 that any class $[x]$ lies in the image of $H^1(\mathscr{E}, f_x(u) \to Z_G(f_x)^\circ)$ and the subgroup $Z_G(f_x)$ (more precisely, the morphism $f_x$) with this property is unique up to $K_{Z_G(f_x)^\circ,G}$-conjugacy. If we take another finite subgroup $A$ with $Z_G(f_x) \subseteq A$ and $Z_G(A) = Z_G(f_x)$ then $[x]$ is also in the image of $H^1(\mathscr{E}, A \to Z_G(f_x)^\circ)$, and one checks easily that the image of $\iota_{[f_x(u)\to G]} \in \bar{Y}_{+,\mathrm{tor}}[f_x(u) \to G]$ in $\bar{Y}_{+,\mathrm{tor}}[A \to G]$ (via the functor $\bar{Y}_{+,\mathrm{tor}}$ on the category $\mathsf{R}^H_{(G)}$

applied to $[f_x(u) \to G] \to [A \to G]$, cf. Proposition 3.5) coincides with $\iota_{[A\to G]}([x])$ (which is well-defined, by Proposition 3.9).

One verifies easily that, for a fixed rigid Newton centralizer $H$ we have a disjoint union decomposition

$$\varinjlim_{[A\to G]\in \mathsf{R}^H_{(G)}} \bar{Y}_{+,\mathrm{tor}}[A \to G] = \bigsqcup_{i\in I_H^\circ} \varinjlim_{A_i} \bar{Y}_{+,\mathrm{tor}}[A_i \to G], \tag{28}$$

where we have fixed set of equivalence class representatives $\{g_i\}_{i\in I_H^\circ}$ for $\sim$ restricted to $K_{N_H(S),G}$ as in Lemma 2.18, denoting ${}^{g_i}H$ by $H_i$ and the direct limits on the right are over $F$-rational finite $A_i$ such that $Z_G(A_i) = H_i$.

We can now give an upgraded version of Definition 3.12; its codomain will be the pointed setoid

$$\bar{\mathfrak{Y}}_{+,\mathrm{tor}}(G) := \bigsqcup_{[H]} \varinjlim \bar{Y}_{+,\mathrm{tor}}[A \to G], \tag{29}$$

where the disjoint union is over the collections of subgroups $[H] = K_{H,G}\cdot H$ and for each $H$ the direct limit is with respect to all finite subgroups $A$ of $H$ such that $Z_G(A) = H$ (as in (28)).

**Definition 3.14.** We define the *rigid Kottwitz map* as the map

$$H^1_{\mathrm{reg}}(\mathscr{E}, G) \xrightarrow{\kappa} \bar{\mathfrak{Y}}_{+,\mathrm{tor}}(G),$$

given by sending $[x]$ to its image in $\bar{Y}_{+,\mathrm{tor}}[f_x(u) \to Z_G(f_x)]$ (for a representative $x$ as in Definition 3.12; in particular $[x]$ is in the image of $H^1_{\mathrm{bas}}(\mathscr{E}, Z_G(f_x)^\circ)$) and then to its image in the direct limit (28) for $H = Z_G(f_x)$.

We have the following immediate analogue of Lemma 3.13:

**Corollary 3.15.** *Two elements $[x],[y] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$ have the same image under $\kappa$ and satisfy $\nu_G([x]) = \nu_G([y])$ if and only if they are equal.*

We will describe the image of $H^1_{\mathrm{reg}}(\mathscr{E}, G)$ via $\kappa$ (equivalently, the image of each $H^1(\mathscr{E}, G)_{H^\circ}$) in the next subsection (Proposition 3.16).

3.4. **Rigid Newton centralizers.** A natural question is which subgroups $H$ of $G$ arise as $H = Z_G(f_x)$ for $[x] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$.

By design, every regular class is in the image of $H^1(\mathscr{E}, T)$ for some $F$-rational maximal torus $T$ of $G$. We can thus compute all possible rigid Newton centralizers in $G$ by ranging over all such $T$ and studying the Newton map of $H^1(\mathscr{E}, T)$: The key insight needed for this approach is due to [Kal16b, §4.1], which shows that, for a fixed finite $Z \subset T$ and $k \gg 0$, the Tate-Nakayama duality isomorphism $\iota_{[Z\to T]}$ fits into the diagram with exact bottom row:

$$\begin{array}{ccccc} H^1(\mathscr{E}, Z \to T) & \longrightarrow & \mathrm{Hom}(\mu_{n_k,\overline{F}}, Z_{\overline{F}}) & & \\ \downarrow\wr & & \downarrow \mathrm{id} & & \\ \bar{Y}_{+,\mathrm{tor}}[Z \to T] := \frac{X_*(T/Z)^{N_{E_k/F}}}{I\cdot X_*(T)} & \longrightarrow & \mathrm{Hom}(\mu_{n_k,\overline{F}}, Z_{\overline{F}}) & \longrightarrow & \frac{X_*(T)^\Gamma}{N_{E_k/F}(X_*(T))}, \end{array} \tag{30}$$

where the first map in the bottom row sends $\bar{\lambda}$ to $[n_k\cdot\bar{\lambda}]|_{\mu_{n_k,\overline{F}}}$ and the second is obtained by, starting with $\nu \in \mathrm{Hom}(\mu_{n_k,\overline{F}}, Z_{\overline{F}})$, lifting to $\bar{\lambda} \in X_*(T/Z)$ such that $[n_k\cdot\bar{\lambda}]|_{\mu_{n_k,\overline{F}}} = \nu$ (such a lift always exists) and then taking $N_{E_k/F}(\bar{\lambda})$. It thus follows that the image of $H^1(\mathscr{E}, Z \to T) \to \mathrm{Hom}(\mu_{n_k}, Z)$ is the kernel of the bottom-right map in (30).

Before continuing with the stated goal of classifying some rigid Newton centralizers, we can use the diagram (30) to describe the image of the rigid Kottwitz map $\kappa$. For a fixed $H \cdot K_{H,G}$ one has (for a choice of $H$) the orbit space

$$\frac{[X_*(S/A)/X_*(S_{\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{\mathrm{sc}})]}/W(G, H^\circ)(F)$$

where $A$ is some finite $F$-rational subgroup of $G$ with $Z_G(A) = H$ and $S$ is an elliptic maximal torus of $H$. Via the diagram (30) we have a map

$$\frac{[X_*(S/A)/X_*(S_{\mathrm{sc}})]^N}{I[X_*(S)/X_*(S_{\mathrm{sc}})]}/W(G, H^\circ)(F) \to \mathrm{Hom}(\mu_{\overline{F}}, S_{\overline{F}})_{H,+}/W(G, H^\circ)(F),$$

where $\mathrm{Hom}(\mu_{\overline{F}}, S_{\overline{F}})_{H,+}$ denotes the set of homomorphisms whose $\Gamma$-orbit has centralizer $H$ (and thus admits an action of $W(G, H^\circ)(F)$, since it factors through $Z(H^\circ)$). We obtain:

**Proposition 3.16.** *The image in $\mathfrak{Y}_{+,\mathrm{tor}}(G)$ of all classes in $H^1(\mathscr{E}, G)_{H^\circ}$ whose centralizers are in $K_{H,G} \cdot H$ for a choice of representative $H$ is exactly all elements of the orbit space (for any $g \in K_{N_{H^\circ}(S),G}$) $\frac{[X_*({}^gS/{}^gA)/X_*({}^gS_{\mathrm{sc}})]^N}{I[X_*({}^gS)/X_*({}^gS_{\mathrm{sc}})]}/W(G, {}^gH^\circ)(F)$ whose image in $\mathrm{Hom}(\mu_{\overline{F}}, T_{\overline{F}})$ has centralizer in $K_{H^\circ,G} \cdot H$.*

Returning to the rigid Newton centralizer question, we now specialize to the case $Z = T[n_k]$ for a fixed $n_k \in \mathbb{N}$ (with $k \gg 0$ so that $E_k/F$ splits $T$); every finite $Z$ is contained in some such subgroup. In this case, the identification $T/T[n_k] \xrightarrow{[n_k],\sim} T$ gives a more concrete description of the bottom row of (30). Under this identification one has $\frac{X_*(T/Z)^{N_{E_k/F}}}{I \cdot X_*(T)} = \frac{X_*(T)^{N_{E_k/F}}}{I \cdot [n_k X_*(T)]}$, the first bottom-row map sends $\lambda \in X_*(T)$ to $\lambda|_{\mu_{n_k,\overline{F}}}$, and the last map sends $\lambda$ to $N_{E_k/F}(\lambda)$. It follows that the kernel of the last bottom-row map gets identified with all $\nu \in \mathrm{Hom}(\mu_{n_k,\overline{F}}, T[n_k]_{\overline{F}})$ which have a lift $\lambda \in X_*(T)$ satisfying $N_{E_k/F}(\lambda) \in n_k[N_{E_k/F}(X_*(T))]$. We therefore deduce:

**Proposition 3.17.** *For $k$ as above, the image of the map $H^1(\mathscr{E}, T[n_k] \to T) \to \mathrm{Hom}(\mu_{n_k,\overline{F}}, T[n_k]_{\overline{F}})$ is exactly those $\nu \in \mathrm{Hom}(\mu_{n_k,\overline{F}}, T[n_k]_{\overline{F}})$ which arise as $\lambda|_{\mu_{n_k,\overline{F}}}$ for $\lambda \in X_*(T)^{N_{E_k/F}}$.*

*Proof.* The above argument shows that any such $\nu$ is the restriction of $\lambda \in X_*(T)$ satisfying $N_{E_k/F}(\lambda) = N_{E_k/F}(n_k\lambda')$ for some $\lambda' \in X_*(T)$. It follows that $\lambda - n_k\lambda' \in X_*(T)^{N_{E_k/F}}$ and has the same restriction to $\mu_{n_k}$ as $\lambda$, since $\lambda'|_{\mu_{n_k,\overline{F}}}$ takes values in $T[n_k]$. □

We can use Proposition 3.17 to show:

**Theorem 3.18.** *Every twisted Levi subgroup $L$ of $G$ containing an elliptic maximal torus of $G$ is a rigid Newton centralizer.*

*Proof.* Fix an elliptic maximal torus $T$ of $G$ contained in $L$. Choose $E/F$ a finite Galois extension splitting $T$ and pick any $\lambda \in X_*(T)$ with $L = Z_G(\lambda)$. We make the obvious but important observation that, setting $R := R(G_{\overline{F}}, T_{\overline{F}})$, the finite set of values

$$\{\langle {}^\sigma\lambda, \alpha\rangle\}_{\alpha \in R,\ \sigma \in \Gamma_{E/F}} \subset \mathbb{Z}$$

is bounded, and so if we take $k \gg 0$ we can assume that $E_k$ contains $E$ and, for any $\alpha \in R$ and any $\gamma \in \Gamma_{E_k/F}$, one has $\sum_{\sigma \in \Gamma_{E/F}} \langle {}^\gamma\lambda - {}^{\gamma\sigma}\lambda, \alpha\rangle \in n_k\mathbb{Z}$ if and only if it equals zero.

Enlarging $k$ so it has the above property, pick some $\sigma \in \Gamma_{E/F}$ with $\lambda - {}^{\sigma}\lambda \neq 0$, which always exists since $T$ is elliptic in $G$. The claim is that $L$ is the centralizer of the homomorphism $u \to T[n_k]$ determined by the identity coordinate $(\sum_{\sigma\in\Gamma_{E/F}} \lambda - {}^{\sigma}\lambda)|_{\mu_{n_k},\overline{F}}$ which, in view of Proposition 3.17, arises as $f_x$ for some $[x] \in H^1(\mathscr{E}, T)$, proving that $L$ is a rigid Newton centralizer.

First, note that $Z_G(\sum_{\sigma\in\Gamma_{E/F}} \lambda - {}^{\sigma}\lambda) = Z_G([\sum_{\sigma\in\Gamma_{E/F}} \lambda - {}^{\sigma}\lambda)|_{\mu_{n_k},\overline{F}})$, since, by the choice of $k$ discussed above, we have $\langle \sum_{\sigma\in\Gamma_{E/F}} \lambda - {}^{\sigma}\lambda, \alpha\rangle \in n_k\mathbb{Z}$ if and only if $\langle \sum_{\sigma\in\Gamma_{E/F}} \lambda - {}^{\sigma}\lambda, \alpha\rangle = 0$, and the same holds for the corresponding centralizers of any $\Gamma$-conjugate of $\sum_{\sigma\in\Gamma_{E/F}} \lambda - {}^{\sigma}\lambda$. It thus suffices to prove the equality

$$\bigcap_{\gamma\in\Gamma} Z_G(\sum_{\sigma\in\Gamma_{E/F}} {}^{\gamma}\lambda - {}^{\gamma\sigma}\lambda) = L$$

using the cocharacters rather than their restrictions to $\mu_{n_k,\overline{F}}$. Since $L$ is $\Gamma$-stable, the containment of $L$ in the left-hand side of the above equation is evident.

For the other containment it suffices to show that $Z_G(\sum_{\sigma\in\Gamma_{E/F}} \lambda - {}^{\sigma}\lambda)$ is contained in $L$ (this is a much stronger statement than we need and implies that each term in the above intersection is exactly $L$). Suppose $\alpha \in R$ satisfies $\langle \sum_{\sigma\in\Gamma_{E/F}} \lambda - {}^{\sigma}\lambda, \alpha\rangle = 0$, which is equivalent to the equality

$$|E : F|\langle\lambda, \alpha\rangle = \langle \sum_{\sigma\in\Gamma_{E/F}} {}^{\sigma}\lambda, \alpha\rangle. \tag{31}$$

Note that $\sum_{\sigma\in\Gamma_{E/F}} {}^{\sigma}\lambda \in X_*(T)^{\Gamma_{E/F}}$, and, since $T$ is elliptic in $G$, this must factor through $Z(G)$ and hence the right-hand side of (31) is zero, which forces $\langle\lambda, \alpha\rangle = 0$, as desired. □

**Remark 3.19.** The above proof shows that, in fact, for any twisted Levi subgroup $L$ containing an elliptic maximal torus of $G$ there is some class $[x] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$ such that the identity coordinate $f_{e,x} \in \mathrm{Hom}(\mu_{\overline{F}}, T_{\overline{F}})$ satisfies $L = Z_G(f_{e,x}) = Z_G(f_x)$; in other words, $L$ comes from a cyclic class, not just a regular class.

There are twisted Levi subgroups of the form $Z_G(f_x)$ for $[x] \in H^1_{\mathrm{reg}}(\mathscr{E}, G)$ which do not contain elliptic maximal tori of $G$:

**Example 3.20.** Let $G = \mathrm{SL}_3$, which contains $\mathrm{GL}_2$ as a maximal proper Levi subgroup in the usual way. For $K/F$ a quadratic extension we have the anisotropic norm-1 torus $S := \mathrm{Res}^1_{K/F}(\mathbb{G}_m) \subset \mathrm{GL}_2 \subset \mathrm{SL}_3$ and we claim that $Z_{\mathrm{SL}_2}(S) = \mathrm{Res}_{K/F}(\mathbb{G}_m) \subset \mathrm{GL}_2 \subset \mathrm{SL}_3$. Evidently $Z_{\mathrm{SL}_2}(S)$ is a twisted Levi subgroup of $G$ and contains $\mathrm{Res}_{K/F}(\mathbb{G}_m)$; it is easy to see that, over $\overline{F}$, there are only three such Levi subgroups: $\mathrm{Res}_{K/F}(\mathbb{G}_m)$ itself, $\mathrm{GL}_2$, or $\mathrm{SL}_3$. Since $S$ is not central in $\mathrm{GL}_2$, the claim follows.

We may then pick a cocharacter $\lambda \in X_*(\mathrm{Res}_{K/F}(\mathbb{G}_m)^{(1)})$ with $Z_{\mathrm{SL}_3}(\lambda) = Z_{\mathrm{SL}_3}(S) = \mathrm{Res}_{K/F}(\mathbb{G}_m)$; since $\lambda$ factors through an anisotropic torus, it is evidently killed by the $\Gamma$-norm, and it follows from the proof of Theorem 3.18 that $Z_{\mathrm{SL}_3}(\lambda) = \mathrm{Res}_{K/F}(\mathbb{G}_m)$ is a rigid Newton centralizer. However, it evidently does not contain an elliptic maximal torus of $G$, since it is a non-elliptic maximal torus.

In fact, the situation is even more general than the above example suggests: There are (connected) rigid Newton centralizers in $G$ which are not twisted Levi subgroups. Indeed, if $T$ is an anisotropic maximal torus of $G$ and $s \in T[n_k](F)$ is a torsion element such that $Z_G(s)$ is not a twisted Levi subgroup, then $Z_G(s)$ is such a subgroup, since one can extend the homomorphism $\mu_{n_k} \to T$ sending a choice of $n_k$-root of unity to $s$ to a cocharacter $\mathbb{G}_{m,\overline{F}} \xrightarrow{\lambda} T_{\overline{F}}$ (via the canonical inclusion $\mu_{n_k} \hookrightarrow T$). The fact that $T$ is anisotropic guarantees that $\lambda$ is killed by $N_{E_k/F}$, and hence by

Proposition 3.17 the group $Z_G(\lambda|_{\mu_{n_k}}) = Z_G(s)$ is defined over $F$ and is thus (cyclic) regular. The following example gives such an element $s$.

**Example 3.21.** Let $G = G_2$ with a choice of $T$ an anisotropic maximal torus split over a quadratic extension $E/F$ on which $\Gamma_{E/F}$ acts by inversion—in particular, we have $T[2](\overline{F}) = T[2](F)$. Let $a, b$ be the short and long (respectively) elements of a root basis corresponding to a choice of Borel subgroup containing $T_{\overline{F}}$, with corresponding fundamental coweights $u, v$ (respectively); the claim is that if we set $s = v(-1) \in T[2](F)$ then $Z_{G_2}(s)$ is not a twisted Levi subgroup of $G_2$.

For any root $\alpha \in R(G_{2,\overline{F}}, T_{\overline{F}})$, one checks that $\alpha(s) = (-1)^n$, where $n$ is the coefficient modulo 2 of $a$ in $\alpha$. The only such $\alpha$ which are positive and have trivial $a$-coefficients modulo 2 are $b$ or $2a + b$, which span a root system of type $A_1 \times A_1$, and so we deduce that $Z_{G_2}(s)_{\overline{F}}$ is isomorphic to $\mathrm{SO}_4$, giving the claim.

## 4. Applications to the local Langlands correspondence

The goal of this section is to use the machinery developed in the previous two sections to give new formulations of the rigid refined local Langlands correspondence. We assume throughout that $G$ is a connected reductive $F$-group, which moreover is quasi-split.

Up until this point we have worked with arbitrary classes in $H^1_{\mathrm{reg}}(\mathscr{E}, G)$, but now we will work with $H^1_{\text{L-reg}}(\mathscr{E}, G)$ for the rest of this paper. We do not believe that this is a strictly necessary restriction, but elect to impose it for two main reasons.

First, by assuming that our rigid Newton centralizers $H = Z_G(f_x)$ are connected we have a clear formulation of the local Langlands conjectures, which are not yet fully developed for disconnected reductive groups. Although there has been substantial recent progress, in [Kal22], towards extending the conjectures to disconnected groups, that work deals only with reductive groups $G$ that are *inner forms* (not in the usual sense, but rather in the sense of [Kal22, §3.2]) of an $F$-group of the form $G^\circ \rtimes \underline{A}$, where $G^\circ$ is a quasi-split connected reductive group and $\underline{A}$ is a constant finite $F$-group scheme. The authors expect that there are disconnected rigid Newton centralizers $H$ that are not of this form.

The second reason for restricting to twisted Levi subgroups is related to $L$-embeddings and will be explained in more detail later in this section (§4.3.5). For now we remark that in order to view an $L$-parameter $\phi\colon W_F \times \mathrm{SL}_2 \to {}^LG$ as an $L$-parameter for a twisted Levi subgroup $M$ of $G$ we use the canonical $\widehat{G}$-conjugacy class of embeddings ${}^LM_\pm \to {}^LG$ constructed in [Kal21a] and consider the classes through which $\phi$ factors. Here ${}^LM_\pm$ is the $L$-group of the double cover $M(F)_\pm \to M(F)$ (relative to $G$) constructed in [Kal21a] to obtain the aforementioned conjugacy class of embeddings. We recall these constructions in detail in §4.1.1.

The problem is that when $H \subseteq G$ contains a maximal torus of $G$ but is not a twisted Levi subgroup, there is not always such a canonical conjugacy class of $L$-embeddings. In fact, there is in general no embedding $\widehat{H} \to \widehat{G}$, a dual version of the phenomenon that endoscopic groups are not always subgroups. For example, if $G = \mathrm{Sp}_8$ and $H = \mathrm{Sp}_4 \times \mathrm{Sp}_4$, which is the centralizer of an $F$-rational element of order two, then there is no embedding of $\widehat{H} = \mathrm{SO}_5(\mathbb{C}) \times \mathrm{SO}_5(\mathbb{C})$ into $\widehat{G} = \mathrm{SO}_9(\mathbb{C})$ because the root system $C_4$ has no closed subsystem of type $C_2 \times C_2$.

To distinguish these twisted Levi subgroups from the more general centralizers studied in §3, we denote them by $M$ rather than $H$.

One advantage of restricting to the class $K_{M,G} \cdot M$ consisting of twisted Levi subgroups of $G$ is:

**Lemma 4.1.** *When some (equivalently, any) $M \in K_{M,G} \cdot M$ is a twisted Levi subgroup, there is always a quasi-split member of $K_{M,G} \cdot M$.*

*Proof.* This follows immediately from [Kal21a, Lemma 6.4] applied to $\xi = \mathrm{id}$. □

Since all members of $K_{M,G} \cdot M$ are inner forms of each other, the quasi-split member is unique up to $F$-rational isomorphism; we fix such a member for each collection of subgroups, and denote it by $M$.

## 4.1. Dual preliminaries.

4.1.1. *Double covers and canonical L-embeddings.* This subsubsection reviews the construction in [Kal21a] of canonical conjugacy classes of $L$-embeddings associated to inclusions of twisted Levi subgroups. In particular, nothing here is original. Fix a dual group $\widehat{G}$ of $G$, an $F$-pinning $(\mathscr{B}_G, \mathscr{T}_G, \{X_{\hat{\alpha}_G}\})$ of $\widehat{G}$, and a $\Gamma$-stable pinning $(B_G, T_G, \{X_{\alpha_G}\})$ of $G$.

Let $T$ be an $F$-rational maximal torus of $G$. We can associate to $T \hookrightarrow G$ a canonical $\widehat{G}$-conjugacy class $J_T$ of embeddings $\widehat{T} \to \widehat{G}$ which is stable under the $\Gamma$-action as follows. Choosing some $g \in G(\overline{F})$ such that ${}^gT = T_G$, we get an isomorphism $\widehat{T} \xrightarrow{\widehat{\mathrm{Ad}(g^{-1})}} \widehat{T_G} \xrightarrow{\sim} \mathscr{T}_G$, where the second isomorphism comes from the duality between $G$ and $\widehat{G}$. As explained in [Kal19, §5.1], the $\widehat{G}$-conjugacy class of this embedding is $\Gamma$-stable and independent of the choices of pinnings.

[Kal21a] constructs a canonical double cover

$$1 \to \{\pm 1\} \to T(F)_\pm \to T(F) \to 1,$$

depending on $G$, as well as a dual group ${}^LT_\pm$ that comes equipped with a canonical $\widehat{G}$-conjugacy class of $L$-embeddings $\eta\colon {}^LT_\pm \to {}^LG$. A key application of the double cover $T(F)_\pm$ is extending the class $J_T$ of admissible embeddings to a canonical $\widehat{G}$-conjugacy of embeddings ${}^LT_\pm \to {}^LG$.

For expository completeness, we briefly review the construction of the above double cover and its $L$-group. Let $S$ be an arbitrary torus defined over $F$, set $\Sigma := \Gamma \times \{\pm 1\}$, and take $R$ an *admissible* $\Sigma$-set, which is a set with a $\Sigma$-action without $-1$-fixed points. Also fix a $\Sigma$-invariant map $R \to X^*(S)$. A $\Sigma$-orbit $O$ is called *symmetric* if it is a single $\Gamma$-orbit; otherwise it is the disjoint union of two $\Gamma$-orbits and is called *asymmetric*. For $\alpha \in R$ set $\Gamma_\alpha = Z_\Gamma(\alpha)$, $\Gamma_{\pm\alpha} = Z_\Gamma(\{\pm\alpha\})$, $F_\alpha := (F^s)^{\Gamma_\alpha}$ and $F_{\pm\alpha} := (F^s)^{\Gamma_{\pm\alpha}}$. So an orbit $O$ is symmetric if and only if $[F_\alpha\colon F_{\pm\alpha}] = 2$ for some (equivalently, every) $\alpha \in O$.

The double cover is constructed from $S$ and the map $R \to X^*(S)$ in stages: we first make a construction for each $\Sigma$-orbit $O$ of $R$, depending on whether the orbit is symmetric or asymmetric, and then combine the constructions together.

For $O$ asymmetric, choose $\alpha \in O$ and define $J_\alpha = \mathrm{Res}_{F_\alpha/F}(\mathbb{G}_m)$. Any other $\beta \in O$ is of the form $\beta = \epsilon\tau\alpha$ for a unique $\epsilon \in \{\pm 1\}$ and for $\tau \in \Gamma$ with uniquely determined coset in $\Gamma_\alpha\backslash\Gamma$ and the map $f_\alpha \mapsto f_\beta$, where $f_\beta(x) = \epsilon f_\alpha(\tau^{-1}x)$, is an isomorphism $X^*(J_\alpha) \to X^*(J_\beta)$ depending only on $\alpha$ and $\beta$. Define $J_O$ to be the inverse limit over all $\alpha \in O$ of $J_\alpha$ and define the canonical split double cover of $J_O$ to be $J_O(F)_\pm = J_O(F) \times \{\pm 1\}$. The map $R \to X^*(S)$ induces a $\Gamma$-equivariant map $X^*(J_\alpha) = \mathbb{Z}[\Gamma_\alpha\backslash\Gamma] \to X^*(S)$ sending $f$ to $\sum_{\tau\in\Gamma_\alpha\backslash\Gamma} f(\tau)\tau^{-1}\bar{\alpha}$, where $\bar{\alpha}$ is the image of $\alpha$ in $X^*(S)$, and the duals $S \to J_\alpha$ assemble to a morphism $S \to J_O$.

For $\alpha \in O$ symmetric, let $\mathrm{Res}^1_{F_\alpha/F_{\pm\alpha}}(\mathbb{G}_m)$ be the norm-one elements in $\mathrm{Res}_{F_\alpha/F_{\pm\alpha}}(\mathbb{G}_m)$ and define $J_\alpha = \mathrm{Res}_{F_{\pm\alpha}/F}(\mathrm{Res}^1_{F_\alpha/F_{\pm\alpha}}(\mathbb{G}_m))$, so that

$$X^*(J_\alpha) = \{f\colon \Gamma \to \mathbb{Z} \mid \forall\sigma \in \Gamma_{\pm\alpha}\colon f(\sigma\tau) = \kappa_\alpha(\sigma)f(\tau)\},$$

where $\kappa_\alpha$ is the sign character of the quadratic extension $\Gamma_{F_\alpha/F_{\pm\alpha}} \xrightarrow{\sim} F_{\pm\alpha}^\times/N(F_\alpha^\times)$. So $J_\alpha(F)$ is simply $F_\alpha^1$, the kernel in $F_\alpha^\times$ of the $F_\alpha/F_{\pm\alpha}$-norm. Any other $\beta \in O$ is of the form $\beta = \tau\alpha$ for some $\tau \in \Gamma$ with unique coset in $\Gamma_\alpha\backslash\Gamma$, so that $\Gamma_{\pm\beta} = \tau\Gamma_{\pm\alpha}\tau^{-1}$, $\Gamma_\beta = \tau\Gamma_\alpha\tau^{-1}$, and $\kappa_\beta = \kappa_\alpha \circ \mathrm{Ad}(\tau)$. It follows that the map $f_\alpha \mapsto f_\beta$, where $f_\beta(x) = f_\alpha(\tau^{-1}x)$, is an isomorphism $X^*(J_\alpha) \to X^*(J_\beta)$ that depends only on $\alpha$ and $\beta$. We may thus define $J_O$ as the inverse limit of $J_\alpha$ over all $\alpha \in O$. At the same time, the map $R \to X^*(S)$ induces a $\Gamma$-equivariant map $X^*(J_\alpha) \to X^*(S)$ sending $f$ to $\sum_{\tau\in\Gamma_{\pm\alpha}\backslash\Gamma} f(\tau)\tau^{-1}\bar{\alpha}$, and the duals of these maps assemble to a morphism $S \to J_O$.

To define the double cover in the symmetric case, given $\alpha \in O$, denote by $\tau_\alpha$ the nontrivial element of $\Gamma_{F_\alpha/F_{\pm\alpha}}$. The map $x \mapsto x/\tau_\alpha(x)$ defines a double cover

$$1 \to \frac{F_{\pm\alpha}^\times}{N(F_\alpha^\times)} \to \frac{F_\alpha^\times}{N(F_\alpha^\times)} \to F_\alpha^1 \to 1,$$

and so identifying $F_{\pm\alpha}^\times/N(F_\alpha^\times)$ with $\{\pm1\}$ (via $\kappa_\alpha$) and $F_\alpha^1$ with $J_\alpha(F)$ gives a double cover $J_\alpha(F)_\pm$ of $J_\alpha$. Any $\tau \in \Gamma_\alpha\backslash\Gamma$ induces an isomorphism $F_\alpha^\times \to F_\beta^\times$, and thus an isomorphism $J_\alpha(F)_\pm \to J_\beta(F)_\pm$, and so it makes sense to define $J_O(F)_\pm$ as the double cover of $J_O$ given by limit over the double covers $J_\alpha(F)_\pm$ for $\alpha \in O$.

We therefore have a $\prod_O\{\pm1\}$-extension of $S(F)$ given by $S(F) \times_{\prod J_O(F)} \prod J_O(F)_\pm$. Define the double cover $S(F)_\pm$ as the pushout of this extension by the product map $\prod_{O\subset R}\{\pm1\} \to \{\pm1\}$.

It remains to define the $L$-group of $S(F)_\pm$. Let $E/F$ be finite Galois such that $\Gamma_E$ acts trivially on $R$ and on $X^*(S)$. Given a gauge $p$ on $R$ (that is, a $\{\pm1\}$-equivariant function $p\colon R \to \{\pm1\}$), we define the corresponding *Tits cocycle* $t_p \in Z^2(\Gamma_{E/F}, \widehat{S})$ by the formula $t_p(\sigma,\tau) = (-1)^{\lambda_p(\sigma,\tau)}$, where $\lambda_p(\sigma,\tau) \in X^*(S) = X_*(\widehat{S})$ is the sum of $\bar{\alpha}$ for $\alpha$ in the subset $\Lambda_p(\sigma,\tau)$of $R$ defined by

$$\Lambda_p(\sigma,\tau) = \{\alpha \in R \mid p(\alpha) = 1, p(\sigma^{-1}\alpha) = -1, p((\sigma\tau)^{-1}\alpha) = 1\}.$$

Given another gauge $q$ there is a canonical cochain $s_{p/q} \in C^1(\Gamma_{E/F}, \widehat{S})$ such that $t_p/t_q = ds_{p/q}$, and for any three gauges $p$, $q$, and $r$ the cochains $s_{p/q}$ and $s_{q/p}$ are cohomologous, as are the cochains $s_{p/q}\cdot s_{q/r}$ and $s_{p/r}$.

To define the $L$-group of $S_\pm$, first define, for any gauge $p$, the twisted product ${}^LS_\pm^{(p)} := \widehat{S} \boxtimes_{t_p} S_\pm$, which is independent of the choice of $p$ up to conjugation by $\widehat{S}$. The Galois form of the $L$-group ${}^LS_\pm$ is then the inverse limit over the system of extensions ${}^LS_\pm^{(p)} := \widehat{S} \boxtimes_{t_p} \Gamma$ over all gauges with the transition isomorphisms given by $s \boxtimes \sigma \mapsto s \cdot s_{p/q} \boxtimes \sigma$. The Weil form of the $L$-group is the pullback along $W_F \to \Gamma$ of the Galois form.

We now construct the aforementioned canonical $\widehat{G}$-conjugacy class of $L$-embeddings

$${}^LT_\pm \to {}^LG,$$

taking $S = T$ and taking $R \to X^*(T)$ to be inclusion of the root system $R(G_{\overline{F}}, T_{\overline{F}}) \subset X^*(T)$.

Set $\Omega := W(\widehat{G}, \mathscr{T}_G)$. Our earlier choice of pinning defines a Tits section $\Omega \to N_{\widehat{G}}(\mathscr{T}_G)$, denoted by $\omega \mapsto n(\omega)$. Choose an isomorphism $\widehat{\jmath}\colon \widehat{T} \to \mathscr{T}_G \subset \widehat{G}$ in the class $J_T$ of embeddings. For any $\sigma \in \Gamma$, the formula $\sigma_{T,G} := \sigma \circ \widehat{\jmath} \circ \sigma^{-1}$ defines an element of $\Omega$ and the $F$-structure on $\mathscr{T}_G$ is defined by the homomorphism $\Gamma \to \Omega \rtimes \Gamma$ sending $\sigma$ to $\sigma_{T,G} \rtimes \sigma$. Let $p$ be the gauge on $R$ which assigns 1 to the $\mathscr{B}_G$-positive roots. It turns out that we can explicitly extend $\widehat{\jmath}\colon \widehat{T} \to \mathscr{T}_G$ via the map

$$\widehat{T} \boxtimes_{t_p} \Gamma \to {}^LG, \qquad t \boxtimes \sigma \mapsto \widehat{\jmath}(t) \cdot n(\sigma_{T,G}) \rtimes \sigma,$$

where the twisted product is formed using the identification of $R$ as a subset of $X^*(\mathscr{T}_G)$ via $\widehat{\jmath}$. Moreover, the $\widehat{G}$-conjugacy class of this extended map is canonical.

The construction of double covers and their $L$-groups extends to arbitrary twisted Levi subgroups (now following [Kal21a, §6]). Let $M$ be a twisted Levi subgroup of $G$ containing the maximal torus $T$. As in the construction of the conjugacy class of embeddings $J_T$, it will be useful for working with dual groups to reduce to the quasi-split case. It follows from [Kal21a, Lemma 6.4] (applied to $\xi = \mathrm{id}$, as in the proof of Lemma 4.1) that there is some $g \in G(\overline{F})$ such that the conjugate $M^* := {}^g M$ is quasi-split and defined over $F$ and the map $\mathrm{Ad}(g)\colon M_{\overline{F}} \to M^*_{\overline{F}}$ is an inner twisting. The embedding $M \hookrightarrow G$ yields a unique $\widehat{G}$-conjugacy of Levi subgroups of $\widehat{G}$ and the fixed pair $(\mathscr{B}_G, \mathscr{T}_G)$ singles out a unique Levi subgroup $\mathscr{T}_G \subseteq \mathscr{M}$ in this conjugacy class, whose base is dual to ${}^{g^{-1}}\Delta_{(B_{M^*}, T_{M^*})} \subseteq \Delta_{(B_G, T_G)}$ under the standard bijection of $\Delta^\vee_{(B_G, T_G)}$ with $\Delta_{(\mathscr{B}_G, \mathscr{T}_G)}$.

Fix $\Gamma$-stable Borel pairs $(B_{M^*}, T_{M^*})$ of $M^*$ and $(B_G, T_G)$ of $G$ along with an element $g \in G(\overline{F})$ such that $B_{M^*} \subseteq {}^g B_G$ and $T_{M^*} = {}^g T_G$, so that $\Delta_{(B_{M^*}, T_{M^*})} \subseteq {}^g \Delta_{(B_G, T_G)}$. Since both pairs are $\Gamma$-stable, for any $\sigma \in \Gamma$ the formula $\sigma_{M,G} := g^{-1} \cdot {}^\sigma g$ defines an element of $W(G_{\overline{F}}, T_{G,\overline{F}})$ and the bijection $R(G_{\overline{F}}, T_{G,\overline{F}}) \to R(\widehat{G}, \mathscr{T}_G)^\vee$ induces an isomorphism $W(G_{\overline{F}}, T_{G,\overline{F}}) \to \Omega$ which lets us view $\sigma_{M,G}$ as an element of $\Omega$. Similarly, denote by $\sigma_G$ the pinned automorphism of $\widehat{G}$ obtained from $\sigma$.

For $S$ a maximal torus of $M$ define $R_S$ as the set of $W(M_{\overline{F}}, S_{\overline{F}})$-orbits in $R(G_{\overline{F}}, S_{\overline{F}}) \setminus R(M_{\overline{F}}, S_{\overline{F}})$. It is shown in [Kal21a, Lemma 6.1] that $R_S$ is a finite admissible $\Sigma$-set, the map $R_S \to X^*(S)$ sending an orbit to the sum of its elements is $\Sigma$-equivariant and factors through the subgroup $X^*(M_{\mathrm{ab}})$, and for any $m \in M(\overline{F})$ that conjugates $S$ to $S'$, the map $\mathrm{Ad}(m)$ induces a canonical, $\Sigma$-equivariant bijection $R_S \to R_{S'}$ compatible with the maps of both sets to $X^*(M_{\mathrm{ab}})$. It thus makes sense to define the $\Sigma$-set $R(M_{\mathrm{ab}}, G)$ as the limit over all $S \subset M$ of $R_S$, which has a map to $X^*(M_{\mathrm{ab}})$.

The admissible $\Sigma$-set $R(M_{\mathrm{ab}}, G)$ and map $R(M_{\mathrm{ab}}, G) \to X^*(M_{\mathrm{ab}})$ determines a double cover $M_{\mathrm{ab}}(F)_\pm \to M_{\mathrm{ab}}(F)$ and we define the double cover $M(F)_\pm \to M(F)$ as the pullback of the diagram $M(F) \to M_{\mathrm{ab}}(F) \leftarrow M_{\mathrm{ab}}(F)_\pm$ with $L$-group ${}^L M_\pm$ given by the push-out of $\widehat{M} \leftarrow Z(\widehat{M})^\circ \to {}^L M_{\mathrm{ab},\pm}$. We conclude by summarizing the main results concerning this construction:

**Proposition 4.2.** *(*[Kal21a, Lemmas 6.11, 6.12]*) For $T \subset M \subset G$ a twisted Levi subgroup with maximal torus $T$:*

1. *Let $T(F)_{M,\pm}$ denote the the double cover of $T$ for the admissible $\Sigma$-set $R(G_{\overline{F}}, T_{\overline{F}}) \setminus R(M_{\overline{F}}, T_{\overline{F}})$. The map $T(F) \to M(F)$ extends to a map $T(F)_{M,\pm} \to M(F)_\pm$.*
2. *The canonical $\widehat{G}$-conjugacy of embeddings $\widehat{M} \xrightarrow{\sim} \mathscr{M} \hookrightarrow \widehat{G}$ (with representative $\widehat{\jmath}_{M,G}$) constructed above extends to a canonical $\widehat{G}$ conjugacy class of $L$-embeddings ${}^L M_\pm \to {}^L G$.*
3. *The canonical $\widehat{M}$-conjugacy class of embeddings $\widehat{T} \to \widehat{M}$ (with representative $\widehat{\jmath}_{T,M}$) extends to a canonical $\widehat{M}$-conjugacy class of $L$-embeddings ${}^L T_\pm \to {}^L M_\pm$.*
4. *The $\widehat{G}$-conjugacy class of $L$-embeddings ${}^L T_\pm \to {}^L G$ constructed above factors as a composition of the two embeddings constructed in the two previous points in this Proposition.*

*Proof.* If $t_{M,G}$ and $t_{T,M}$ denote the Tits cocycles corresponding to the maps $\sigma \mapsto n(\sigma_{(M,G)})$ and $\sigma \mapsto n(\sigma_{(T,M)})$ respectively, then the claimed maps are given by

$$\widehat{M} \boxtimes_{t_{M,G}} \Gamma \to \widehat{G} \rtimes W_F, \quad \widehat{\jmath}_{M,G}(m) \boxtimes \sigma \mapsto m \cdot n(\sigma_{M,G}) \rtimes \sigma$$

and

$$\widehat{T} \boxtimes_{t_{T,M}} \Gamma \to \widehat{M} \boxtimes_{t_{M,G}} \Gamma, \quad \widehat{\jmath}_{T,M}(t) \boxtimes \sigma \mapsto t \cdot n(\sigma_{T,M}) \boxtimes \sigma.$$

The compatibility of these with ${}^LT_\pm \to {}^LG$ follows from the identity $n(\sigma_{T,M})\cdot n(\sigma_{M,G}) = n(\sigma_{T,G})$. We refer the reader to [Kal21a] for the details and the proof of the first statement. □

In fact, we will need the following strengthening of the above result; this does not appear as stated in [Kal21a] but follows from analogous arguments:

**Corollary 4.3.** *The analogue of Proposition 4.2 holds if one replaces the chain of inclusions $T \subset M \subset G$ with $M \subset N \subset G$, where $M$ and $N$ are arbitrary twisted Levi subgroups of $G$.*

4.1.2. *Definitions and setup.* Recall that, for a quasi-split connected reductive group $M$, there is a short exact sequence of $F$-group schemes

$$1 \to \underline{\mathrm{Inn}}_M \to \underline{\mathrm{Aut}}_M \to \underline{\mathrm{Out}}_M \to 1$$

which is split via a homomorphism $s_{\mathscr{P}_M}$ corresponding to a choice of $\Gamma$-stable pinning $\mathscr{P}_M$ for $M$. This splitting gives an identification

$$\frac{\underline{\mathrm{Aut}}_M(F)}{\underline{\mathrm{Inn}}_M(F)} = \underline{\mathrm{Out}}_M(F) \xrightarrow{\sim} \frac{\mathrm{Aut}_\Gamma(\widehat{M})}{\mathrm{Inn}_\Gamma(\widehat{M})},$$

as explained in [Kal23, §2.3.4], and also an inclusion

$$s_{\mathscr{P}_M}(\underline{\mathrm{Out}}_M(F)) \hookrightarrow \mathrm{Aut}_\Gamma(\widehat{M}), \tag{32}$$

where every automorphism in the image of (32) preserves the fixed pinning $\mathscr{P}_{\widehat{M}}$. When $M$ is in addition a twisted Levi subgroup of $G$, we can compose (32) with the map $W(G,M)(F) \to \underline{\mathrm{Out}}_M(F)$ to obtain an action of the group $W(G,M)(F)$ by $\Gamma$-stable automorphisms of $\widehat{M}$ which preserve the pinning $\mathscr{P}_{\widehat{M}}$. Denote the $F$-rational automorphism of $M$ (resp. $\Gamma$-equivariant automorphism of $\widehat{M}$) corresponding to $w \in W(G,M)(F)$ by $\theta_w$ (resp. $\theta_w^\vee$). Whenever we speak of the $W(G,M)(F)$-action on $M$ (resp. $\widehat{M}$), we always mean via the automorphisms $\theta_w$ (resp. $\theta_w^\vee$).

**Lemma 4.4.** *Let $M$ be a quasi-split twisted Levi subgroup of $G$. For any $w \in W(G,M)(F)$, pre-composing any embedding in $J_M$ by $\theta_w^\vee$ gives another element of $J_M$.*

Here $J_M$ is the canonical $\widehat{G}$-conjugacy class $J_M$ of embeddings $\widehat{M} \to \widehat{G}$ constructed in §4.1.1 that corresponds to the inclusion $M \hookrightarrow G$.

*Proof.* Let $(B_M, T_M)$ and $(B_G, T_G)$ be the Borel pairs in the pinnings $\mathscr{P}_M$ and $\mathscr{P}_G$ and let $(\mathscr{B}_M, \mathscr{T}_M)$ and $(\mathscr{B}_G, \mathscr{T}_G)$ be the Borel pairs in the $\Gamma$-stable pinnings $\mathscr{P}_{\widehat{M}}$ and $\mathscr{P}_{\widehat{G}}$. Choose $g \in G(\overline{F})$ such that ${}^g(B_M, T_M) \subseteq (B_G, T_G)$. Via the bijection $\Delta(B_G, T_G) \to \Delta(\mathscr{B}_G, \mathscr{T}_G)^\vee$, this choice singles out a subset $\Delta(\mathscr{B}_G, \mathscr{T}_G)_M$ of $\Delta(\mathscr{B}_G, \mathscr{T}_G)$ corresponding to the Levi subgroup $\mathscr{M}$, the image of $\widehat{M}$ under some element of $J_M$.

The $F$-rational lift $\theta_w$ in $\underline{\mathrm{Aut}}_M$ of $w \in \underline{\mathrm{Out}}_M(F)$ is the unique preimage of $w$ in $\underline{\mathrm{Aut}}_M(\overline{F})$ which preserves the pinning $\mathscr{P}_M$. Since an arbitrary lift $g_w' \in N_G(M)(\overline{F})$ can be translated by an element of $M(\overline{F})$ to ensure that it preserves $\mathscr{P}_M$, the element $w \in W(G,M)(F)$ has a lift $\mathrm{Ad}(g_w)$ with $g_w \in N_G(M)(\overline{F})$ such that $\mathrm{Ad}(g_w)|_M$ is defined over $F$ and preserves the pinning $\mathscr{P}_M$. The element $g_w$ is unique up a $Z(M)(\overline{F})$-translate.

Replacing $g$ by $gg_w$ (for any $g_w$ as above) and repeating the above construction preserves the $\widehat{G}$-conjugacy class of embeddings $\widehat{M} \to \widehat{G}$ and corresponds to pre-composing the embedding $\widehat{M} \to \widehat{G}$ constructed in the first paragraph by the automorphism $\theta_w^\vee$, giving the result. □

**Definition 4.5.** Given a finite central subgroup $A \to M$, the isogeny $M \to M/A$ dualizes to $\widehat{M/A} \to \widehat{M}$. Set

$$\widehat{\bar{M}} := \varprojlim_{A \subset_{\text{fin}} Z(M)} \widehat{M/A} \to \widehat{M}.$$

As in [Kal18], it is sometimes useful to take a more explicit, totally ordered exhaustive system of finite central $A$ by taking, for $n \geq 1$, the finite central subgroup $Z_{M,n}$ to be the preimage of $(Z(M)/Z(M_{\text{der}}))[n]$ in $Z(M)$ and the isogenous quotient to be $M_n := M/Z_{M,n}$. All choices of a totally-ordered exhaustive system of finite central subgroups define canonically-isomorphic inverse limits.

We fix some additional notation building on Definition 4.5:

**Notation 4.6.** For a subgroup $V \subset \widehat{M}$, denote by $V^+$ its preimage in $\widehat{\bar{M}}$.

We warn the reader that this notation is different from the one used in [Kal16b], where it is used to denote the preimage (for $\Gamma$-stable $V$) of $V^\Gamma$ in $\widehat{\bar{M}}$. In our notation, the latter is denoted by $V^{\Gamma,+}$.

For any $M$ there is a canonical $\Gamma$-equivariant embedding $Z(\widehat{G}) \to \widehat{M}$ whose construction we briefly recall. We have the isomorphism $\widehat{T_M} \xrightarrow{\sim} \mathscr{T}_M$ given as part of the construction of $\widehat{M}$ and taking $g \in G(\overline{F})$ sending $(B_M, T_M)$ into $(B_G, T_G)$ (by which we mean ${}^g B_M \subseteq B_G$) gives an isomorphism

$$X^*(\mathscr{T}_M) \xrightarrow{\sim} X_*(T_M) \to X_*(T_G) \to X^*(\mathscr{T}_G)$$

which induces an isomorphism $\mathscr{T}_M \to \mathscr{T}_G$ depending on all choices up to $\widehat{G}$-conjugacy, and hence induces a canonical ($\Gamma$-equivariant) embedding $Z(\widehat{G}) \to \mathscr{T}_M \to \widehat{M}$.

**Notation 4.7.** We will denote the image of $Z(\widehat{G})^\Gamma$ in $\widehat{M}$ via the above embedding by $Z(\widehat{G})^\Gamma_{(M)}$, similarly for any subgroup of $Z(\widehat{G})$. In particular, if $V$ is a subgroup of $Z(\widehat{G})$, the notation $V^+_{(M)}$ denotes the preimage of $V_{(M)}$ in the infinite cover $\widehat{\bar{M}} \to \widehat{M}$.

We will now need to make some choices of sections. For each class $K_{M,G} \cdot M$ fix for once and for all a $\widehat{M}_{\text{ad}} \rtimes W(G, M)(F)$-equivariant (set-theoretic) section

$$s_M \colon \frac{\widehat{M}}{Z(\widehat{G})^{\Gamma,\circ}} \longrightarrow \widehat{\bar{M}} \tag{33}$$

of the surjective composition $\widehat{\bar{M}} \to \widehat{M} \to \widehat{M}/Z(\widehat{G})^{\Gamma,\circ}$. We can choose the section to be equivariant for the action of the group $\widehat{M}_{\text{ad}} \rtimes W(G, M)(F)$ because the map $\widehat{\bar{M}} \to \widehat{M}/Z(\widehat{G})^{\Gamma,\circ}$ is equivariant for this group action and the group acts freely on the source and target of the map.

4.1.3. *Genuine characters and the Langlands correspondence.* Continue with the notation as in §4.1.1; call a character $M(F)_\pm \xrightarrow{\chi} \mathbb{C}^\times$ *genuine* (where $M(F)_\pm$ is viewed as a topological group in the usual way) if $\chi(ab) = a\chi(b)$ for $a \in \{\pm 1\}$. As with §4.1.1, the purpose of this subsection is to summarize important ideas in [Kal21a].

Recall the following notion from the theory of $L$-embeddings, continuing with a fixed torus $T$ and admissible $\Sigma$-set $R \to X^*(T)$:

**Definition 4.8.** A (set of) *$\chi$-data* is a collection of characters $\{F_\alpha^\times \xrightarrow{\chi_\alpha} \mathbb{C}^\times\}_{\alpha \in R}$ which satisfy $\chi_{\sigma\alpha} = \chi_\alpha \circ \sigma^{-1}$ for all $\sigma \in \Gamma$, $\chi_{-\alpha} = \chi_\alpha^{-1}$, and $\chi_\alpha|_{F^*_{\pm\alpha}} = \kappa_\alpha$.

There is a canonical way to construct a genuine character from a $\chi$-datum $\{\chi_\alpha\}$ by constructing characters $\chi_{O,\pm}$ of each $J_O(F)_\pm$ for each $\Sigma$-orbit $O \subseteq R$. More precisely, choosing any $\alpha \in R$ gives a canonical identification $J_O(F) = J_\alpha(F)$, and when $O$ is asymmetric one defines $\chi_O = \chi_\alpha$ on $J_O(F)$ (via the aforementioned identification) and then extending it to a genuine character of $J_O(F)_\pm$ in the obvious way. For $\alpha \in O$ asymmetric one uses the chain of identifications

$$J_O(F)_\pm \xrightarrow{\sim} J_\alpha(F)_\pm \xrightarrow{\sim} \frac{F_\alpha^\times}{N_{F_\alpha/F_{\pm\alpha}}(F_\alpha^\times)}; \tag{34}$$

by construction $\chi_\alpha$ factors through the rightmost term of (34) and we define $\chi_{O,\pm}$ to be the character of $J_O(F)_\pm$ induced by $\chi_\alpha$ and the composition (34).

Since the product character $(\chi_{O,\pm})_{O\subseteq R}$ of $\prod_O J_O(F)_\pm$ is trivial on the kernel of $\prod_O\{\pm1\} \to \{\pm1\}$, it induces a genuine character $S(F)_\pm \to \mathbb{C}^\times$.

On the Galois side, define a *Langlands parameter* for $S(F)_\pm$ to be a continuous homomorphism $W_F \to {}^L S_\pm$ commuting with the maps of both sides to $\Gamma$, and consider these up to $\widehat{S}$-conjugacy. This definition generalizes to continuous homomorphisms $W_F \times \mathrm{SL}_2 \to {}^L M_\pm$ in the obvious way.

**Theorem 4.9.** *(*[Kal21a, Theorem 3.16]*) There is a canonical bijection between the set of genuine characters of $S(F)_\pm$ and equivalence classes of Langlands parameters $W_F \to {}^L S_\pm$.*

*Proof.* Fix $(\chi_\alpha)$ a set of $\chi$-data for $R \to X^*(S)$ with corresponding character $\chi$ as constructed above, to which one can associate a parameter $W_F \xrightarrow{\varphi_\chi} {}^L S_\pm$; see [Kal21a, Definition 3.18] for the precise formula.

Then for a genuine character $S(F)_\pm \xrightarrow{\theta'} \mathbb{C}^\times$ there is a character $\theta$ of $S(F)$ such that $\theta \cdot \chi = \theta'$, and $\theta$ corresponds, via the Langlands correspondence for tori, to a parameter $W_F \xrightarrow{\varphi_\theta} \widehat{S}$. The claimed bijection is the map $\theta \mapsto [\varphi_\chi \cdot \varphi_\theta]$ and does not depend on the choice of $\chi$-data. □

4.2. **Enhancements.** We now give a non-basic generalization of an enhanced $L$-parameter, which will require some further setup. First, in §4.2.1 we give a slight modification of the (basic) rigid refined local Langlands correspondence. The rest of this subsection (§4.2.2-4.2.4) is devoted to defining non-basic rigid enhancements—for expository purposes, we break the latter up into three parts. Continue with the notation from §4.1.

4.2.1. *Basic enhancements for double covers.* First recall the rigid refined local Langlands correspondence for quasi-split $G$: For an $L$-parameter $W_F \times \mathrm{SL}_2 \xrightarrow{\phi} {}^L G$ and fixed finite $Z \subseteq Z(G)$ we set $S_\phi^+$ as the preimage of $Z_{\widehat{G}}(\phi)$ in $\widehat{G/Z}$ and fix a Whittaker datum $\mathfrak{w}$ for $G$.

**Conjecture 4.10.** There is a commutative diagram with horizontal bijections

$$\begin{array}{ccc} \Pi_\phi^Z & \xrightarrow{\iota_{\mathfrak{w}}} & \mathrm{Irr}(\pi_0(S_\phi^+)) \\ \downarrow & & \downarrow \\ H^1(\mathscr{E}, Z \to G) & \longrightarrow & \pi_0(Z(\widehat{G/Z})^{\Gamma,+})^* , \end{array}$$

where $\Pi_\phi^Z$ is a (finite) set of isomorphism classes of representations $(G', z, \pi)$ of $Z$-rigid inner twists of $G$ (as in [Kal16b, §5.1]), $Z(\widehat{G/Z})^{\Gamma,+}$ denotes the preimage of $Z(\widehat{G})^\Gamma$ in $\widehat{G/Z}$, the bottom map is the one from Theorem 3.4, the left-hand column extracts the underlying torsor, and the right-hand column is induced by taking central characters. As the notation indicates, the top horizontal bijection depends on $\mathfrak{w}$.

The map $\iota_{\mathfrak{w}}$ is expected to satisfy many additional properties, such as the endoscopic character identities (cf. [Kal16b, §5.4]). The goal is to extend Conjecture 4.10 to double covers (of twisted Levi subgroups of a fixed group $G$).

For a parameter $W_F \times \mathrm{SL}_2 \xrightarrow{\phi_\pm} {}^L M_\pm$, define $S_{\phi_\pm} := Z_{\widehat{M}}(\phi_\pm)$ and for some fixed finite $Z \subset Z(M)$ define $S^+_{\phi_\pm}$ as the preimage of $S_{\phi_\pm}$ in $\widehat{M/Z}$. We need:

**Lemma 4.11.** *For any inner twist $M \xrightarrow{\psi} M'$ the induced map $M_{\mathrm{ab}} \to M'_{\mathrm{ab}}$ is defined over $F$.*

*Proof.* This follows immediately from the fact that for any $\gamma \in \Gamma$ the map $\psi^{-1} \circ {}^\sigma\psi$ is an inner automorphism of $M$. □

Lemma 4.11 allows one to define a double cover for $M'$ as follows: We have the admissible $\Sigma$-set $R(M_{\mathrm{ab}}, G) \to X^*(M_{\mathrm{ab}})$ defined in §4.1.1 and define a new admissible $\Sigma$-set, denoted by $R(M'_{\mathrm{ab}}, G)$, which is just $R(M_{\mathrm{ab}}, G)$ equipped with the composition

$$R(M_{\mathrm{ab}}, G) \to X^*(M_{\mathrm{ab}}) \xrightarrow{\psi^{-1}} X^*(M'_{\mathrm{ab}}),$$

where the first map is the one associated to the $\Sigma$-set $R(M_{\mathrm{ab}}, G)$. We warn the reader that the notation $R(M'_{\mathrm{ab}}, G)$ does not mean that we are embedding $M'$ into $G$. We then define $M'_{\mathrm{ab}}(F)_\pm$ using the above $\Sigma$-set and $M'(F)_\pm$ as the pullback of the diagram $M'(F) \to M'_{\mathrm{ab}} \leftarrow M'_{\mathrm{ab}}(F)_\pm$.

This observation motivates the following definition:

**Definition 4.12.** (1) A *rigid inner twist* of $M(F)_\pm$ is a pair $(M'(F)_\pm, z)$, where $(M', z)$ is a rigid inner twist of $M$ and the double cover $M'(F)_\pm$ is as defined above.

(2) An *isomorphism* $(M_1(F)_\pm, z_1) \to (M_2(F)_\pm, z_2)$ is a pair $(f, m)$ where

$$(M_1, z_1) \xrightarrow{(f,m)} (M_2, z_2)$$

is an isomorphism of rigid inner twists; recall that this means that $M_1 \xrightarrow{f} M_2$ is an $F$-rational isomorphism and $m \in M(\overline{F})$ satisfies $\psi_1 \circ \mathrm{Ad}(m) = f \circ \psi_2$ and twisting $z_1$ by the coboundary of $m$ gives $z_2$.

The follow elementary lemma justifies the previous definition:

**Lemma 4.13.** *An isomorphism*

$$(M_1(F)_\pm, z_1) \xrightarrow{(f,m)} (M_2(F)_\pm, z_2)$$

*induces an isomorphism of extensions*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \{\pm 1\} & \longrightarrow & M_1(F)_\pm & \longrightarrow & M_1(F) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f_\pm} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \{\pm 1\} & \longrightarrow & M_2(F)_\pm & \longrightarrow & M_2(F) & \longrightarrow & 1. \end{array} \tag{35}$$

*Proof.* By construction, the map $X^*(M_{1,\mathrm{ab}}) \xrightarrow{f^{-1}} X^*(M_{2,\mathrm{ab}})$ induced by $f^{-1}$ is compatible with the maps of both groups to $X^*(M_{\mathrm{ab}})$ and hence (using [Kal21a, §5]) the isomorphism $M_1(F) \to M_2(F)$ lifts canonically to an isomorphism $M_1(F)_\pm \xrightarrow{f_\pm} M_2(F)_\pm$ making the diagram (35) commute. □

**Definition 4.14.** A *genuine representation* of the rigid inner twist $(M'(F)_\pm, z)$ is a triple

$$(M'(F)_\pm, z, \pi)$$

where $\pi$ is a genuine representation of $M'(F)_\pm$ (recall that $\pi$ is genuine if and only if $\pi(-1) = -\mathrm{id}$). An *isomorphism* $(M_1(F)_\pm, z_1, \pi_1) \xrightarrow{(f,m)} (M_2(F)_\pm, z_2, \pi_2)$ between two such representations is an isomorphism of rigid inner twists such that $f_\pm$ (as in Lemma 4.13) identifies the genuine representations $\pi_1$ and $\pi_2$.

We can now re-formulate Conjecture 4.10 in the double cover (relative to $G$) setting, fixing a *quasi-split* twisted Levi subgroup $M$ of $G$, a finite central subgroup $Z \subset M$, and a Whittaker datum $\mathfrak{w}_M$ for $M$:

**Theorem 4.15.** *Assume that Conjecture 4.10 holds for $M$ and is compatible with twists by co-central characters. Then given an $L$-parameter $W_F \times \mathrm{SL}_2 \xrightarrow{\phi_{M,\pm}} {}^L M_\pm$ there is a finite subset $\Pi_{\phi_{M,\pm}}$ of genuine representations of rigid inner twists of $M(F)_\pm$ and a commutative diagram with horizontal bijections*

$$\begin{array}{ccc} \Pi^Z_{\phi_{M,\pm}} & \xrightarrow{\iota_{\mathfrak{w},\pm}} & \mathrm{Irr}(\pi_0(S^+_{\phi_{M,\pm}})) \\ \downarrow & & \downarrow \\ H^1(\mathscr{E}, Z \to M) & \longrightarrow & \pi_0(Z(\widehat{M/Z})^{\Gamma,+})^*, \end{array}$$

*where the bottom map is the one from Theorem 3.4, the left-hand column extracts the underlying torsor, and the right-hand column is induced by taking central characters.*

*Proof.* This proof will mostly follow the arguments of [Kal21a, Remark 6.14]. Choose $\chi$-data $\{\chi_\alpha\}$ for $R(M_{\mathrm{ab}}, G)$, which determines a genuine character $\chi_M$ of $M_{\mathrm{ab}}(F)_\pm$ (that we can pull back to a genuine character of $M(F)_\pm$) and an isomorphism ${}^L M \xrightarrow{\iota_{\chi_M}} {}^L M_\pm$ (by [Kal21a, Fact 6.13]). It follows that $\phi_{M,\pm}$ and $\iota_{\chi_M}$ determine an $L$-parameter $W_F \times \mathrm{SL}_2 \xrightarrow{\phi} {}^L M$ and an isomorphism $S^+_{\phi_M} \xrightarrow{s_\chi} S^+_{\phi_{M,\pm}}$.

Given a representation $(M', z, \pi) \in \Pi^Z_{\phi_M}$, the isomorphism $\psi_z$ transfers the $\chi$-data $\{\chi_\alpha\}$ to $\chi$-data for the $\Sigma$-set $R(M'_{\mathrm{ab}}, G) \to X^*(M'_{\mathrm{ab}})$ which corresponds to a genuine character $\chi_{M'}$ of $M'_{\mathrm{ab}}(F)_\pm$. Twisting $\pi$ by $\chi_{M'}$ gives a genuine representation of $M'(F)_\pm$, and we thus obtain an element $(M', z, \pi \cdot \chi_{M'}) \in \Pi^Z_{\phi_{M,\pm}}$ (one checks easily that if we take an isomorphic representation of $(M', z, \pi)$ then the resulting genuine representations of rigid inner twists remain isomorphic).

Via twisting by each $\chi_{M'}$ we obtain a bijection $\Pi^Z_{\phi_M} \xrightarrow{\cdot \chi_-^{-1}} \Pi^Z_{\phi_{M,\pm}}$ and thus can define $\iota_{\mathfrak{w}_M,\pm}$ as the composition $s_\chi \circ \iota_{\mathfrak{w}} \circ (\cdot \chi_-^{-1})$. The assumption on the compatibility with twisting implies that this composition does not depend on the choice of $\chi$-data. Commutativity of the diagram follows from the above construction and the analogous commutativity in Conjecture 4.10. □

In the notation of the above theorem, for a fixed rigid inner form $M'(F)$ of $M$, denote by $\Pi^{Z}_{\phi_{M,\pm}}(M')$ the subset of $\Pi^{Z}_{\phi_{M,\pm}}$ consisting of all isomorphism classes representations of rigid inner twists which have a representative $(M'(F)_{\pm}, z, \pi)$. There is also a version of Theorem 4.15 obtained by splicing together all possible $Z$:

**Corollary 4.16.** *Using the same assumptions and notation as Theorem 4.15, there is a commutative diagram with horizontal bijections*

$$\begin{array}{ccc} \Pi_{\phi_{M,\pm}} & \xrightarrow{\iota_{\mathfrak{w}_{M,\pm}}} & \mathrm{Irr}(\pi_0(S^{+}_{\phi_{M,\pm}})) \\ \big\downarrow & & \big\downarrow \\ H^1_{\mathrm{bas}}(\mathscr{E}, M) & \longrightarrow & \pi_0(Z(\widehat{\bar{M}})^{\Gamma,+})^{*}, \end{array}$$

*where the bottom map is the one from Theorem 3.4, the left-hand column extracts the underlying torsor, and the right-hand column is induced by taking central characters.*

**Remark 4.17.** One can state a version of the endoscopic character identities in the context of Theorem 4.15. We leave this for a future paper.

4.2.2. *Non-basic enhancements I: $\phi$-minimal subgroups.* Continue with the notation of the previous subsections; in particular, whenever we write $N$, we mean the unique (up to $F$-rational isomorphism) quasi-split representative of some set $K_{N,G} \cdot N$ of rigid Newton centralizers in $G$. The enhancements will be valued in twisted extended quotients, whose definition we recall now.

Let $\mathsf{G}$ be a finite group acting on a set $X$ and for $x \in X$ denote by $\mathsf{G}_x$ the stabilizer of $x$. Fix an $X$-indexed collection of cocycles $\natural := \{\natural_x\}_{x \in X}$ where $\natural_x \in Z^2(\mathsf{G}_x, \mathbb{C}^{\times})$ such that $g_{*}\natural_x$ is cohomologous to $\natural_{gx}$ for all $g \in \mathsf{G}$, where $g_{*}\natural_x := \natural_x \circ \mathrm{Ad}(g)$. For a given $x$, the twisted group algebra $\mathbb{C}[\mathsf{G}_x, \natural_x]$ is the $\mathbb{C}$-algebra with basis of formal symbols $\{[g]\}_{g \in \mathsf{G}}$ which satisfy $[g] \cdot [g'] = \natural_x(g, g')[gg']$. Define

$$\tilde{X} := \{(x, \rho) \mid x \in X, \rho \in \mathrm{Irr}(\mathbb{C}[\mathsf{G}_x, \natural_x])\},$$

where by $\mathrm{Irr}(\mathbb{C}[\mathsf{G}_x, \natural_x])$ we mean simple $\mathbb{C}[\mathsf{G}_x, \natural_x]$-modules which are finite-dimensional $\mathbb{C}$-vector spaces.

Assume now that there is, for each $(g, x) \in \mathsf{G} \times X$, a "gluing" isomorphism

$$\mathbb{C}[\mathsf{G}_x, \natural_x] \xrightarrow{\varphi_{g,x}} \mathbb{C}[\mathsf{G}_{gx}, \natural_{gx}] \tag{36}$$

which is inner if $g \in \mathsf{G}_x$ and satisfies, for $g, g' \in \mathsf{G}$, the identity $\varphi_{g',gx} \circ \varphi_{g,x} = \varphi_{g'g,x}$.

With the above setup, there is an action of $\mathsf{G}$ on $\tilde{X}$ defined by $g \cdot (x, \rho) = (gx, \rho \circ \varphi^{-1}_{g,x})$, and the *twisted extended quotient* is the resulting quotient space $\tilde{X}/\mathsf{G}$, which is usually denoted by $(X^{+}_{\phi}(\widehat{G}) /\!\!/ \mathsf{G})_{\natural}$.

Now we specialize the above construction to our situation. In the context of the above definitions, we take $\mathsf{G} = \pi_0(S_\phi) = S_\phi / Z(\widehat{G})^{\Gamma,\circ}$, where $S_\phi := Z_{\widehat{G}}(\phi(W'_F))$ and the second equality follows from the fact that $\phi$ is discrete. Roughly speaking, the set $X$ on which $\pi_0(S_\phi)$ acts (defined precisely in Definition 4.30) will involve characters of the component group of the preimage of $Z(\widehat{G})^{\Gamma,\circ}$ in central extensions (cf. (40)) of certain Levi subgroups of $\widehat{G}$ which are normalized by $\phi$—the difficulty inherent in this context is then how to relate such characters for two different Levi subgroups of $\widehat{G}$. We choose to approach this problem by considering all of the minimal such Levi subgroups (defined precisely in Definition 4.20), which is the focus of this subsection (§4.2.2) and then defining an equivalence relation on these characters by using a dual interpretation of the rigid Newton map (the focus of §4.2.3). Finally, §4.2.4 defines the family of cocycles $\{\natural_x\}_{x\in X}$ and all other remaining data.

Before going into the details of the aforementioned construction, we record the following basic result, which describes the behavior of elliptic maximal tori of $G$, which we will call *$G$-elliptic* for expository clarity:

**Lemma 4.18.** *Let $M$ be a twisted Levi subgroup of $G$ containing a $G$-elliptic maximal torus. Then any $K_{M,G}$-conjugate $M'$ of $M$ (which is a fortiori an inner form of $M$) also contains a $G$-elliptic maximal torus.*

*Proof.* We claim that a twisted Levi subgroup $M$ contains a $G$-elliptic maximal torus if and only if $Z(M)^\circ/Z(G)^\circ$ is anisotropic. One direction is obvious; conversely, if this quotient is anisotropic let $T'$ be an elliptic maximal torus of $M_{\mathrm{der}}$ and set $T := T' \cdot Z(M)^\circ$, a maximal torus of $M$ which is anisotropic modulo $Z(G)$, using the surjection

$$\frac{T'}{Z(G)\cap M_{\mathrm{der}}} \times \frac{Z(M)^\circ}{Z(G)} \to \frac{T}{Z(G)}.$$

The statement of the lemma follows immediately from this claim, since inner twists of $M$ restrict to $F$-rational isomorphisms on $Z(M)$. □

We are interested in the twisted Levi subgroups $M$ through which the fixed discrete $L$-parameter factors (via the canonical conjugacy class of embeddings ${}^LM_\pm \to {}^LG$). The following result puts a severe restriction on such $M$:

**Corollary 4.19.** *If a discrete $L$-parameter $\phi$ factors through the $\widehat{G}$-conjugacy class of embeddings ${}^LM_\pm \to {}^LG$ corresponding to the inclusion of a twisted Levi subgroup $M \hookrightarrow G$ then $M$ contains a $G$-elliptic maximal torus.*

*Proof.* This follows immediately from the proof of Lemma 4.18 and the standard fact that for a reductive group $H$ the split rank of $Z(H)$ equals the rank of $Z(\widehat{H})^{\Gamma,\circ}$ (applied to $H = G$ and $H = M$ separately). □

**Definition 4.20.** We say that a Levi subgroup $\mathscr{M} \subseteq \widehat{G}$ is *$\phi$-minimal* if

(1) there is a twisted Levi subgroup $M$ of $G$ such that $\phi$ factors through an embedding ${}^LM_\pm \to {}^LG$ (in the canonical $\widehat{G}$-conjugacy class of such embeddings) that takes $\widehat{M}$ to $\mathscr{M}$, and
(2) $\mathscr{M}$ is minimal for this property with respect to inclusion of Levi subgroups of $\widehat{G}$.

Being $\phi$-minimal is a property specific to a chosen representative $\phi \in [\phi]$; each such $\mathscr{M}$ yields a $\widehat{G}$-conjugacy class of $\phi'$-minimal subgroups for each $\phi' \in [\phi]$.

For a fixed embedding $\eta$ in the conjugacy class of embeddings ${}^LM_\pm \to {}^LG$ with image $\mathscr{M}$ through which $\phi$ factors, there is a uniquely determined parameter $W_F' \xrightarrow{\phi_{\mathscr{M},\eta}} {}^LM_\pm$ which when post-composed with $\eta$ gives $\phi$. By definition, any $\phi$-minimal $\mathscr{M}$ is equipped with a $N_{\widehat{G}}(\mathscr{M})$-conjugacy class of isomorphisms $\widehat{M} \xrightarrow{\sim} \mathscr{M}$ and one obtains a quasi-split $F$-rational reductive group $\widehat{\mathscr{M}}$ along with a $K_{M,G}$-conjugacy class of embeddings $\widehat{\mathscr{M}}_{\overline{F}} \hookrightarrow G_{\overline{F}}$, using the classification of quasi-split connected reductive groups. This $K_{M,G}$-conjugacy class contains a (non-empty) subset of $F$-rational such embeddings, which are unique up to $K_{Z(M),G}$-conjugacy. Conversely, given a $(K_{Z(M),G}) \cap N_G(M)(\overline{F})$-conjugacy class of $F$-rational isomorphisms $\widehat{\mathscr{M}} \to M$, one obtains a canonical $\widehat{M}_{\mathrm{ad}}^{\Gamma} \rtimes W(G,M)(F)$-orbit $J_M$ (sometimes also denoted by $J_{M,\mathscr{M}}$ if we want to emphasize $\mathscr{M}$) of isomorphisms $\widehat{M} \to \mathscr{M}$ (where $W(G,M)(F)$ acts via $\theta(w)^\vee$).

Consider a fixed isomorphism $\widehat{M} \xrightarrow{h_M^\vee} \mathscr{M}$ in the $\widehat{M}_{\mathrm{ad}}^{\Gamma} \rtimes W(G,M)(F)$-class of isomorphisms inside the larger $\widehat{G}$-conjugacy class corresponding to the standard embedding $M \to G$. Inside the canonical $\widehat{G}$-conjugacy class of $L$-embeddings ${}^L M_\pm \to {}^L G$ constructed in §4.1.1 there is a canonical $Z(\mathscr{M})$-conjugacy class consisting of the embeddings whose restriction to the composition

$$\widehat{M} \to {}^L M_\pm \to {}^L G$$

(where the first map is $m \mapsto m \boxtimes 1$) factors as the isomorphism $\widehat{M} \xrightarrow{h_M^\vee} \mathscr{M}$ followed by the inclusion $\mathscr{M} \hookrightarrow \widehat{G}$.

If we choose a $g \in K_{Z(M),G}$-conjugate ${}^g M$ then $\theta^\vee_{\mathrm{Ad}(g)}$ (which denotes the isomorphism $\widehat{M} \to \widehat{{}^g M}$ induced by $\mathrm{Ad}(g)$ and a choice of $\Gamma$-pinnings for $M$ and ${}^g M$) induces an isomorphism ${}^L({}^g M)_\pm \xrightarrow{\sim} {}^L M_\pm$ which identifies $J_{M,\mathscr{M}}$ with $J_{{}^g M,\mathscr{M}}$.

A fixed $\phi$-minimal subgroup $\mathscr{M}$ of $\widehat{G}$ is automatically normalized by $\phi$. For a fixed quasi-split twisted Levi subgroup $M$ corresponding to $\mathscr{M}$ (in the sense explained above) and $h_M^\vee \in J_M$, we have the cover $\widehat{\widehat{M}} \to \widehat{M}$ containing the subgroups $S^+_{\phi_{\mathscr{M}},\eta}$ (for any choice of $\eta$ in the $Z(\mathscr{M})$-conjugacy class of embeddings ${}^L M_\pm \to \widehat{G}$ associated to $h_M^\vee$) and the group $Z(\widehat{G})^{\Gamma,\circ,+}_{(M)}$, the latter being the preimage of the canonically embedded $Z(\widehat{G})^{\Gamma,\circ}_{(M)} \subseteq \widehat{M}$ constructed in §4.1.2.

We have the following dual interpretation of the diagram (30) from §3.4:

**Proposition 4.21.** *There is a canonical injective homomorphism*

$$X^*(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(M)})) \to \mathrm{Hom}(\mu_{\overline{F}}, Z(M)_{\overline{F}}) \tag{37}$$

*which makes the following diagram commute*

$$\begin{array}{ccc} X^*(\pi_0(Z(\widehat{M})^{\Gamma,+})) & \longrightarrow & X^*(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(M)})) \\ \Big\downarrow{\scriptstyle \iota^{-1}} & & \Big\downarrow \\ H^1(\mathscr{E}, Z(M) \to M) & \longrightarrow & \mathrm{Hom}(\mu_{\overline{F}}, Z(M)_{\overline{F}}), \end{array} \tag{38}$$

*where the top horizontal map is given by restriction, the bottom by taking the identity coordinate of $f_x$, and the left-hand by Tate-Nakayama duality.*

*Proof.* We first fix some $k \in \mathbb{N}$ and prove the analogous result for $Z(\widehat{M})^{\Gamma,+}$ replaced by the preimage of $Z(\widehat{G})^\Gamma$ in $\widehat{M/Z_{M,n_k}}$ and $\mu$ by $\mu_{n_k}$, keeping the "+" notation for preimages of subgroups of $Z(\widehat{M})$ in $\widehat{M/Z_{M,n_k}}$, by abuse of notation. Write $Z(\widehat{G})$ rather than $Z(\widehat{G})_{(M)}$ in this proof in order to simplify notation—there will be no danger of confusion locally.

In this case, the cover $\widehat{M/Z_{M,n_k}} \to \widehat{M}$ may be identified with the map $\widehat{M}_{\mathrm{sc}} \times Z(\widehat{M})^\circ \to \widehat{M}$ given by the usual map on the first factor and the $n_k$-power map on the second, with kernel $\widehat{Z_{M,n_k}}$; evidently the subgroup $Z(\widehat{G})^{\Gamma,\circ} = \{\mathrm{id}\} \times Z(\widehat{G})^{\Gamma,\circ}$ is contained in $Z(\widehat{G})^{\Gamma,\circ,+}$ and maps surjectively via $[n_k]$ onto $Z(\widehat{G})^{\Gamma,\circ}$, since it's a torus and, moreover, that $Z(\widehat{G})^{\Gamma,\circ,+}/Z(\widehat{G})^{\Gamma,\circ}$ is finite, and hence $[Z(\widehat{G})^{\Gamma,\circ,+}]^\circ = Z(\widehat{G})^{\Gamma,\circ}$.

In particular, we have the identification

$$\frac{\widehat{Z_{M,n_k}}}{\{\mathrm{id}\} \times (Z(\widehat{G})^{\Gamma,\circ})[n_k]} \xrightarrow{\sim} \pi_0(Z(\widehat{G})^{\Gamma,\circ,+})$$

induced by the inclusion $\widehat{Z_{M,n_k}} \hookrightarrow Z(\widehat{G})^{\Gamma,\circ,+}$.

The desired map (at "level $k$") is then given by the composition

$$X^*(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+})) \xrightarrow{\sim} X^*(\widehat{Z_{M,n_k}}/(Z(\widehat{G})^{\Gamma,\circ})[n_k]) \hookrightarrow X^*(\widehat{Z_{M,n_k}}) \xrightarrow{\sim} \mathrm{Hom}(\mu_{n_k,\overline{F}}, Z_{M,n_k,\overline{F}}), \quad (39)$$

where the right-most map is given by the identifications

$$X^*(\widehat{Z_{M,n_k}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(X^*(Z_{M,n_k}), \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\mu_{n_k,\overline{F}}, Z_{M,n_k,\overline{F}}).$$

It is clear that the map (39) is compatible with the projection maps as $k$ varies, giving a well-defined injection $X^*(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(M)})) \to \mathrm{Hom}(\mu_{\overline{F}}, Z(M)_{\overline{F}})$ as claimed.

It suffices to check compatibility with the Tate-Nakayama isomorphism at the $k$-level as well, using the same notation as in the first part of the proof. Following the diagram (38) down and then to the right is, by the diagram (30) (and the ensuing discussion), the map (choosing an elliptic maximal torus $T$ contained in $M$ and $k \gg 0$)

$$X^*(\pi_0(Z(\widehat{M})^{\Gamma,+})) \xrightarrow{\sim} \frac{[X_*(T/Z_{M,n_k})/X_*(T_{M,\mathrm{sc}})]^N}{I \cdot [X_*(T)/X_*(T_{M,\mathrm{sc}})]} \to \frac{X_*(T/Z_{M,n_k})}{X_*(T)}$$

and following the same diagram in the other direction corresponds to the composition

$$X^*(\pi_0(Z(\widehat{M})^{\Gamma,+})) \to X^*(\widehat{Z_{M,n_k}}/(Z(\widehat{G})^{\Gamma,\circ})[n_k]) \to X^*(\widehat{Z_{M,n_k}}) \xrightarrow{\sim} \frac{X_*(T/Z_{M,n_k})}{X_*(T)},$$

where the first map is restriction (using that $\widehat{Z_{M,n_k}} \cap (Z(\widehat{M})^{\Gamma,+,\circ}) = (Z(\widehat{G})^{\Gamma,\circ})[n_k]$) and the last map is the canonical identification used in (39). The two preceding displayed equations are the same map, giving the result. □

We will also need:

**Corollary 4.22.** *Let $N$ be a twisted Levi subgroup of $G$ containing $M$; then $\chi \in X^*(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(M)}))$ factors through the quotient map*

$$\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(M)}) \to \pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(N)})$$

*if and only if its image via* (37) *is in the subgroup* $\mathrm{Hom}(\mu_{\overline{F}}, Z(N)_{\overline{F}})$.

*Proof.* The image of $\chi$ lies in $\mathrm{Hom}(\mu_{\overline{F}}, Z(M)_{\overline{F}})$ if and only if, by the commutativity of the diagram (38), it has a preimage $\tilde{\chi} \in X^*(\pi_0(Z(\widehat{M})^{\Gamma,+}))$ whose image $[x] \in H^1(\mathscr{E}, Z(M) \to M)$ via Tate-Nakayama lies in the subset $H^1(\mathscr{E}, Z(N) \to M)$. We obtain the desired result by applying functoriality of the Tate-Nakayama duality isomorphism. □

4.2.3. *Non-basic enhancements II: Highest weights.* Continue with the same notation as in §4.2.2.

**Definition 4.23.** We say that a character $\chi \in X^*(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(M)}))$ is *Levi-suitable* if its image $\nu_\chi \in \mathrm{Hom}(\mu_{\overline{F}}, Z(M)_{\overline{F}})$ under the map (37) has a corresponding morphism $u \xrightarrow{f_\chi} Z(M)$ with $Z_G(f_\chi)$ a twisted Levi subgroup of $G$. Denote the set of all Levi-suitable characters by $X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(M)}))$.

For the fixed $\phi$-minimal subgroup $\mathscr{M}$ of $\widehat{G}$ and corresponding quasi-split twisted Levi subgroup $M$ of $G$ choosing a different $h_M^\vee \in J_M$ or even a different $f_{M'}^\vee \in J_{M'}$ for $M'$ a $g \in K_{Z(M),G}$-conjugate of $M$, we have a $\Gamma$-equivariant isomorphism $\widehat{M} \xrightarrow{\theta^\vee_{\mathrm{Ad}(g)}} \widehat{M'}$ which sends the $Z(\mathscr{M})$-conjugacy class of embeddings of $\widehat{M}$ in $\widehat{G}$ determined by $h_M^\vee$ to the class of embeddings of $\widehat{M'}$ in $\widehat{G}$ determined by $f_{M'}^\vee$. It is clear that this isomorphism also induces a bijection from $X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(M)}))$ to $X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(M')}))$. In particular, $\widehat{M}^\Gamma_{\mathrm{ad}} \rtimes W(G,M)(F)$ acts on the set $X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(M)}))$ for a fixed $M$ (but allowing $h_M^\vee$ to vary).

There is an analogue of the cover $\widehat{\bar{N}} \to \widehat{N}$ that can be defined intrinsically to any Levi subgroup $\mathscr{N}$ of $\widehat{G}$: Denote by

$$\widetilde{\mathscr{N}} = \mathscr{N}_{\mathrm{sc}} \times \varprojlim_k Z(\mathscr{N})^\circ, \tag{40}$$

where the transition maps are the $n_l/n_k$-power maps. We have a map $\widetilde{\mathscr{N}} \to \mathscr{N}$ via the usual map $\mathscr{N}_{\mathrm{sc}} \to \mathscr{N}$ on the left-hand direct factor and projection to the first coordinate on the right-hand factor. For any subgroup $V \subseteq \mathscr{N}$, denote by $V^+_{(\mathscr{N})}$ the preimage of $V$ in $\widetilde{\mathscr{N}}$. If $M$ is a choice of quasi-split twisted Levi subgroup of $G$ determined by the $\phi$-minimal $\mathscr{M}$ along with some $h_M^\vee \in J_M$, then the group $(S_\phi \cap \mathscr{M})^+_{(\mathscr{M})}$ is isomorphic to $S^+_{\phi_{\mathscr{M}},\eta}$ (for any choice of $\eta$ associated to $h_M^\vee$) and at level of characters, after restricting to $Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})}$, the image of $X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(M)}))$ is canonical (that is, independent of the choice of $h_M^\vee \in J_M$ and of $M \in K_{Z(M),G} \cdot M$) and denoted by $X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})}))$.

The set on which $\mathsf{G} = \pi_0(S_\phi)$ acts arises as the quotient of the union

$$\bigsqcup_{\mathscr{M}} X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})})), \tag{41}$$

where the above union is over all $\phi$-minimal subgroups of $\widehat{G}$.

To define the equivalence relation, we need the following construction. Fix a character $\chi \in X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})}))$ and Levi subgroup $\mathscr{N}$ of $\widehat{G}$ containing $\mathscr{M}$. Denoting by $\mathscr{M}^+_{(\mathscr{N})} \to \mathscr{M}$ the pullback of the composition $\widetilde{\mathscr{N}} \to \mathscr{N} \hookrightarrow \widehat{G}$ to $\mathscr{M}$ (similarly for any subgroup $V \subseteq \mathscr{M}$), there is an induced surjection

$$Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})} \to Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{N})}; \tag{42}$$

we are interested in the case where $\widetilde{\mathscr{N}} \to \mathscr{N}$ is such that $\chi$ factors through the map

$$\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})}) \to \pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{N})})$$

induced by (42). We now construct for each $\chi$ a canonical such $\mathscr{N}$.

Choose any quasi-split twisted Levi subgroup $M$ corresponding to $\mathscr{M}$ along with an isomorphism $h_M^\vee \in J_M$, determining isomorphisms (for some choice of $\eta$ associated to $h_M^\vee$)

$$(S_\phi \cap \mathscr{M})^+_{(\mathscr{M})} \xrightarrow{h_M^{\vee,-1,\sim}} S^+_{\phi_{\mathscr{M}},\eta}, \;\; Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})} \xrightarrow{\sim} Z(\widehat{G})^{\Gamma,\circ,+}_{(M)};$$

we can use the right-hand isomorphism to transfer $\chi$ to an element of $X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(M)}))$ and then, via (37), obtain homomorphisms $f_{\chi,h^\vee_M,e}$ and $u \xrightarrow{f_{\chi,h^\vee_M}} Z(M)$, setting $N' := Z_G(f_{\chi,h^\vee_M})$, a twisted Levi subgroup of $G$. To make things easier, we have:

**Lemma 4.24.** *The fixed quasi-split representative $N$ of $K_{N',G} \cdot N'$ is always $K_{M,G}$-conjugate to $N'$. Moreover, the conjugating element of $K_{M,G}$ can be chosen so that the corresponding conjugate of $M$ is also quasi-split.*

*Proof.* By Lemma 4.1, we have the fixed $K_{N',G}$-conjugate of $N$ which is quasi-split; choosing such a $g$ gives a class $dg$ in $Z^1(F, N')$; it follows from Lemma 4.18 that $dg$ is $N'$-cohomologous to a cocycle in $Z^1(F, M)$, which gives $g' \in K_{M,G}$ such that $N := {}^gN'$ is quasi-split. We now have an inclusion ${}^gM \hookrightarrow N$ of a twisted Levi subgroup, and now since $N$ is quasi-split we may use [Kal21a, Lemma 6.4] to find some $n \in N(\overline{F})$ such that ${}^{ng}M$ is $F$-rational, quasi-split, and ${}^gM \xrightarrow{\mathrm{Ad}(n)} {}^{ng}M$ is an inner twisting, which means that $n \in K_{{}^gM,N}$. It follows that $ng \in K_{M,G}$ and both ${}^{ng}M$ and ${}^{ng}N' = N$ are quasi-split. □

Because of the above Lemma, we can and do always choose, for each $\chi$, the $M$ and $h^\vee_M \in J_M$ such that both $M$ and $N = Z_G(f_{\chi,h^\vee_M})$ are quasi-split, and $N$ is the fixed quasi-split representative in the class $K_{N,G} \cdot N$ (we do not assume that $M$ is though).

According to the discussion in §4.1.1 the containment $M \subseteq G$ dualizes to a canonical $\widehat{G}$-conjugacy class of embeddings $J_{N,G,\pm}$ of ${}^LN_\pm$ into ${}^LG$ and the containment $M \subseteq N$ dualizes to a canonical $\widehat{N}$-conjugacy class of embeddings $J_{M,N,\pm}$ of ${}^LM_\pm \to {}^LN_\pm$; we are interested in pairs $j^\vee_M \in J_{M,N,\pm}$ and $j^\vee_N \in J_{N,G,\pm}$ such that $j^\vee_N \circ j^\vee_M = \eta$.

**Lemma 4.25.** *For two pairs $(j^\vee_M, j^\vee_N)$ and $(j^{\vee,'}_M, j^{\vee,'}_N)$ as above, there is some $n \in \widehat{N}$ such that $j^\vee_N = j^{\vee,'}_N \circ \mathrm{Ad}(n)$. In particular, the Levi subgroup $j^\vee_N(\widehat{N})$ of $\widehat{G}$ is canonically associated to $\eta$.*

*Proof.* Since $j^\vee_M$ and $j^{\vee,'}_M$ are $n' \in \widehat{N}$-conjugate we may replace $(j^{\vee,'}_M, j^{\vee,'}_N)$ by $(\mathrm{Ad}(n') \circ j^{\vee,'}_M, j^{\vee,'}_N \circ \mathrm{Ad}(n'^{-1}))$—again a pair as above—to assume $j^\vee_M = j^{\vee,'}_M$. We also know that $j^\vee_N$ and $j^{\vee,'}_N$ are $g \in \widehat{G}$-conjugate, and necessarily $g \in Z_{\widehat{G}}(\mathscr{M}) = Z(\mathscr{M})$, so that setting $n = j^{\vee,-1}_N(g) \cdot n'^{-1}$ gives the desired element. □

Replacing $h^\vee_M \in J_M$ with a $\widehat{M}^\Gamma_{\mathrm{ad}} \times [K_{Z(M),M}/M(\overline{F})]$-conjugate (where $[K_{Z(M),M}/M(\overline{F})]$ acts on the disconnected groupoid consisting of all possible ${}^L({}^gM)_\pm$ for all choices of quasi-split twisted Levi subgroups $M$ via a choice of pinning for each ${}^gM$) replaces $f_{\chi,h^\vee_M}$ with ${}^gf_{\chi,h^\vee_M}$ for $g \in K_{Z(M),G}$—since we are always assuming that $Z_G(f_{\chi,h^\vee_M}) = N$, we further have $g \in (K_{Z(M),G}) \cap N_G(N)(\overline{F})$—and we may take $j^\vee_{{}^gM} = \theta^\vee_{w(g)} \circ \mathrm{Ad}(n) \circ j^\vee_M \circ \theta^\vee_{\mathrm{Ad}(g^{-1})}$ and $j^\vee_{{}^gN} = j^\vee_N \circ \mathrm{Ad}(n^{-1}) \circ \theta^{\vee,-1}_{w(g)}$ for some $n \in \widehat{N}$ (see the beginning of §4.1.2 for the "$\theta^\vee_{w(g)}$" notation, which we are abusively also using to denote the induced maps on the $L$-groups of the relevant double covers). Even though ${}^gN = N$, we still write the former in order to emphasize that we are working with ${}^gM$ rather than $M$. It follows that $\mathscr{N}$ is the Levi subgroup of $\widehat{G}$ canonically associated to any such conjugate of the original choice of $h^\vee_M$.

**Remark 4.26.** One checks using identical arguments (cf. also the proof of Lemma 4.4) as above that replacing $h^\vee_M$ by $h^\vee_{M'}$ such that $f_{h^\vee_{M'}} = {}^gf_{h^\vee_M}$ for a more general $g \in K_{Z(M),G}$ (not necessarily normalizing $N$) with ${}^gN$ quasi-split such that $\mathrm{Ad}(g)|_N$ is defined over $F$ replaces $\mathscr{N}$ with a $g^\vee \in \widehat{G}$-conjugate Levi subgroup $\mathscr{N}'$ such that, via the embeddings of $\widehat{N}$ and $\widehat{{}^gN}$ constructed above, $\mathrm{Ad}(g^\vee)$ recovers the isomorphism $\mathrm{Ad}(g)^\vee$ from $\widehat{N}$ to $\widehat{{}^gN}$. Since we are already fixing some quasi-split representative $N \in K_{N,G}$ to begin with, we do not explain in full detail the effect of this change (other than the case where $g \in N_G(N)(\overline{F})$, which we covered above). Our construction (Theorem 4.41) will be invariant under $W(G,N)(F)$-automorphisms, and hence any two choices of $N$ yield constructions which can be canonically identified via any choice of $g$ as above.

We record some more basic results about these subgroups:

**Lemma 4.27.** *The subgroup $\mathscr{N}$ as above is normalized by $\phi$.*

*Proof.* This follows immediately from the fact that $\phi$ factors through the conjugacy class of embeddings ${}^LN_\pm \to {}^LG$. □

**Definition 4.28.** For $\chi \in X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})}))$, call the Levi subgroup $\mathscr{N} = \mathscr{N}_\chi$ of $\widehat{G}$ as above the *Newton Levi subgroup* of $\chi$.

**Lemma 4.29.** *For $\mathscr{N}$ the Newton Levi subgroup of $\chi$, the character $\chi$ factors through $\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{N})})$ (as in* (42)*).*

*Proof.* This is straightforward after using Corollary 4.22. □

Define an equivalence relation on (41) as follows: We declare $\chi_1 \in X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M}_1)}))$ and $\chi_2 \in X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M}_2)}))$ equivalent and write $\chi_1 \simeq \chi_2$ if $\mathscr{N}_{\chi_1} = \mathscr{N}_{\chi_2}$ and the induced characters of $\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{N}_{\chi_i})})$ are equal. This is evidently a well-defined equivalence relation on the union (41).

**Definition 4.30.** The set $X$ on which $\mathsf{G} = \pi_0(S_\phi)$ acts, denoted by $X^+_\phi(\widehat{G})$, is then the corresponding quotient of (41). To justify this action, it is clear that $\pi_0(S_\phi)$ acts on (41) (because $S_\phi$ acts on $\phi$-minimal subgroups of $\widehat{G}$ and $S^\circ_\phi$ acts trivially, by the discreteness assumption on $\phi$), and one checks easily that this action preserves the above equivalence classes.

4.2.4. *Non-basic enhancements III: The family $\{\natural_x\}$.* Continue with the same notation as in §§4.2.2, 4.2.3.

Having defined the set $X = X^+_\phi(\widehat{G})$, one now needs to define a family of cocycles $\{\natural_x\}_{x\in X^+_\phi(\widehat{G})}$, which requires some caution because of all of the choices involved; choose for $x = [\chi] \in X$ a representative $\chi \in X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})}))$, so we have also chosen some $\mathscr{M}$ along with some quasi-split $M$ and $h^\vee_M \in J_M$ which, as explained above, yields a quasi-split twisted Levi subgroup $N$ of $G$ and a Levi subgroup $\mathscr{N}$ of $\widehat{G}$. Recall that in §4.1.2 we defined an $\widehat{N}_{\mathrm{ad}} \rtimes W(G,N)(F)$-equivariant section (33)

$$\frac{\widehat{N}}{Z(\widehat{G})^{\Gamma,\circ}} \xrightarrow{s_N} \widehat{\widehat{N}}.$$

By construction, there is a canonical $\widehat{N}$-conjugacy class of isomorphisms $\mathscr{N} \to \widehat{N}$ whose restrictions to $Z(\widehat{G})$ are the inverse of the canonical embedding of $Z(\widehat{G})$ in $\widehat{N}$ discussed previously. It follows that we can use any isomorphism $\xi$ in this $\widehat{N}$-conjugacy class to transfer the section $s_N$ to a section of $\widetilde{\mathscr{N}} \to \mathscr{N}/Z(\widehat{G})^{\Gamma,\circ}$ via $\xi^{+,-1} \circ s_N \circ \xi$, where $\xi^+$ is the isomorphism $\widetilde{\mathscr{N}} \to \widehat{\bar{N}}$ induced by $\xi$.

The section $\mathscr{N}/Z(\widehat{G})^{\Gamma,\circ} \to \widetilde{\mathscr{N}}$ defined in the previous paragraph does not depend on the choice of $\xi$ in its $\widehat{N}$-conjugacy class because of the $\widehat{N}_{\mathrm{ad}}$-equivariance of $s_N$. Moreover, if we choose a different $M'$ and $h^\vee_{M'} \in J_{M'}$ (allowing $M' = M$ but with a different $h^\vee_M$) yielding ${}^g f_{\chi,h^\vee_M}$ for $g \in (K_{Z(M),G}) \cap N_G(N)(\overline{F})$ with the same twisted Levi subgroup $N$ of $G$ and ${}^g M$ quasi-split, post-composing by the automorphism $\theta^\vee_{w(g)}$ identifies the two $\widehat{N}$-conjugacy classes of isomorphisms corresponding $h^\vee_M$ to $h^\vee_{M'}$ and gives the same section of $\widetilde{\mathscr{N}} \to \mathscr{N}$ as the one obtained using $h^\vee_M$, by the $\widehat{N}_{\mathrm{ad}} \rtimes W(G,N)(F)$-equivariance of $s_N$.

Choosing a different representative $\chi' \in X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})}))$ of $[\chi]$ for a different $\phi$-minimal $\mathscr{M}$ yields the same section as $\chi$ for each Newton Levi subgroup, since the choice of section only depends on $N$.

The key input for defining the family of cocycles $\{\natural_x\}_{x \in X}$ is the following:

**Proposition 4.31.** *For $[\chi] \in X^+_\phi(\widehat{G})$, the stabilizer $S_{\phi,[\chi]}$ is contained in the Newton Levi subgroup $\mathscr{N} = \mathscr{N}_\chi$.*

*Proof.* Fix a representative $\chi \in X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})}))$ for $[\chi]$ along with $M$ and $h^\vee_M \in J_M$, yielding the (quasi-split) twisted Levi subgroup $N$ containing $M$ and let $s \in S_{\phi,[\chi]}$. Also fix a minimal Levi subgroup $T_M$ of $M$, a maximal torus $\mathscr{T}_M$ of $\widehat{M}$ which is part of a $\Gamma$-pinning $\mathscr{P}_M$, and set $\mathscr{T}_{\mathscr{M}} := h^\vee_M(\mathscr{T}_M)$.

By assumption, ${}^s\mathscr{N} = \mathscr{N}$ and hence some left $\mathscr{N}$-translate $n's$ of $s$ normalizes $\mathscr{T}_{\mathscr{M}}$, and thus so does $s(s^{-1}n's)$, where $s^{-1}n's =: n \in \mathscr{N}$. Note that $sn$ (acting on $Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{N})}$) preserves $\chi$—this makes sense to say since $\chi$ factors through $Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{N})}$ (by Lemma 4.29). Since $s$ commutes with $\phi$, it also stabilizes each ${}^{\phi(w)}\chi$ for any $w \in W_F$—it is easy to see that ${}^{\phi(w)}\chi$ has the same Newton Levi subgroup $\mathscr{N}$ as $\chi$. It follows that $sn$, in addition to stabilizing $\chi$, also stabilizes each $W_F$-conjugate ${}^{\phi(w)}\chi$. It evidently suffices to show that $sn \in \mathscr{N}$ (we do not claim that $sn \in S_\phi$).

Denote by $\overline{sn}$ the image of $sn$ in $W(\widehat{G}, \mathscr{T}_{\mathscr{M}})$; the isomorphism $h^\vee_M$ gives an identification

$$W(\widehat{G}, \mathscr{T}_{\mathscr{M}}) \xrightarrow{\sim} W(G_{\overline{F}}, T_{M,\overline{F}}) \tag{43}$$

such that the action of $W(\widehat{G}, \mathscr{T}_{\mathscr{M}})$ on $\mathrm{Hom}(\mu_{\overline{F}}, T_{M,\overline{F}})$ via (43) coincides with the action given by acting on $X^*(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T}_{\mathscr{M}})}))$ and then mapping to $\mathrm{Hom}(\mu_{\overline{F}}, T_{M,\overline{F}})$ via $h^{\vee,-1}_M$ and (37); it follows that (43) identifies the stabilizer of each ${}^{\phi(w)}\chi$ in $W(\widehat{G}, \mathscr{T}_{\mathscr{M}})$ (for any $w \in W_F$) with the stabilizer of ${}^w f_{\chi,h^\vee_M,e}$ in $W(G_{\overline{F}}, T_{M,\overline{F}})$ and identifies $W(\mathscr{N}, \mathscr{T}_{\mathscr{M}})$ with $W(N_{\overline{F}}, T_{M,\overline{F}})$ (this last statement follows from how $\mathscr{N}$ was constructed).

By the first paragraph one has that $\overline{sn}$ stabilizes each ${}^{\phi(w)}\chi$ and thus its image under (43) stabilizes each ${}^w f_{\chi,h^\vee_M,e}$, which exactly means that it lies in the subgroup $W(N_{\overline{F}}, T_{M,\overline{F}})$. We deduce from the previous paragraph that $\overline{sn} \in W(\mathscr{N}, \mathscr{T}_{\mathscr{M}})$ and therefore $sn \in \mathscr{N}$, as desired. □

We can finally define the family of cocycles. Given $[\chi] \in X^+_\phi(\widehat{G})$, let $\mathscr{N}$ be the associated Newton Levi subgroup of $\widehat{G}$. The restriction

$$s\colon \pi_0(S_{\phi,[\chi]}) = \frac{S_{\phi,[\chi]}}{Z(\widehat{G})^{\Gamma,\circ}} \longrightarrow \widetilde{\mathscr{N}}$$

of the section $s_N$ of (33) yields an element of $Z^2(\pi_0(S_{\phi,[\chi]}), Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{N})})$ defined by the usual formula $(x,y) \mapsto s(x)s(y)s(xy)^{-1}$. Pushing forward this 2-cocycle along the character $\chi$, on which $\pi_0(S_{\phi,[\chi]})$ acts trivially, yields the desired cocycle $\natural_x \in Z^2(\pi_0(S_{\phi,[\chi]}), \mathbb{C}^\times)$. The above discussion implies that $\natural_x$ does not depend on any choices (such as the representative $\chi$ for $[\chi]$, the quasi-split twisted Levi subgroup $M$, or the isomorphism $h^\vee_M \in J_M$) other than the choice of section $s_N$ of (33), whose effect will be discussed later (cf. Proposition 4.38).

For the family $\{\natural_x \mid x \in X^+_\phi(\widehat{G})\}$ to define a twisted extended quotient, we need to check that for any $s \in \pi_0(S_\phi)$ and $x \in X^+_\phi(\widehat{G})$, the cocycle $s_*\natural_x = \natural_x \circ \mathrm{Ad}(s^{-1})$ is cohomologous to $\natural_{^s x}$ in $Z^2(\pi_0(S_{\phi,[^s\chi]}), \mathbb{C}^\times)$—we will show that they are equal. Choose a representative $\chi \in X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})}))$ for $[\chi]$ along with $M$ and $h^\vee_M \in J_{M,\mathscr{M}}$, so that $^s\chi$ is a representative for $[^s\chi]$. In particular, we have that $\mathrm{Ad}(s^{-1}) \circ h^\vee_M \in J_{^s(\mathscr{M}),M}$, so that $\natural_{[^s\chi]}$ may be formed using the isomorphism $\mathrm{Ad}(s^{-1}) \circ h^\vee_M$, which transfers the character $^s\chi$ to the same character of $\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})})$ as the one determined by $\chi$ and $h^\vee_M$. In particular, we obtain the same twisted Levi subgroup $N$ and deduce that $s_*\natural_x = \natural_{^s x}$, as claimed.

The argument of the previous paragraph also implies that for $(s,[\chi]) \in \pi_0(S_\phi) \times X^+_\phi(\widehat{G})$ we may define the "gluing" algebra isomorphism (36) as the map induced by $\mathrm{Ad}(s)$, finishing the construction of the desired twisted extended quotient $(X^+_\phi(\widehat{G}) \mathbin{/\!/} \pi_0(S_\phi))_\natural$. Moreover, for a fixed $([\chi],\rho) \in \widetilde{X^+_\phi(\widehat{G})}_\phi$ for our representative $\phi \in [\phi]$, $g^\vee \in \widehat{G}_{\mathrm{ad}}$ sends $([\chi],\rho)$ to $([\chi'],\rho') \in \tilde{X}_{\mathrm{Ad}(g^\vee)\circ\phi}$ such that $[\chi']$ has Newton Levi subgroup $\mathrm{Ad}(g^\vee)(\mathscr{N}_\chi)$, and it is straightforward to check that this action is compatible with the equivalence relations on $X^+_\phi(\widehat{G})$, $X^+_{\mathrm{Ad}(g^\vee\circ\phi}(\widehat{G})$, sends $S_\phi$-conjugacy to $S_{\mathrm{Ad}(g^\vee)\circ\phi}$-conjugacy, and induces a *canonical* identification

$$(X_\phi \mathbin{/\!/} \pi_0(S_\phi))_\natural \to (X_{\mathrm{Ad}(g^\vee)\circ\phi} \mathbin{/\!/} \pi_0(S_{\mathrm{Ad}(g^\vee)\circ\phi}))_\natural.$$

We denote the resulting limit over $\widehat{G}_{\mathrm{ad}}$ by $[(X^+_\phi(\widehat{G}) \mathbin{/\!/} \pi_0(S_\phi))_\natural]$. This allows us to give the main definition:

**Definition 4.32.** A *rigid enhancement* of $\phi$ is an element $[x]$ of the set $[(X^+_\phi(\widehat{G}) \mathbin{/\!/} \pi_0(S_\phi))_\natural]$ defined above. Concretely, it has representatives given by pairs $([\chi],\rho)$ consisting of a "highest weight" $\chi \in X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})}))$ and $\rho$ an irreducible representation of the twisted algebra $\mathbb{C}[\pi_0(S_{\phi,[\chi]}), \natural_{[\chi]}]$, which can be thought of as the "non-abelian" part of the data.

We give an example of non-basic enhancements, continuing the notation from the previous subsections. First, we note that when there is a maximal torus $\mathscr{T}$ of $\widehat{G}$ which is the unique $\phi$-minimal subgroup of $\widehat{G}$, then the equivalence relation used to define the set $X^+_\phi(\widehat{G})$ collapses to equality of characters in $X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T})})$.

**Example 4.33.** In place of a concrete example, we first give a broad class of examples in which non-basic enhancements $[(X^+_\phi(\widehat{G}) /\!\!/ \pi_0(S_\phi))_\natural]$ take a more concrete form. Assume that there is a maximal torus $\mathscr{T}$ of $\widehat{G}$ which is the unique $\phi$-minimal subgroup of $\widehat{G}$, and further that $S_\phi \subseteq \mathscr{T}$, which evidently means $S_\phi = \mathscr{T}^\Gamma$ (with respect to the $\phi$-twisted action, as usual).

Choosing a $W(G,T)(F)$-equivariant section $s_T$ of the surjection $\widehat{\bar{\tilde{T}}} \to \widehat{T}/Z(\widehat{G})^{\Gamma,\circ}_{(T)}$ as in (33), we obtain sections of any intermediate quotient

$$\widehat{\bar{\tilde{T}}} \to \widehat{\tilde{T}} \to \frac{\widehat{T}}{Z(\widehat{G})^{\Gamma,\circ}_{(T)}}$$

by composing $s_T$ with the projection $\widehat{\bar{\tilde{T}}} \to \widehat{\tilde{T}}$. It is an easy inductive exercise to check that for each quasi-split twisted Levi subgroup $M$ containing $T$ we may choose the sections $s_M$ so that when restricted to the image of $\widehat{T}/Z(\widehat{G})^{\Gamma,\circ}_{(T)}$ they recover the section induced by $s_T$ as above.

It then follows from Clifford's theorem on representations of finite groups that each pair $(\chi,\rho) \in \widetilde{X^+_\phi(\widehat{G})}$ (where $\widetilde{X^+_\phi(\widehat{G})}$ is formed with respect to the sections from the above paragraph) determines a representation $\tilde{\rho}$ of the profinite group $\pi_0(\mathscr{T}^{\Gamma,+}_{(\mathscr{T})})$ (which factors through $\pi_0(\mathscr{T}^{\Gamma,+}_{(\mathscr{M})})$, where $\mathscr{M}$ is the image of $\widehat{M}$ and is the Newton Levi subgroup associated to $\chi$).

We conclude that in this class of examples, non-basic enhancements are equivalent to characters of $\pi_0(\mathscr{T}^{\Gamma,+}_{(\mathscr{T})})$ which factor through $\pi_0(\mathscr{T}^{\Gamma,+}_{(\mathscr{M})})$ and have restriction to $\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T})})$ which yields (via the duality map (37)) a (quasi-split) twisted Levi subgroup $M$ of $G$. In Section 4.4 we will see more specific instances of this class of examples.

## 4.3. **The Langlands correspondence.**

Continue with the notation of §4.2. We now explain how to generalize the basic local Langlands correspondence for double covers discussed in §4.2.1 to the non-basic setting using the notion of an enhancement from Definition 4.32.

4.3.1. *Construction of basic enhancements for twisted Levi subgroups.* Fix a pair $[([\chi],\rho)] \in (X^+_\phi(\widehat{G}) /\!\!/ \pi_0(S_\phi))_\natural$ with representative $([\chi],\rho)$ along with $\chi \in X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})}))$ for $\mathscr{M} \subseteq \widehat{G}$ a $\phi$-minimal subgroup. As explained in §4.2, we can canonically associate to $[\chi]$ (up to $F$-rational isomorphism) a quasi-split twisted Levi subgroup $N$ containing $M$ for some choice of quasi-split twisted Levi subgroup $M$ of $G$ corresponding to $\mathscr{M}$ (cf. the discussion immediately following Definition 4.20). The first order of business is obtaining from $([\chi],\rho)$ an enhanced parameter for $N(F)_\pm$ using Clifford theory and Theorem 4.15. Because of the ambiguity in the choices of $M$, the embedding $h^\vee_M \in J_M$, and the representative $\phi$ of $[\phi]$, we will actually obtain a $W(G,N)(F)$-orbit of parameters, as will be explained precisely below.

By construction (cf. §4.2), for a fixed $M$ and $h^\vee_M$ with corresponding quasi-split twisted Levi subgroup $N$ there are embeddings $j^\vee_M \colon {}^LM_\pm \to {}^LN_\pm$ and $j^\vee_N \colon {}^LN_\pm \to {}^LG$ whose composition restricts to $\widehat{M}$ as $h^\vee_M$; let $\phi_{M,\pm}$ and $\phi_{N,\pm}$ denote the corresponding parameters, uniquely determined by $j^\vee_M$ and $j^\vee_N$ up to $Z(\mathscr{M})$-conjugacy. By construction, $\chi \circ (j^{\vee,-1}_N|_{\widetilde{\mathscr{M}}})$ (observing that $j^\vee_N$ induces an isomorphism $j^\vee_M(\widehat{M})^+_{(j^\vee_M(\widehat{M}))} \xrightarrow{\sim} \widetilde{\mathscr{M}})$ factors through $\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(N)})$—denote the corresponding character by $\chi_{\widehat{N}}$. Since (by Proposition 4.31) $S_{\phi,[\chi]} \subset j^\vee_N(\widehat{N}) =: \mathscr{N}$, we have an isomorphism

$$S_{\phi,[\chi]} \xrightarrow{j^{\vee,-1}_N} S_{\phi_{N,\pm},\chi_{\widehat{N}}},$$

and the section $s_N$ gives a section $\pi_0(S_{\phi_{N,\pm}}) \to S^+_{\phi_{N,\pm}}$ and thus also a section

$$\pi_0(S_{\phi_{N,\pm}}) \to \pi_0(S^+_{\phi_{N,\pm}}) \tag{44}$$

which, by restricting and then composing with $\chi_{\widehat{N}}$, yields a 2-cocycle in $Z^2(\pi_0(S_{\phi_{N,\pm}})_{\chi_{\widehat{N}}}, \mathbb{C}^\times)$ (we can take stabilizers before or after applying the functor $\pi_0$) which is identified with $\natural_{[\chi]}$ via $j_N^\vee$ (essentially by how we set things up).

One has the short exact sequence of finite groups

$$0 \to \pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(N)}) \to \pi_0(S^+_{\phi_{N,\pm}}) \to \pi_0(S_{\phi_{N,\pm}}) \to 1. \tag{45}$$

According to Clifford's theorem (see e.g. [AMS18, Proposition 1.1]), giving an irreducible representation of $\pi_0(S^+_{\phi_{N,\pm}})$ is equivalent to giving a character $\Xi$ of $\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(N)})$ and a representation $\tilde{\rho}$ of the twisted group algebra $\mathbb{C}[\pi_0(S_{\phi_{N,\pm}})_\Xi, \theta]$, where $\theta$ is the 2-cocycle valued in $\mathbb{C}^\times$ determined by the section (44) and the character $\Xi$. We can thus set $\Xi = \chi_{\widehat{N}}$ and $\tilde{\rho}$ the representation induced by $\rho$ and $j_N^\vee$ to obtain a basic enhancement $\rho_{\chi,h_M^\vee}$ for $\phi_{N,\pm}$.

Recall from the discussion of Newton Levi subgroups in §4.2.3 that replacing $h_M^\vee$ with $h_{M'}^\vee$ replaces $f_{\chi,h_M^\vee}$ with ${}^g f_{\chi,h_M^\vee}$ for $g \in (K_{Z(M),G}) \cap N_G(N)(\overline{F})$, and we may take $j^\vee_{{}^gM} = \theta^\vee_{w(g)} \circ \mathrm{Ad}(n) \circ$ $\circ j_M^\vee \circ \theta^\vee_{\mathrm{Ad}(g^{-1})}$ for some $n \in \widehat{N}$ and $j^\vee_{{}^gN} = j_N^\vee \circ \mathrm{Ad}(n^{-1}) \circ \theta^{\vee,-1}_{w(g)}$ (as in §4.2.2, even though ${}^gN = N$ we still write ${}^gN$ to reflect the change in embeddings in our notation). We track the effect of these changes on the above construction (keeping the preceding notation): First, $\chi_{\widehat{{}^gN}} = \chi_{\widehat{N}} \circ \theta^{\vee,-1}_{g(w)}$ and the representation of $\mathbb{C}[\pi_0(S_{\phi_{{}^gN,\pm}})_{\chi_{\widehat{{}^gN}}}, (\theta \circ \theta^{\vee,-1}_{w(g)})]$ is obtained by applying $\mathrm{Ad}(n) \circ \theta^\vee_{g(w)}$ to the representation of $\mathbb{C}[\pi_0(S_{\phi_{N,\pm}})_{\chi_{\widehat{N}}}, \theta]$ from the previous paragraph.

We have thus proved:

**Proposition 4.34.** *Using the above notation, replacing $h_M^\vee$ with another $h_{M'}^\vee$ (allowing $M' = M$ but choosing a different isomorphism in $J_M$), replaces the representation $\rho_{\chi,h_M^\vee}$ of $\pi_0(S^+_{\phi_{N,\pm}})$ with the representation $\rho_{\chi,h_M^\vee} \circ \mathrm{Ad}(n^{-1}) \circ \theta^{\vee,-1}_{w(g)}$ of $\pi_0(S^+_{\theta^\vee_{w(g)} \circ \mathrm{Ad}(n) \circ \phi_{N,\pm}})$ for $g \in (K_{Z(M),G}) \cap N_G(N)(\overline{F})$.*

It is now necessary to investigate how the above construction behaves after replacing $\chi \in X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})}))$ by an equivalent $\chi' \in X^*_{\mathrm{Lev}}(\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M}')}))$ for another $\phi$-minimal subgroup $\mathscr{M}' \subseteq \widehat{G}$, yielding another basic enhancement $\underline{\rho}_{\chi',j'}$ for the same twisted Levi subgroup $N$ containing some quasi-split twisted Levi subgroup $M'$ for the parameter $\phi_{N,\pm}$.

Choose $M$ and $M'$ as above with corresponding $h_M^\vee \in J_M$ and $h_{M'}^\vee \in J_{M'}$, along with pairs $(j_M^\vee, j_N^\vee)$, $(j_{M'}^\vee, \tilde{j}_N^\vee)$ associated to $h_M^\vee$ and $h_{M'}^\vee$ as in §4.2.3. By construction, there is some $g^\vee \in \widehat{G}$ such that ${}^g j_N^\vee = \tilde{j}_N^\vee$, and the torsion cocharacters $f_{\chi,h_M^\vee,e}$, $f_{\chi',h_{M'}^\vee,e}$ associated to $\chi$ and $\chi'$ (respectively) are related by $f_{\chi',h_{M'}^\vee,e} = {}^g f_{\chi,h_M^\vee,e}$ for some $g \in N_G(N)(\overline{F})$ (unique up to $N_{\mathrm{ad}}(\overline{F})$). We claim that in fact $g \in K_{N,G}$ as well; to see this, note that for any $\sigma \in \Gamma$ we have (working exclusively with the subgroup $Z(\widehat{G})^{\Gamma,\circ,+}_{(N)}$)

$${}^{\sigma g} f_{\chi,h_M^\vee,e} = \chi \circ j_N^\vee \circ \mathrm{Ad}({}^\sigma g^{-1}) = \chi \circ \mathrm{Ad}({}^{\phi(\sigma)} g^{\vee,-1}) \circ j_N^\vee = \chi \circ \mathrm{Ad}(g^{\vee,-1}) \circ j_N^\vee = f_{\chi',h_{M'}^\vee,e},$$

where the penultimate equality is the key one and follows from the fact that the $\widehat{G}$-coordinate of $\phi(\sigma)$ lies in $\mathscr{N}$ and therefore conjugates both $\chi$ and $\mathrm{Ad}(g^\vee) \circ \chi$ (which factor through $Z(\widetilde{\mathscr{N}})$) trivially.

In particular, each ${}^{\sigma}g^{-1}g$ fixes $f_{\chi,h^\vee_M,e}$ so that in fact ${}^{\sigma}g^{-1}g \in N(\overline{F})$, giving the claim that $g \in K_{N,G}$. Moreover, any such $\mathrm{Ad}(g^\vee)|_{\mathscr{N}}$ differs from $\theta^\vee_{w(g)}$ by $\mathscr{N}_{\mathrm{ad}}$, and thus the parameter $\theta^\vee_{w(g)} \circ \phi_{N,\pm,h^\vee_M}$ for ${}^LN$ is $\widehat{N}$-conjugate to $\phi_{N,\pm,h^\vee_{M'}}$.

The analogue of Proposition 4.34 is:

**Corollary 4.35.** *For $\chi$ and $\chi'$ as above with choices of $M, M'$ and $h^\vee_M \in J_M$ and $h^\vee_{M'} \in J_{M'}$ and $g \in (K_{N,G}) \cap N_G(N)(\overline{F})$ as in the preceding discussion, one has an isomorphism $S_{\theta^\vee_{w(g)} \circ \phi_{N,\pm,h^\vee_M}} \xrightarrow{\sim} S_{\phi_{N,\pm,h^\vee_{M'}}}$ of groups induced by $\mathrm{Ad}(n)$ for some $n \in \widehat{N}$ and an isomorphism of representations*

$$\rho_{\chi',h^\vee_{M'}} \circ \mathrm{Ad}(n^{-1}) \simeq \rho_{\chi,h^\vee_M} \circ \theta^{\vee,-1}_{w(g)}.$$

*Proof.* This result follows from the above discussion and the fact that both enhancements are constructed using the same twisted group algebra representation $\rho$. □

Putting all of the above results together gives:

**Theorem 4.36.** *For $[([\chi],\rho)] \in [(X^+_\phi(\widehat{G}) /\!\!/ \pi_0(S_\phi))_\natural]$ with representative $([\chi],\rho)$ we obtain a canonical $W(G,N)(F)$-orbit of basic enhancements*

$$\{[\tilde{\rho} \circ \theta^{\vee,-1}_w]\}_{w \in W(G,N)(F)}$$

*where $\tilde{\rho}$ is some fixed element of the orbit; if $\tilde{\rho}$ is an enhancement of the parameter $\phi_{N,\pm}$ of ${}^LN_\pm$, then $\tilde{\rho} \circ \theta^{\vee,-1}_w$ is an enhancement for the parameter $\theta^\vee_w \circ \phi_{N,\pm}$ (and each enhanced parameter $(\theta^\vee_{w(g)} \circ \phi_{N,\pm}, \rho \circ \theta^{\vee,-1}_w)$ is viewed up to the $\widehat{N}_{\mathrm{ad}}$-action, as usual).*

*Proof.* The only part of this statement not argued explicitly above is independence of the choice of element in $\tilde{X}$ representing the $\pi_0(S_\phi)$-orbit, which is straightforward (studying the above construction of basic enhancements, one sees that taking $\pi_0(S_\phi)$-conjugates replaces, after possibly taking a $\widehat{N}$-conjugate, each $\pi_0(S^+_{\phi_{N,\pm}})$-representation with a $W(G,N)(F)$-conjugate—actually, this will be a $W(G,N)_\phi$-conjugate, see Remark 4.37 below for the definition of this notation). □

**Remark 4.37.** One can work, for a fixed $\phi \in [\phi]$, with $(X^+_\phi(\widehat{G}) /\!\!/ \pi_0(S_\phi))_\natural$ rather than the $\widehat{G}$-conjugacy classes $[(X^+_\phi(\widehat{G}) /\!\!/ \pi_0(S_\phi))_\natural]$ and obtain from the preceding arguments a $W(G,N)_\phi$-orbit of enhancements of a fixed parameter $\phi_{N,\pm}$ for $N(F)_\pm$ which factors via some standard embedding to give the representative $\phi$, where $W(G,N)_\phi$ denotes all automorphisms $\theta^\vee_w$ for $w \in W(G,N)(F)$ such that the image of $\theta^\vee_w$ in $N_{\widehat{G}}(\mathscr{N})/\mathscr{N}$ (via the identification of $\widehat{N}$ and $\mathscr{N}$) lies in the image of $S_\phi \cap N_{\widehat{G}}(\mathscr{N})$.

Recall that we picked a section $s_N$ of each covering $\widehat{\bar{N}} \to \widehat{N}/Z(\widehat{G})^{\Gamma,\circ}_{(N)}$ which were used to construct the family of cocycles $\{\natural_{[\chi]}\}_{[\chi] \in X^+_\phi(\widehat{G})}$.

**Proposition 4.38.** *Any two choices of section $s_N$ as in* (33) *yield the same $W(G,N)(F)$-orbits of basic enhancements from Theorem 4.36 (up to isomorphism).*

*Proof.* For any $[\chi]$ as in Theorem 4.36, by examining the construction of the cocycles $\natural_{[\chi]}$ in §4.2.4 it is clear that different choices of section yield cohomologous cocycles of $\pi_0(S_\phi)_{[\chi]}$, and therefore for a given $([\chi],\rho) \in \tilde{X}$ (with respect to the section $s_M$), we may twist by a coboundary to identify the representations of the twisted stabilizer algebras and obtain isomorphic representations of each $\pi_0(S^+_{\phi_{N,\pm}})$ in the $W(G,N)(F)$-orbit. □

4.3.2. *A correspondence.* Let $N$ be a twisted Levi subgroup of $G$, which we assume is quasi-split (cf. Lemmas 4.1, 4.24).

Recall from §3.2 that the group $W(G,N)(F)$ acts on the cohomology set $H^1_{\mathrm{bas}}(\mathscr{E},N)$ in a way that preserves the fibers of the map $H^1_{\mathrm{bas}}(\mathscr{E},N)\to H^1(\mathscr{E},G)$. In this vein, we have:

**Lemma 4.39.** *If* $[y]=w\cdot[x]\in H^1_{\mathrm{bas}}(\mathscr{E},N)$ *for* $w\in W(G,N)(F)$ *then* $[x]$ *and* $[y]$ *have the same image in* $H^1(F,N_{\mathrm{ad}})$.

*Proof.* By twisting we can reduce to the case when $[x]$ is the neutral class and $[y]$ has representative $dg_w$ for $g_w\in G(\overline{F})$ lifting $w$. The fact that the short exact sequence of $F$-group schemes

$$1\to\underline{\mathrm{Inn}}_N\to\underline{\mathrm{Aut}}_N\to\underline{\mathrm{Out}}_N\to 1$$

is split (using that $N$ is quasi-split, as explained at the beginning of §4.1.2) implies that any $w\in W(G,N)(F)\subset\mathrm{Out}_N(F)$ can be lifted to $\theta_w\in\mathrm{Aut}_N(F)$.

The morphism $\mathrm{Ad}(g_w)\in\underline{\mathrm{Aut}}_N(\overline{F})$ has the same image in $\underline{\mathrm{Out}}_N(F)$ as $\theta_w$ and therefore there is some $n\in N(\overline{F})$ such that $\mathrm{Ad}(g_wn)$ is defined over $F$. But then we may use $d(g_wn)$ as the representative for $[dg_w]$ and it has trivial image in $H^1(F,N_{\mathrm{ad}})$. □

Recall that for the quasi-split representative $N\in K_{N,G}\cdot N$ we have fixed a pinning which determines a Whittaker datum $\mathfrak{w}_N$ (as in [Kal16a, §1.3], after choosing an additive character $F\to\mathbb{C}^\times$)—we will prove shortly that the construction will not depend on these choices (in the sense that there is a canonical way to identify the constructions corresponding to different choices).

One additional discussion is needed: Consider an $L$-parameter $W_F'\xrightarrow{\varphi_\pm}{}^LN_\pm$; recall that we have the family of $F$-rational automorphisms of $N$ (resp. automorphisms of ${}^LN_\pm$) given by acting by elements of $W(G,N)(F)$ via the automorphisms $\theta_w$ (resp. $\theta_w^\vee$) defined in §4.1.2 corresponding to the choice of pinning of $N$.

Now let $\dot\pi_\pm=(N',\psi,z,\pi_\pm)$ be a representation of the rigid inner twist of $N(F)_\pm$ (as in Definition 4.12). As explained in [Kal23, §2.3.4] (for the version of the local Langlands conjecture that does not involve double covers, as in Theorem 4.10) each $\theta_w$ acts on this datum by the equation

$$\theta_w\cdot(N',\psi,z,\pi_\pm)=(N',\psi\circ\theta_w^{-1},\theta_w(z),\pi_\pm).$$

One then has:

**Conjecture 4.40.** ([Kal23, Conjecture 2.12]) In the above notation, we have

$$\Pi_{\theta_w\circ\varphi_\pm}=\theta_w\cdot\Pi_{\varphi_\pm}:=\{\theta_w\cdot\dot\pi,\dot\pi\in\Pi_{\varphi_\pm}\},$$

and also

$$\iota_{\mathfrak{w}_N,\pm}(\theta_w\cdot\dot\pi)=(\theta_w^\vee\circ\varphi,\rho\circ\theta_w^{\vee,-1}),$$

where $(\varphi,\rho)=\iota_{\mathfrak{w}_N,\pm}(\dot\pi)$ is the enhanced parameter corresponding to $\dot\pi$ (in the above equation we are using that $\theta_w(\mathfrak{w}_N)=\mathfrak{w}_N$ by how we set things up; the more general formula given loc. cit. is slightly different to account for the Whittaker datum changing).

Note that $W(G,N)(F)$ acts on the set $\Pi^{\mathrm{rig}}(N)$ of isomorphism classes of representations of rigid inner twists of $N(F)_\pm$ (via a choice of pinning). For a fixed parameter $[\varphi_\pm]$ for ${}^LN_\pm$, denote by $[\![\Pi_{\varphi_\pm}]\!]_{W(G,N)(F)}$ the image of the $L$-packet $\Pi_{\varphi_\pm}$ in the quotient space $\Pi^{\mathrm{rig}}(N)/W(G,N)(F)$. We observe that $W(G,N)(F)$ acts on the subset $\Pi^{\mathrm{rig}}(N,N')\subset\Pi^{\mathrm{rig}}(N)$ of isomorphism classes of representations which have a representative whose underlying group is $N'$ for a fixed inner form $N'$ of $N$ as well (by Lemma 4.39). For $[x]\in H^1(\mathscr{E},G)$ with quasi-split representative $N$ denote (following the notation of [BMO23]) by $G_{[x]}$ the corresponding inner form of $N$ (uniquely defined up to $F$-rational isomorphism, also by Lemma 4.39).

Suppose that $\phi$ is an $L$-parameter for ${}^LG$ factoring through the conjugacy class of embeddings ${}^LN_\pm \to {}^LG$ associated to a twisted Levi subgroup $N$, and let $\phi_{N,\pm}$ be a parameter for ${}^LN_\pm$ which yields $[\phi]$ via this composition. Denote by $H^1_{\text{bas}}(\mathscr{E}, N)_+$ the subset of all $[x] \in H^1_{\text{bas}}(\mathscr{E}, N)$ such that $Z_G(f_x) = N$ (for any representative $x$). We are interested in the subset

$$\Pi^+_{\phi_{N,\pm}} := \{\dot{\pi}_\pm = (N', \psi, z, \pi_\pm) \in \Pi_{\phi_{N,\pm}} \mid [z] \in H^1_{\text{bas}}(\mathscr{E}, N)_+\} \subseteq \Pi_{\phi_{N,\pm}};$$

one then defines $\Pi^+_{\phi_{N,\pm}}(G_{[x]})$ in the obvious way. These classes of subsets of $\Pi^{\text{rig}}(N)$ and $\Pi^{\text{rig}}(N, N')$ are preserved by the $W(G,N)(F)$-action, and so we may write $[\![\Pi^+_{\phi_{N,\pm}}(N')]\!]_{W(G,N)(F)}$.

We can now state the main result of this paper:

**Theorem 4.41.** *Let $G$ be a quasi-split connected reductive group and $W'_F \xrightarrow{\phi} {}^LG$ a discrete $L$-parameter. Assume that the basic rigid local Langlands conjecture (Conjecture 4.10) holds for $G$ and all inner forms of its (quasi-split) twisted Levi subgroups. Then there is a bijection*

$$\bigsqcup_{[N]} [\![\Pi^+_{\phi_{N,\pm}}]\!]_{W(G,N)(F)} \xrightarrow{\sqcup \iota_{\mathfrak{w}_{N,\pm}}} [(X^+_\phi(\widehat{G}) /\!\!/ \pi_0(S_\phi))_\natural], \tag{46}$$

*where the disjoint union is over all ($K_{N,G}$-conjugacy classes of) twisted Levi subgroups $N$ such that $\phi$ factors through an (uniquely determined up to $\widehat{G}$-conjugacy) $L$-parameter $W'_F \xrightarrow{\phi_{N,\pm}} {}^LN_\pm \to {}^LG$ via the canonical conjugacy class of $L$-embeddings from Proposition 4.2.*

*Proof.* This is an immediate consequence of combining Theorem 4.36 with Theorem 4.15 (which depends on Conjecture 4.10) and the preceding discussion (including Conjecture 4.40). □

**Remark 4.42.** There is a canonical map

$$\bigsqcup_{[N]} [\![\Pi^+_{\phi_{N,\pm}}]\!]_{W(G,N)(F)} \to H^1_{\text{L-reg}}(\mathscr{E}, G)$$

given by sending the $W(G,N)(F)$-orbit of a point $(N', z_\psi, \pi_\pm)$ to the image of $[z_\psi] \in H^1_{\text{bas}}(\mathscr{E}, N)$ in $H^1(\mathscr{E}, G)$. We can thus break up the left-hand side of (46) according to their image in $H^1(\mathscr{E}, G)$ and re-write Theorem 4.41 as a bijection

$$\bigsqcup_{[x] \in H^1_{\text{L-reg}}(\mathscr{E}, G)_\phi} [\![\Pi_{\phi_{N,\pm}}(G_{[x]}, [x])]\!]_{W(G,N)(F)} \to [(X^+_\phi(\widehat{G}) /\!\!/ \pi_0(S_\phi))_\natural],$$

where for $[x']$ an arbitrary preimage of $[x]$ in $H^1_{\text{bas}}(\mathscr{E}, N)$ the set $\Pi_{\phi_{N,\pm}}(G_{[x]}, [x'])$ denotes the subset of $\Pi_{\phi_{N,\pm}}$ consisting of all isomorphism classes which have a representative with a rigidification cohomologous to $[x']$ and we denote its image in the quotient $\Pi^{\text{rig}}(N)/W(G,N)(F)$ by $[\![\Pi_{\phi_{N,\pm}}(G_{[x]}, [x])]\!]_{W(G,N)(F)}$ (since the set $[\![\Pi_{\phi_{N,\pm}}(G_{[x]}, [x])]\!]_{W(G,N)(F)}$ does not depend on the choice of $[x']$) and $H^1_{\text{L-reg}}(\mathscr{E}, G)_\phi$ is the subset of Levi-regular classes such that $\phi$ factors through ${}^LN_\pm$ for the rigid Newton centralizer $N$.

Recall that we have fixed a quasi-split representative $N$ in each class $K_{N,G} \cdot N$ of subgroups—it is unique up to acting by $g \in K_{N,G}$ such that $\text{Ad}(g)|_N$ is defined over $F$ and this $g$ is unique up to $F$-rational automorphisms of $N$ induced by $W(G,N)(F)$ and a choice of pinning (cf. Remark 4.26). Since the construction of the correspondence in Theorem 4.41 is invariant under the action of each Weyl group $W(G,N)(F)$ induced by the pinning, if we choose any $g$ as above which gives an isomorphism from $N$ to ${}^gN$ and fix a pinning of ${}^gN$, then $\mathrm{Ad}(g)$ induces an identification of all data appearing in Theorem 4.41 that does not depend on the choice of $g$.

**Remark 4.43.** For a discrete parameter $\phi$, one combines Theorem 3.18 with Corollary 4.19 to see that every twisted Levi subgroup $N$ of $G$ such that $\phi$ factors through ${}^LN_\pm \to {}^LG$ in the canonical conjugacy class arises from a class in $H^1(\mathscr{E},G)_{(G),\phi}$. In other words, the map (46) is not "missing" any twisted Levi subgroups of $G$ (or their inner forms).

Although the definition of $\Pi^+_{\phi_{N,\pm}}$ may seem restrictive, we have:

**Proposition 4.44.** *For a fixed $[x] \in H^1_{\text{L-reg}}(\mathscr{E},G)$, every smooth irreducible, genuine representation $\pi_\pm$ of $G_{[x]}(F)_\pm$ whose image under the local Langlands correspondence for $G_{[x]}(F)_\pm$ is an L-parameter for ${}^LN_\pm$ which is discrete after composing with the embedding ${}^LN_\pm \to {}^LG$ lies in $\Pi^+_{\phi_{N,\pm}}$ for some L-parameter $\phi$ as in Theorem 4.41.*

We are abusively using the term "lies in" to refer to $\pi_\pm$ being the representation component of an element $\dot\pi_\pm \in \Pi^+_{\phi_{N,\pm}}(G_{[x]})$.

*Proof.* Let $\pi_\pm$ be such a representation of $G_{[x]}$; by construction, $[x]$ has a preimage $[y] \in H^1_{\text{bas}}(\mathscr{E},N)_+$, and we may thus enrich $\pi_\pm$ to a representation of a rigid inner twist $(G_{[x]},\psi,y,\pi_\pm)$. Then Theorem 4.15 yields an enhanced parameter $(\phi_{N,\pm},\rho_N)$ for $N(F)_\pm$. In particular, we have $\rho_N$ an irreducible representation of $\pi_0(S^+_{\phi_{N,\pm}})$ and we can apply Clifford theory to the short exact sequence (45) to obtain from $\rho_N$ a character $\tilde\chi$ of $\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(N)})$ and a simple $\mathbb{C}[\pi_0(S_{\phi_{N,\pm}})_{\tilde\chi},\theta]$-module $\tilde\rho$ (where $\theta$ is as in §4.3.1).

By assumption, the composition of $\phi_{N,\pm}$ with any embedding ${}^LN_\pm \to {}^LG$ in the canonical conjugacy class yields a discrete $L$-parameter $\phi$ and picking an arbitrary $\phi_{N,\pm}$-minimal subgroup $\mathscr{M}_N$ of $\widehat{N}$ with corresponding quasi-split twisted Levi subgroup $M$ of $N$, the image $\mathscr{M}$ under this embedding is evidently $\phi$-minimal. We may then define $\chi$ to be the character of $\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})})$ obtained by transporting (via this choice of embedding) the character of $\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M}_N)})$ defined by post-composing the projection

$$\pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M}_N)}) \to \pi_0(Z(\widehat{G})^{\Gamma,\circ,+}_{(N)})$$

with the character $\tilde\chi$.

One verifies readily (using that $[y] \in H^1_{\text{bas}}(\mathscr{E},N)_+$) that the Newton Levi subgroup associated to $\chi$ is exactly the image of $\widehat{N}$ under the embedding ${}^LN_\pm \to {}^LG$ chosen above, and hence this embedding also induces an identification (cf. Proposition 4.31) of $\pi_0(S_{\phi_{N,\pm}})_{\tilde\chi}$ with $\pi_0(S_\phi)_{[\chi]}$ (compatible with the relevant 2-cocycles of both stabilizers) which lets us transport $\tilde\rho$ to a simple module $\rho$ over the twisted algebra $\mathbb{C}[\pi_0(S_\phi)_{[\chi]},\natural_{[\chi]}]$. We thus obtain the pair $([\chi],\rho)$ and we can take its image in $(X /\!\!/ \pi_0(S_\phi))$ to obtain a non-basic enhancement for $\phi$. It is then straightforward to verify that the construction in §4.3.1 recovers the $W(G,N)(F)$-orbit of $(\phi_{N,\pm},\rho_N)$. □

4.3.3. *Relation to the rigid Kottwitz map.* Continue with the notation of the previous subsection. We want to re-write Theorem 4.41 in an analogous form to Conjecture 4.10, which requires defining an analogue of the bottom row in the relevant diagram, the first step of which is to give a dual interpretation of the codomain of the rigid Kottwitz map $\kappa$ from Definition 3.14.

Fix a quasi-split twisted Levi subgroup $N$ of $G$ containing a maximal torus $T$ which is elliptic in $G$ (and thus also in $N$).

**Lemma 4.45.** *For $N$ as above, there is a canonical identification*

$$\varinjlim_n \bar{Y}_{+,\mathrm{tor}}[Z_{N,n} \to G] \xrightarrow{\sim} X^*(\pi_0(Z(\widehat{G})^{\Gamma,+}_{(N)}))/W(G,N)(F),$$

*where $Z(\widehat{G})^{\Gamma,+}_{(N)}$ is the preimage of $Z(\widehat{G})^{\Gamma}$ in $\bar{\widehat{N}}$ (cf. Definition 4.5) via a choice of embedding in the $\widehat{G}$-conjugacy class of embeddings $\{\widehat{N} \to \widehat{G}\}$ canonically associated to the inclusion $N \hookrightarrow G$.*

The fact that we are quotienting by $W(G,N)(F)$-action means that the above map is independent of the choice of embedding.

*Proof.* This essentially follows from the proof of [Kal16b, Proposition 5.3], which we summarize here. Choose embeddings $\widehat{T} \to \widehat{N}$ and $\widehat{N} \to \widehat{G}$ in their respective canonical conjugacy classes.

As explained loc. cit., for any finite central subgroup $A$ in $N$ there is an isomorphism

$$\varinjlim_{K/F} \frac{[X_*(T/A)/X_*(T_{\mathrm{sc}})]^{N_{K/F}}}{I_{K/F}\cdot [X_*(T)/X_*(T_{\mathrm{sc}})]} \xrightarrow{\sim} X^*(\pi_0(Z(\widehat{G})^{\Gamma,+,A})),$$

where the limit is over a cofinal system of finite Galois extensions of $F$ which all split $T$ (so that writing $N_{K/F}$ and $I_{K/F}$ makes sense) and the superscript "+, $A$" means that we are taking the preimage in $\widehat{T/A}$ via the map $\widehat{T/A} \to \widehat{N} \to \widehat{G}$. Taking the quotient by the $W(G,N)(F)$-action on both sides and then applying the colimit over all $A = Z_{N,n}$ gives the result. □

Given a non-basic enhancement $[([\chi],\rho)$ for $\phi$ such that there is an $L$-parameter $\phi_{N,\pm}$ factoring through the conjugacy class of embeddings $\{{}^L N_{\pm} \to {}^L G\}$ to yield $[\phi]$ we can produce (Theorem 4.41) a $W(G,N)(F)$-orbit of basic enhancements which combine with the $W(G,N)(F)$-orbit of $\phi_{N,\pm}$ to give an orbit of enhanced parameters for $N(F)_{\pm}$. Restricting each representation of $\pi_0(S^+_{\phi_{N,\pm}})$ in this orbit to $\pi_0(Z(\widehat{G})^{\Gamma,+}_{(N)})$ (via a choice of embedding $\widehat{N} \to \widehat{G}$ as above) yields a uniquely-determined $W(G,N)(F)$-orbit of characters of $\pi_0(Z(\widehat{G})^{\Gamma,+}_{(N)})$. Combining this discussion with the previous lemma gives the final version of the main theorem, whose setup we re-state in its entirety for expository completeness:

**Theorem 4.46.** *Let $G$ be a quasi-split connected reductive group and $W'_F \xrightarrow{\phi} {}^L G$ a discrete $L$-parameter. Assume that the basic local Langlands correspondence (Conjecture 4.10) holds for $G$ and all inner forms of its (quasi-split) twisted Levi subgroups. Then there is a commutative diagram*

$$\begin{array}{ccc} \bigsqcup_{[N]} [\![\Pi^+_{\phi_{N,\pm}}]\!]_{W(G,N)(F)} & \xrightarrow{\sqcup \iota_{\mathfrak{w}_{N,\pm}}} & [(X^+_{\phi}(\widehat{G}) /\!\!/ \pi_0(S_{\phi}))_{\natural}], \\ \downarrow & & \downarrow \\ H^1_{\text{L-reg}}(\mathscr{E}, G) & \xrightarrow{\kappa} & \bigsqcup_{[N]} X^*(\pi_0(Z(\widehat{G})^{\Gamma,+}_{(N)}))/W(G,N)(F) \end{array}$$

*where the disjoint union is over all ($K_{N,G}$-conjugacy classes of) twisted Levi subgroups $N$ such that $\phi$ factors through an (uniquely determined up to $\widehat{G}$-conjugacy) $L$-parameter $W'_F \xrightarrow{\phi_{N,\pm}} {}^L N_{\pm} \to {}^L G$ via the canonical conjugacy class of $L$-embeddings from Proposition 4.2, the lefthand map is the one from Remark 4.42, and the bottom map is obtained by composing $\kappa$ with the identification* *from Lemma 4.45. The top map is a bijection but the bottom map is in general only injective (cf. Proposition 3.16).*

4.3.4. *Examples continued.* We give another example of the non-basic correspondence.

**Example 4.47.** We use the following example from [Kal16b, §5.4], originally from [She79]: Let $G = \mathrm{SL}_2$ so that $\widehat{G} = \mathrm{PGL}_2(\mathbb{C})$ and fix $E = F(\sqrt{\alpha})/F$ a quadratic extension along with a character $E^\times \xrightarrow{\theta} \mathbb{C}^\times$ such that $\theta^{-1} \cdot (\theta \circ \sigma)$ is a character of order 2, where $\sigma$ is the nontrivial element of $\Gamma_{E/F}$ with a fixed preimage $\sigma^\circ \in W_{E/F}$.

One then defines a parameter $W_{E/F} \xrightarrow{\phi} \mathrm{PGL}_2(\mathbb{C})$ by setting, for $e \in E^\times$, the value $\phi(e) = \begin{pmatrix} \theta(e) & 0 \\ 0 & \theta(\sigma(e)) \end{pmatrix}$ and $\phi(\sigma^\circ) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We record the following basic facts relevant to the parameter $\phi$: First, it normalizes the standard maximal torus $\mathscr{T}$ of $\mathrm{PGL}_2(\mathbb{C})$, and this is the unique $\phi$-minimal subgroup of $\widehat{G}$. Second, we have

$$S_\phi = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$$

(the Klein four group $V_4$) and third, we have

$$S_\phi^+ = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\},$$

which is isomorphic to the quaternion group.

We evidently have $\mathscr{T} \cap S_\phi = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\}$, its preimage in $\mathrm{SL}_2(\mathbb{C})$ is

$$(S_\phi \cap \mathscr{T})^+ = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\},$$

and its preimage $(S_\phi \cap \mathscr{T})^+_{(\mathscr{T})}$ in $\widetilde{\mathscr{T}}$ is $\varprojlim_n \mu_{2n}$ (with the obvious identifications), and the map $(S_\phi \cap \mathscr{T})_{(\mathscr{T})} \to (S_\phi \cap \mathscr{T})^+$ is projection onto the $\mu_4$-term. Finally, we have $Z(\widehat{G})^{\Gamma,\circ,+} = \{\mathrm{id}, -\mathrm{id}\}$ and $Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T})} = \varprojlim_n \mu_n$ (mapping into $\varprojlim_n \mu_{2n}$ in the obvious way).

Since $\phi$ normalizes a unique maximal torus $\mathscr{T}$ (the standard one) of $\mathrm{PGL}_2(\mathbb{C})$, the only (quasi-split) twisted Levi subgroups $M$ of $\mathrm{SL}_2$ that should appear (up to $F$-rational isomorphism) are $\mathrm{SL}_2$ and $T = \mathrm{Res}_{E/F}(\mathbb{G}_m)^{(1)}$ (which is anisotropic), embedded in the usual way.

As discussed above, $Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T})} = \varprojlim_n \mu_n$, and hence giving a character $\chi \in X^*(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T})})$ (the "highest weight" component of the enhancement $x \in (X^+_\phi(\widehat{G}) /\!\!/ S_\phi)_\natural$) is equivalent to choosing some $n \in \mathbb{N}$ and an $n$th root of unity $\tilde{\zeta}_n \in \mathbb{C}^\times$; denote the character by $\chi_{\tilde{\zeta}_n}$. The homomorphism $\mu_{\overline{F}} \to \mathrm{Res}_{E/F}(\mathbb{G}_m)^{(1)}_{\overline{F}}$ corresponding to such a character via (37) is then (fixing a compatible system of $m$th roots of unity $(\zeta_m)_m$ once and for all) determined by

$$\varprojlim_k (\zeta_{n_k}) \mapsto \zeta_n \mapsto \begin{pmatrix} \frac{\tilde{\zeta}_n + \tilde{\zeta}_n^{-1}}{2} & \frac{\sqrt{\alpha}(\tilde{\zeta}_n - \tilde{\zeta}_n^{-1})}{2} \\ \frac{\tilde{\zeta}_n - \tilde{\zeta}_n^{-1}}{2\sqrt{\alpha}} & \frac{\tilde{\zeta}_n + \tilde{\zeta}_n^{-1}}{2} \end{pmatrix} \in T[n](\overline{F}) \tag{47}$$

which is killed by $N_{E_k/F}$ for any $k$ such that $E_k$ has degree divisible by $n$ over $E$, which splits $T[n]$.

To compute the centralizer of the corresponding morphism $u \xrightarrow{f_{\tilde{\zeta}_n}} T \subseteq \mathrm{SL}_2$ (which we will do over $\overline{F}$) it will be convenient to change coordinates in order to identify $T_{\overline{F}}$ with the standard maximal torus. From this perspective, it is clear that the morphism (47) has image $\begin{pmatrix} \tilde{\zeta}_n & 0 \\ 0 & \tilde{\zeta}_n^{-1} \end{pmatrix}$, and that the $\Gamma$-action is the usual action composed with inversion (based on whether or not $\sigma \in \Gamma$ has trivial or nontrivial image in $\Gamma_{E/F}$). From this description, it is clear that $Z_{\mathrm{SL}_2}(f_{\tilde{\zeta}_n}) = \mathrm{SL}_2$ if $\tilde{\zeta}_n = \pm 1$ and $Z_{\mathrm{SL}_2}(f_{\tilde{\zeta}_n}) = T$ if $\tilde{\zeta}_n$ is anything else. In particular, we see that in this example every character of $Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T})}$ is Levi suitable.

We pick the sections $s_T$ and $s_G$ of the coverings $\widehat{\overline{T}} \to \widehat{T}$ and $\widehat{\overline{G}} \to \widehat{G}$ as in (33) (we do not write out these sections explicitly—any two choices will produce isomorphic representations, cf. Proposition 4.38). According to §4.2.2, the next step is to pair each $\chi_{\tilde{\zeta}_n}$ with a simple module over the twisted algebra $\mathbb{C}[S_{\phi,\chi_{\tilde{\zeta}_n}}, \natural_{\chi_{\tilde{\zeta}_n}}]$. When $\tilde{\zeta}_n = \pm 1$, the character $\chi_{\tilde{\zeta}_n}$ factors through $Z(\widehat{G})^{\Gamma,\circ,+} = \mu_2 \subseteq \mathrm{SL}_2(\mathbb{C})$ and we see that in these cases $S_{\phi,\chi_{\tilde{\zeta}_n}} = S_\phi$. The section $s_G$ restricts to give a section of the surjection $S_\phi^+ \to S_\phi$ (which has kernel $\mu_2$) and so it's clear (by Clifford theory for finite groups) that there are six possible pairs $(\chi_{\tilde{\zeta}_n}, \rho)$ for $\tilde{\zeta}_n = \pm 1$ (giving five $S_\phi$-conjugacy classes), which correspond to the five irreducible representations of $S_\phi^+$ (the quaternion group). In this case, Theorem 4.41 simply produces the basic rigid $L$-packet for the parameter $\phi$.

Similarly, pairs $(\chi_{\tilde{\zeta}_n}, \rho)$ with $\tilde{\zeta}_n \neq \pm 1$ correspond to irreducible representations of $\mathscr{T}^{\Gamma,+}_{(\mathscr{T})} = \varprojlim_n \mu_{2n}$ which factor through a finite level, which are evidently just the same as a choice of a root of unity $\zeta_m \in \mathbb{C}^\times$. However, we must exclude those such that $\zeta_m^2 = \pm 1$, since then one has $\chi_{\tilde{\zeta}_n} = \pm 1$, which we assume is not the case. We thus obtain all possible characters of $\mathscr{T}^{\Gamma,+}_{(\mathscr{T})} = \varprojlim_n \mu_{2n}$ other than those whose image is 4-torsion.

We omit the construction of the double cover $T(F)_\pm \to T(F)$; there is an $L$-parameter $W_F \xrightarrow{{}^L\phi_{T,\pm}} {}^LT_\pm$ such that when composed with any embedding ${}^LT_\pm \to \mathrm{PGL}_2(\mathbb{C}) \rtimes W_F$ in the canonical conjugacy class one recovers $[\phi]$. Moreover, the action of $W(G,T)(F) = \mathbb{Z}/2\mathbb{Z}$ on $T$ sends $\begin{pmatrix} a & \alpha b \\ b & a \end{pmatrix}$ to $\begin{pmatrix} a & -\alpha b \\ -b & a \end{pmatrix}$ which dualizes to the action of $\mathbb{Z}/2\mathbb{Z}$ on ${}^LT = \mathbb{C}^\times \rtimes W_F$ given by the automorphism $\theta_w^\vee$ which is the identity on $W_F$ and inversion on $\mathbb{C}^\times$. We thus obtain from $\mathscr{T}$ and the torus $T$ a canonical $\mathbb{Z}/2\mathbb{Z}$-orbit of $L$-parameters $\{\phi_{T,\pm}, \theta_w^\vee \circ \phi_{T,\pm}\} = \{\phi_{T,\pm}\}$ (because $N_{S_\phi}(\mathscr{T}) \to W(G,T)(F)$ is surjective in this example, cf. Remark 4.37). Then [Kal21a, Theorem 3.16] associates to $\phi_{T,\pm}$ the genuine character $\chi_T$ of $T(F)_\pm$.

Tying the the two preceding paragraphs together, we obtain for each non-basic enhancement $(\chi_{\tilde{\zeta}_n}, \rho)$ with $\tilde{\zeta}_n \neq \pm 1$ a character $\zeta_m$ of $\mathscr{T}^{\Gamma,+}_{(\mathscr{T})} = \varprojlim_n \mu_{2n}$ (which we are identifying with a root of unity $\zeta_m$ for $m \neq 1, 2, 4$), which yields the $W(G,T)(F)$-orbit of representations of rigid inner twists of $T(F)_\pm$ given by

$$\{(\chi_T, z_{\zeta_m}), (\chi_T, z_{\zeta_m^{-1}})\},$$

where $z_{\zeta_m}$ denotes a choice of cocycle in the class in $H^1(\mathscr{E}, T)$ corresponding to the character $\zeta_m$ via Tate-Nakayama duality (choosing a different cocycle representative gives the same isomorphism class); denote by $[z_{\zeta_m}]_G$ its image in $H^1(\mathscr{E}, G)$.

In particular, we have that the non-basic contribution to the left-hand side of Theorem 4.41) is:

$$[\![\Pi^+_{\phi_{T,\pm}}]\!]_{W(G,T)(F)} = \bigsqcup_{[\zeta_m]_G,\ m\neq 1,2,4} [\![\Pi^+_{\phi_{T,\pm}}(T,[z_{\zeta_m}]_G)]\!]_{W(G,T)(F)} = \bigsqcup_{\zeta_m,\ m\neq 1,2,4} \{(\chi_T, z_{\zeta_m}), (\chi_T, z_{\zeta_m^{-1}})\},$$

where $\zeta_m$ ranges over all possible primitive $m$th roots of unity (evidently $G_{[z_{\zeta_m}]_G} = T$).

4.3.5. *Non-Levi subgroups.* In this short subsection we roughly explain how the above framework applies to the case where $H = Z_G(f_x)$ is connected but not a twisted Levi subgroup of $G$. Let us simplify things even further and assume that $H$ is the centralizer of a semisimple element of $G(\overline{F})$. Expanding on the relevant discussion from the beginning of this section, there is in general no embedding $\widehat{H} \to \widehat{G}$ (precluding the existence of any kind of $L$-embedding), so the chief difficulty is transferring an $L$-parameter $\phi$ for $G$ to an $L$-parameter for $H$.

Since we assume that $\phi$ factors through a parameter $\phi_{M,\pm}$ of the double cover $M(F)_\pm$ (the double cover taken with respect to $G$) for a twisted Levi subgroup $M$ of $G$, one may hope that if $H$ contains $M$ as a twisted Levi subgroup then $\phi_{M,\pm}$ can be transferred via some canonical $L$-embedding ${}^LM_\pm \to {}^LH$. In fact, since the definition of the admissible $\Sigma$-set used to define the double cover $M(F)_\pm$ for a twisted Levi subgroup $M$ makes sense for $H$ as well (since it's $F$-rational and full-rank), one may hope for the existence of a double cover $H(F)_\pm \to H(F)$ (with respect to $G$) and a canonical conjugacy class of embeddings ${}^LM_\pm \to {}^LH_\pm$. This turns out not to be the case—for one, the relevant $\Sigma$-set for $H$ is in general not admissible (as defined at the beginning of §4.1.1), and even when it is, the authors do not expect such a canonical class of embeddings to exist.

Nevertheless, a choice of $\chi$-data $\underline{\chi}_G$ (with respect to $G$) for $M_{\mathrm{ab}}$ determines an isomorphism ${}^LM_\pm \xrightarrow{\sim} {}^LM$ and a choice of $\chi$-data $\underline{\chi}_H$ with respect to $H$ for $M_{\mathrm{ab}}$ determines an isomorphism ${}^LM_{H,\pm} \xrightarrow{\sim} {}^LM$, where by ${}^LM_{H,\pm}$ we mean the $L$-group of the double cover $M(F)_{H,\pm}$ of $M$ determined by the inclusion $M \hookrightarrow H$ as a twisted Levi subgroup. These choices allow us to pass from $\phi$ to an $L$-parameter $\phi_{(\underline{\chi}_G,\underline{\chi}_H)}$ for $H$. This parameter depends of course on the choices of $\chi$-data, but also on the choice of $\phi$-minimal $M$ contained in $H$. It would be interesting to find an explicit example of two such $M$, $M'$ such that there are no choices of $\chi$-data for $M$ and $M'$ (relative to $G$ and $H$) resulting in the same parameter for ${}^LH$.

If one assumes that there is a unique $\phi$-minimal subgroup $\mathscr{M}$ of $\widehat{G}$ (this occurs for $\mathscr{M}$ a torus in many examples) with a corresponding choice of twisted Levi subgroup $M$, then one can attempt to define a larger version of the twisted extended quotient $(X^+_\phi(\widehat{G}) /\!\!/ \pi_0(S_\phi))_\natural$ as follows: Consider the partial order on the set of all connected rigid Newton centralizers of the form $Z_{G_{\overline{F}}}(s)$ defined by inclusion of twisted Levi subgroups, which has a finite set of maximal elements $\{K_{H_i,G} \cdot H_i\}_i$ from which we can choose representatives $\{H_i\}$. As in the twisted Levi subgroup case, we can assume that each $H_i$ is quasi-split (the proof of the relevant result [Kal21a, Lemma 6.4] holds because of the assumption that $H_i$ is connected and arises as the centralizer of an element of $G(\overline{F})$). Choose $\chi$-data $\underline{\chi}_G$ for $M$ relative to $G$ and also $\chi$-data $\underline{\chi}_{H_i}$ for $M$ relative to each $H_i$, determining an $L$-parameter $\phi_i := \phi_{(\underline{\chi}_G,\underline{\chi}_{H_i})}$ for each $H_i$ as explained above.

Another technicality now appears: Because there is no $L$-embedding ${}^LH_i \to {}^LG$ which sends $\phi_i$ to $\phi$, it is not clear that $\phi_i$ is a discrete $L$-parameter, which is a crucial assumption in §§4.2, 4.3.

**Question 4.48.** In the above notation, is $\phi_i$ automatically a discrete parameter for $H_i$?

In any case, we will assume that each $\phi_i$ is discrete for $H_i$ so that we can apply the above framework. The set $X$ is defined identically as in §4.2.3, except now allowing $\chi \in X^*_{L,H_i}(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})})$ rather than just $X^*_{\mathrm{Lev}}(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})})$, where $X^*_{L,H_i}(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{M})})$ denotes those characters whose image under (37) have centralizers which are twisted Levi subgroups of some $K_{H_i,G}$-conjugate of $H_i$ (one needs to choose an embedding of the unique $\phi$-minimal Levi subgroup $\mathscr{M}$ into $\widehat{H_i}$ which is the image of $\widehat{M}$ under one of the embeddings in the canonical $\widehat{H_i}$-conjugacy class ${}^LM_{H,\pm} \to {}^LH_i$, but the resulting constructions will not depend on this choice).

Rather than acting on the set $X$ by $\pi_0(S_\phi)$, one needs to act by the groupoid $\mathfrak{S}_\phi := \sqcup_i \pi_0(S_{\phi_i})$ via conjugation. One defines the family of cocycles $\{\natural_x\}_{x\in X}$ as in §4.2.4 to obtain the twisted extended quotient $(X /\!\!/ \mathfrak{S}_\phi)_\natural$ (where now we are quotienting by the action of a groupoid). Theorem 4.46 then holds verbatim in this context and yields packets of representations of inner forms of twisted Levi subgroups of each $H_i$. However, many aspects of this construction are unclear to the authors, for example if there is an explicit way to describe how the resulting local Langlands correspondence changes as one varies the $\chi$-data $\underline{\chi}_G$ and $\{\underline{\chi}_{H_i}\}_i$.

4.4. **Toral supercuspidal $L$-parameters.** In this subsection we explain how to construct in our formalism compound $L$-packets for many of Kaletha's regular supercuspidal $L$-parameters [Kal19], the "toral parameters". We retain the assumptions of [Kal19, 2.1] (see also [Kal24]) on $p$ and $G$: in particular, $G$ is tame and $p$ is odd, good for $G$, and prime to $\#\pi_1(G_{\mathrm{der}})$ and $\#\pi_0(Z(G))$. In addition, in our construction we will eventually assume that $p$ does not divide the order of the Weyl group of $G$.

Our construction is merely a repackaging of the construction of regular supercuspidal $L$-packets in [Kal19]. Since this earlier construction uses, at least implicitly, the $L$-groups of double covers that feature in our work, it should not be surprising that our enhancements are well adapted to this case. The main obstacle for us is the possibility that a regular supercuspidal parameter normalize several different maximal tori, which would then require an analysis of the equivalence relation defining $X^+_\varphi(\widehat{G})$. We do not overcome this obstacle in this work and we will soon restrict our attention to a class of regular $L$-parameters that normalize a unique maximal torus. As proof that there is an obstacle, however, we offer the following example.

**Example 4.49** (A strongly regular depth-zero $L$-parameter that normalizes two tori)**.** Let $G = G_2$ and let $\widehat{T} \subseteq \widehat{G} = G_2(\mathbb{C})$ be a maximal torus. Suppose that $q \equiv -1 \bmod 6$. The Weyl group of $G_2$ is the dihedral group of order 12, generated by a reflection and a rotation of order 6. It turns out [Adr22, MO4] that the Weyl group also lifts to $G_2$, meaning that there is a homomorphic section $W(\widehat{G},\widehat{T}) \to N_{\widehat{G}}(\widehat{T})$ of the natural projection. Let $W \subseteq G_2(\mathbb{C})$ denote the image of some such section. Define the $L$-parameter $\varphi\colon W_F \to G_2$ to be trivial on wild inertia, send a generator $s$ of tame inertia to an order-6 rotation in $W$, and send a Frobenius element $f$ to a reflection in $W$. Identify $s$ and $f$ with their images in $G_2$. We claim that

(1) $\varphi$ is strongly regular, meaning in this case that $\varphi$ is discrete and $s$ is strongly regular, and
(2) $\varphi$ normalizes two tori, $\widehat{T}$ and $Z_{\widehat{G}}(s)$.

Since $G_2$ is simply connected strongly regular is the same as regular [Ste75, 2.15] and it suffices to show that $s$ is regular. For this, we use [Ree10, Lemma 5.2], which applies because $s$ is elliptic for $\widehat{T}$ and has two orbits on the root system, the long roots and the short roots. Hence $s$ is regular. Since $fsf = s^{-1}$ and $Z_{\widehat{G}}(s) = Z_{\widehat{G}}(s^{-1})$, the torus $\widehat{S} = Z_{\widehat{G}}(s)$ is also normalized by $\varphi$, proving the second claim. For the first claim, it remains to show that $\varphi$ is discrete. Here it is easier to work with $\widehat{S}$. It suffices to show that the image of $f$ in $W(\widehat{G}, \widehat{S})$ is the elliptic element $-1$. Indeed, the element $-1$ takes $s$ to $s^{-1}$ and since $s$ is strongly regular there is a unique such element of the Weyl group.

For the convenience of the reader, and since the theory evolved in [FKS23] from the original treatment of [Kal19], we briefly summarize the construction of regular supercuspidal $L$-packets.

On the automorphic side, let $(T, \theta)$ be a pair consisting of a maximal torus of $G$ and a character $\theta$. [Kal19] isolates (in Definition 3.7.5) a class of *regular* torus-character pairs and explains (in Proposition 3.7.8) how to construct from them an input into Yu's construction of supercuspidal representations [Yu01]. Let $\pi^{\mathrm{old}}_{(T,\theta)}$ denote the resulting supercuspidal representation. As our subscript "old" is meant to indicate, this construction is not entirely optimal. [FKS23] constructs (in Theorem 4.1.13 and the preceding discussion) a certain quadratic character $\epsilon = \epsilon_T$ of the compact-open subgroup from which $\pi_{(T,\theta)}$ is induced. Let

$$\pi_{(T,\theta)} := \pi^{\mathrm{old}}_{(T,\epsilon\theta)}.$$

On the Galois side, let $\varphi\colon W_F \to {}^LG$ be a regular supercuspidal $L$-parameter [Kal19, Definition 5.2.3]. Define the maximal torus

$$\mathscr{T} := Z_{Z_{\widehat{G}}(\varphi(P_F))^\circ}(Z_{\widehat{G}}(\varphi(I_F))^\circ)$$

of $\widehat{G}$, a torus normalized by $\varphi$. The conjugation action of $\varphi$ on $Z_{\widehat{G}}(P_F)$ yields a Galois action on this torus, and from there, an $F$-torus $T$ with a stable conjugacy class of embeddings $T \hookrightarrow G$. Moreover, $\varphi$ factors through some $L$-embedding $\eta\colon {}^LT_\pm \to {}^LG$ in the canonical conjugacy class of such embeddings. Write $\varphi = \eta \circ \varphi^{T_\pm}$ for this factorization and let $\theta_{\varphi,\pm}\colon T(F)_\pm \to \mathbb{C}^\times$ be the resulting character dual to $\varphi^{T_\pm}$. As explained in [Kal21a, §3.2], a choice of $\chi$-data $\chi$ for $R(G_{\overline{F}}, T_{\overline{F}})$ gives rise to a genuine character $\zeta_\chi\colon T(F)_\pm \to \mathbb{C}^\times$. Write $\theta_{\varphi,\pm} \cdot \zeta_\chi$ for the resulting character of $T(F)$, from which the true product of these two characters is inflated. The restriction $\theta_{0+}$ of $\theta_{\varphi,\pm} \cdot \zeta_\chi$ to $T(F)_{0+}$ does not depend on the choice of $\chi$ provided that $\chi$ is ramified. Let $\chi''$ be the $\chi$-data constructed from $T$ and $\theta_{0+}$ in [FKS23, Notation 4.3.4] and the discussion preceding it. Given an admissible embedding $j\colon T \to G$, define

$$\pi_{\varphi,j} := \pi_{(T,\theta)}, \qquad \theta := \theta_{\varphi,\pm} \cdot \zeta_{\chi''}.$$

The basic rigid compound $L$-packet for $\varphi$ is then the set of $\pi_{\varphi,j}$ as $j$ ranges over representatives of rational conjugacy classes of admissible embeddings of $T$ into rigid inner forms of $G$.

For the precise parameterization of this $L$-packet, it follows from the definition of $\mathscr{T}$, and the assumption that this formula defines a torus, that $S_\varphi \subseteq \mathscr{T}$, and from there it is easy to see that $S_\varphi$ is identified with $\widehat{T}^\Gamma$. Completing the parameterization amounts to giving a bijection between $X^*(\widehat{T}^{\Gamma,+})$ and conjugacy classes of admissible embeddings $j$. Since the latter set is a torsor for a certain action of the former group, it is enough to give a distinguished admissible embedding. By [FKS23, Theorem 4.4.2], a choice of Whittaker datum $\mathfrak{w}$ does precisely this: there is a unique $j$ for which $\pi_{\varphi,j}$ is $\mathfrak{w}$-generic. All in all, given $\chi \in X^*(\widehat{T}^{\Gamma,+})$ write

$$\pi_{\varphi,\chi}$$

for the resulting regular supercuspidal representation, the dependence on $\mathfrak{w}$ left implicit. We refer the reader to [Kal19, §5.3] and [FKS23, §4.4] for more details of the parameterization.

Moreover, [Kal21a, Remark 6.15] explains how to canonically extend the construction of regular supercuspidal $L$-packets to Kaletha's double covers.

Next we will focus our attention on the following large class of regular supercuspidal $L$-parameters. These parameters are adapted to our methods because, as Lemma 4.51 shows, they normalize a unique maximal torus.

**Definition 4.50.** An $L$-parameter $\varphi\colon W_F \to {}^L G$ is *toral supercuspidal* if

(1) (the projection from ${}^L G$ to $\widehat{G}$ of) $\varphi(P_F)$ is contained in a torus and
(2) $Z_{\widehat{G}}(P_F)$ is a maximal torus of $\widehat{G}$.

This terminology is Chan–Oi's (cf. [CO21, Definition 3.7]) (we warn the reader that this disagrees with Kaletha's [Kal19, 6.1.1]). A toral supercuspidal $L$-parameter is automatically strongly regular.

**Lemma 4.51.** *Let $\varphi$ be a toral supercuspidal $L$-parameter and let $\mathscr{M}$ be a Levi subgroup of $\widehat{G}$ normalized by $\varphi$. Suppose $p \nmid \#W_G$.*

(1) *The $L$-parameter $\varphi$ normalizes a unique maximal torus of $\mathscr{M}$, the torus $Z_{\widehat{G}}(P_F)$.*
(2) *The torus $Z_{\widehat{G}}(P_F)$ is the unique $\varphi$-minimal Levi subgroup of $\widehat{G}$.*

Here $W_G = W(G_{\overline{F}}, T_{\overline{F}})$ is the Weyl group of $G$.

*Proof.* Here we work with the "minimal form" of the $L$-group of $G$, the semidirect product $\widehat{G} \rtimes \mathrm{Gal}(E/F)$ where $E/F$ is the (tame, Galois) splitting field of the quasi-split inner form of $G$. With this convention, we may identify $\varphi(P_F)$ with a subgroup of $\widehat{G}$.

For the first part, let $\mathscr{T}$ be a maximal torus of $\widehat{G}$ normalized by $\varphi$. Then $\varphi$ induces a map $P_F \to W(\widehat{G}, \mathscr{T})$, which must be trivial because $p$ does not divide the order of the target. So $\varphi(P_F) \subseteq \mathscr{T}$, forcing $\mathscr{T} = Z_{\widehat{G}}(\varphi(P_F))$. It remains to show that $Z_{\widehat{G}}(\varphi(P_F)) \subseteq \mathscr{M}$. Since $W(\widehat{G}, \mathscr{M})$ can be identified with a subgroup of $W(\widehat{G}, \mathscr{T})$ for any maximal torus $\mathscr{T}$ of $\mathscr{M}$, so that $p \nmid \#W(\widehat{G}, \mathscr{M})$, we again have $\varphi(P_F) \subseteq \mathscr{M}$. Moreover, since $\varphi(P_F)$ is supersolvable, [SS70, Chapter II, Theorem 5.16] shows that $\varphi(P_F)$ normalizes a maximal torus $\widehat{T}$ of $\mathscr{M}$. Since $p \nmid \#W_{\mathscr{M}}$ this torus must contain $\varphi(P_F)$, so that $\mathscr{T} = Z_{\widehat{G}}(\varphi(P_F))$.

For the second part, if $\mathscr{M}$ is $\varphi$-minimal then $\varphi$ normalizes $\mathscr{M}$, which contains $Z_{\widehat{G}}(P_F)$ by the first part. [Kal21a, Proposition 4.1] then shows that $\varphi$ factors through an $L$-embedding ${}^L T_\pm \to {}^L G$ for $T$ a maximal torus of $G$ dual to $Z_{\widehat{G}}(P_F)$, whose $F$-rational structure is obtained from $\varphi$. □

For the remainder of this section, assume $p \nmid \#W_G$.

Let $\varphi$ be a toral supercuspidal $L$-parameter of $G$ with associated torus $T$. By Lemma 4.51,

$$X^+_\varphi(\widehat{G}) = X^*_{\mathrm{Lev}}(Z(\widehat{G})^{\Gamma,\circ,+}_{(\mathscr{T})}).$$

Moreover, if ${}^L N_\pm \to {}^L G$ is an $L$-embedding through which $\varphi$ factors then the resulting parameter of $N(F)_\pm$ is toral, hence regular, and therefore $S_{\varphi_{N,\pm}}$ is contained in the preimage of $\mathscr{T}$ under the embedding of $\widehat{N}$.

The construction of rigid compound $L$-packets is mostly completed by Example 4.33. Given $\chi \in X^+_\varphi(\widehat{G})$, let $N_\chi$ denote the corresponding quasi-split twisted Levi subgroup of $G$. To $\chi$ we attach the $W(G, N_\chi)(F)$-orbit

$$[\![\pi_{\varphi_{N_\chi,\pm},\chi}]\!]_{W(G,N_\chi)(F)}$$

of toral supercuspidal representations of rigid inner twists of double covers.

## Appendix A. Examples

### A.1. Examples in $\mathrm{PGL}_2$.

**Lemma A.1.** *Let $G = \mathrm{PGL}_2$ over a field $F$ not of characteristic two.*

1. *There is a subgroup $A$ of $G$ that is isomorphic to $\mu_2^2$ and is not contained in any torus of $G$.*
2. *Any two such subgroups $A$ are $G(\overline{F})$-conjugate.*
3. *The subgroup $A$ normalizes exactly three maximal tori of $G$, those of the form $Z_G^\circ(s)$ for $s \in A$ of order two.*
4. *$Z_G(A) = A$*
5. *If $\sqrt{-1} \in F$ then $N_G(A)/A \simeq S_3 \simeq \mathrm{GL}_2(\mathbb{F}_2)$ and $N_G(A)$ is generated by the groups $T[4]$ where $T$ is a torus normalized by $A$.*

If $\sqrt{-1} \notin F$ then $N_G(A)/A$ will instead be some form of $S_3$. One can further show that $N_G(A)$ is a form of $S_4$, generalizing the classical geometry of the tetrahedron inscribed in the (Riemann) sphere.

*Proof.* For the first part, to start with, note that every maximal torus $T$ of $\mathrm{PGL}_2$ is of the form $\mathrm{Res}_{E/F}(\mathbb{G}_m)/\mathbb{G}_m$, where $E/F$ is a separable quadratic extension, and the normalizer of this torus is $T \rtimes \mathrm{Gal}(E/F)$. The subgroup of $N_G(T)$ generated by the two-torsion element of $T$ and $\mathrm{Gal}(E/F)$ is the desired copy of $\mu_2^2$ in $\mathrm{PGL}_2$. Since every torus of $\mathrm{PGL}_2$ has rank one, and thus contains a unique point of order two, which is an $F$-point, this subgroup is not contained in a torus.

For the second part, a simple matrix calculation shows that the centralizer in $\mathrm{PGL}_2$ of the order-two element $\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$ is the normalizer of the diagonal maximal torus. It follows that the assignment $s \mapsto Z_G^\circ(s)$ is a bijection between order-two elements of $\mathrm{PGL}_2$ and maximal tori of $\mathrm{PGL}_2$; the inverse sends a torus to its unique element of order two. Consequently, there is a unique way to enlarge any order-two element of $\mathrm{PGL}_2(F)$ to a copy of $\mu_2^2$. The fact that all tori of $\mathrm{PGL}_2$ are $G(\overline{F})$-conjugate implies that all copies of $\mu_2^2$ are $G(\overline{F})$-conjugate.

For the third part, it is clear that $A$ normalizes the given tori, so the problem is to show that no more are normalized. For this, we may assume that $A$ is generated by $s = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$ and $t = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$. The space of maximal tori is isomorphic to the variety $X$ of two-element subsets of $\mathbb{P}^1$ and the conjugation action on tori becomes the action of $\mathrm{PGL}_2$ on $\mathbb{P}^1$ by Möbius transformations. In this model, $s$ corresponds to the transformation $1/z$ and $t$ to the transformation $-z$. A short calculation shows that the only points of $X$ fixed by this group of Möbius transformations are $\{0, \infty\}$, $\{\pm 1\}$ and $\{\pm\sqrt{-1}\}$.

For the fourth part, working again in the setting of the previous paragraph, $Z_G(A)$ is the centralizer of $s$ in $Z_G(t)$, which one easily computes to be $A$. Take $a \in Z_G^\circ(t)$ of order 4, so that $a^2 = t$. Then $a$ commutes with $t$ and $asa^{-1} = a^2 s = ts$, so that $a$ realizes the permutation $(s, ts)(t)$ of the 3-element set $\{s, t, ts\}$. Replacing $s$ by $t$ and $ts$ in turn, one finds the other two 2-cycles in $S_3$. □

Now we return to the setting of the paper, where $F$ is a nonarchimedean local field of characteristic zero.

**Lemma A.2.** *Let $G = \mathrm{PGL}_2$ over $F$. Assume $\sqrt{-1} \in F$.*

1. *Any form of $\mu_2^2$ that splits over an unramified extension of $F$ embeds as a subgroup of $G$.*
2. *There is a cocycle $z \in Z^1(\mathcal{E}, G)$ and a form $A$ of $\mu_2^2$ in $G$ such that the image of $f_z$ is $A$.*

*Proof.* For the first part, it suffices to show that for any element $n \in N_G(A)$ there is $g \in \mathrm{PGL}_2(F^{\mathrm{unr}})$ such that $g^{-1} \cdot \mathrm{Frob}(g) = n$; then the conjugate $g^{-1}Ag$ of $A$ is defined over $F$ and has Galois action sending Frob to the image of $n$ in $\mathrm{Aut}(A) = S_3$. For this, let $\mathcal{G}$ be the standard maximal parahoric integral model of $\mathrm{PGL}_2$, so that $\mathcal{G}(\mathcal{O}_F)$ is the elements of $\mathrm{PGL}_2(\mathcal{O})$ whose determinant has even valuation. Then $\left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]$ extends to a copy $A$ of $\mu_2^2$ contained in $\mathcal{G}$, and since $\sqrt{-1} \in F$, the three tori normalized by $A$ are split. Hence $N_G(A)$ embeds in $\mathcal{G}$. The existence of $g$ now follows from the facts that $H^1(F^{\mathrm{unr}}/F, \mathcal{G}) = 1$ and that $n$ is a finite-order element of $\mathcal{G}(\mathcal{O}_F)$.

For the second part, let $E/F$ be an unramified cubic extension of $F$ and take $A = \mathrm{Res}_{E/F}(\mu_2)/\mu_2$. Let $\widehat{A} = \mathrm{Hom}(X^*(A), \mathbb{Q}/\mathbb{Z})$, isomorphic to $\widehat{A} \simeq \mathbb{F}_2^2$ with Frobenius action $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. As [Dil24, Section 5] explains, we may identify $\mathrm{Hom}_F(u, Z)$ with $\widehat{A}$ and the image of $H^1(\mathcal{E}, A)$ in $\mathrm{Hom}_F(u, Z)$ with the kernel of the map from $\widehat{A}$ to its Galois coinvariants. Since $\widehat{A}$ has trivial Galois coinvariants, every element of $\mathrm{Hom}_F(u, Z)$, in particular, the natural projection $u \to A$, extends to a cocycle $z$. By construction, the image of $f_z$ is $A$. □

## A.2. **Examples in $G_2$.**

**Lemma A.3.** *Let $G = G_2$ over an algebraically closed field $F$ not of characteristic two.*

(1) *There is a unique $G(F)$-conjugacy class of subgroups $A$ isomorphic to $\mu_2^3$.*
(2) $Z_G(A) = A$ *and* $N_G(A)/A \simeq \mathrm{Aut}(A) \simeq \mathrm{GL}_3(\mathbb{F}_2)$.
(3) *There is a unique $N_G(A)$-conjugacy class of subgroups $B$ of $N_G(A)$ that contain $A$ and have order* 168.
(4) *The group $B$ does not normalize a maximal torus of $G$.*

*Proof.* For the first part, any subgroup isomorphic to $\mu_2^2$ is contained in a maximal torus $T$ [Ste75, Theorem 2.27]. Since $N_G(T) \simeq T \rtimes W(G,T)$ [Adr22, Proposition 3.17], a lift $s$ of $-1 \in W(G,T)$ of order two together with the group $T[2] \simeq \mu_2^2$ generates a subgroup isomorphic to $\mu_2^3$.

For the second part, the description of $A$ given above shows that $Z_G(A) = A$. It follows that $N_G(A)/A \subseteq \mathrm{Aut}(A)$. To show that this inclusion is an equality, one can use the interpretation of $G_2$ as the automorphism group of the octonions and describe $N_G(A)$ as the automorphisms of the multiplication table of the octonions [Cox46].

For the third part, since $\mathrm{GL}_3(\mathbb{F}_2)$ has order $168 = 8 \cdot 3 \cdot 7$, it contains a unique conjugacy class of subgroups of order 7, the Sylow 7-subgroups, which can be identified with the maximal tori $\mathbb{F}_8^\times$. Using the general description of normalizers of elliptic tori in $\mathrm{GL}_n$ one sees that the normalizer of an $\mathbb{F}_8^\times$ is a semidirect product $\mathbb{F}_8^\times \rtimes \mathrm{Gal}(\mathbb{F}_8/\mathbb{F}_2)$, of order $3 \cdot 7$.

For the fourth part, suppose $B$ normalizes a maximal torus $T$ of $G$ and let $C$ be a Sylow 7-subgroup of $B$. Since $W(G,T)$ is the dihedral group of order 12, the image of $C$ in $W(G,T)$ is trivial, so that $C \subseteq T$. The same argument shows that there is a subgroup $A' \subseteq A$, of order at least two, such that $A' \subseteq T$, as $8 \nmid 12$. Hence some nontrivial subgroup of $A$ commutes with $C$. At the same time, recalling from the third part that the conjugation action of $C$ on $A$ can be identified with the multiplication action of $\mathbb{F}_8^\times$ on $\mathbb{F}_8$, we see that $C$ acts simply transitively on $A \setminus \{1\}$, a contradiction. □

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