APPENDIX 2: ERGODICITY

In this appendix we present a brief review of the elementary definitions and results of the theory of ergodicity. For (much) more information, with an emphasis on the connection between ergodic theory and classical systems, the interested reader may consult, for example, the book by V.Arnold and A. Avez ([Ar8]).

Consider a flow T_t which preserves the probability measure μ on the space X ($\mu(X) = 1$). Because of its fundamental implications in statistical physics, the following question originally stimulated the development of ergodic theory:
Given a function φ on the space X, under what conditions will the time average

$$<\varphi>_T(x^*) = 1/T \int_0^T \varphi(T_t(x^*))\, dt$$

(of the function φ over the trajectory $T_t(x^*)$ with initial condition x^*) converge to the space average $\varphi^* = \int_X f\, d\mu$ of φ as T approaches infinity? Here, one can think of T_t as the time t map of a system of ODE's; the same kind of questions arise in the discrete framework for the iteration of a given transformation. Since the present book is concerned with differential equations however, we shall mainly state the definitions and results for flows.

By taking φ to be the characteristic function χ_A of a set A in X which is invariant under the flow T_t, one immediately shows that a necessary condition for answering this question in the affirmative is that all subsets invariant under the flow have measure 0 or 1. Flows with this property are said to be *ergodic*. Birkhoff's ergodic theorem says that this property is also sufficient:

<u>Birkhoff's Ergodic Theorem</u>:

For any function φ in $L^1(X, \mu)$, and for almost all x, the temporal average

$$<\varphi>_T = 1/T \int_0^T \varphi \circ T_s(x)\, ds$$

converges, when $T \to \infty$, to a function $M(\varphi)$ in $L^1(X, \mu)$ which is invariant under the flow and which has the same space average as φ.

In particular, if the flow is ergodic, the temporal average $<\varphi>_T(x)$ of φ over the trajectory with initial condition x converges in $L^1(X, \mu)$, for almost all x, to the spatial average φ^* of φ.

It is then interesting to formulate an analogous question in the space $L^2(X, d\mu)$ of square integrable functions, with its Hilbert space structure, and to change perspective slightly, by considering, not trajectories of the flow on X, but rather the evolution of functions in $L^2(X, d\mu)$, a change which is equivalent to passing from Schrödinger's to Heisenberg's representation in quantum mechanics. In this way one associates to the flow T_t a one parameter group of evolution operators $\{ U_t , t \in \mathbb{R} \}$ on $L^2(X, \mu)$ defined by:

(1) $\quad U_t [f](x) = f[T_t(x)].$

One easily verifies that measure preservation under the flow is equivalent to unitarity of the operators U_t . This point of view is largely due to Koopman, and it has the advantage of reducing many aspects of the problem to the study of *linear* operators on a Hilbert space (see also [Re], Chapter VII).

In this framework, the first question which arises is the following: Under what conditions does the time average

$$<\varphi>_T (x) = 1/T \int_0^T U_s[f] \, ds(x)$$

converge in L^2 norm to the constant function $\varphi^* = \int_X f \, d\mu$? This question is answered by Von Neumann's ergodic theorem, which may be stated as follows:

Von Neumann's Ergodic Theorem:

Let V be the vector subspace of $L^2(X, \mu)$ consisting of functions invariant under the flow T_t ; that is, V is the eigenspace associated to the eigenvalue 1 of the operator U_1 . Let P be the orthogonal projection onto V. Then as $T \to \infty$, the time average of an arbitrary function φ in $L^2(X, \mu)$ converges in the L^2 sense to the function $P\varphi$:

(2) $\quad \lim_{T\to\infty} \| 1/T \int_0^T U_s[\varphi] \, ds - P\varphi \| = 0.$

In fact, the time average $\lim_{T\to\infty} <\varphi>_T$ and the space average φ^* coincide (in L^2 norm) if and only if $P\varphi = \int_X \varphi \, d\mu$. Since the space average is simply the scalar product of φ and the constant function 1, the necessary and sufficient condition for equality of the averages is that the multiplicity of the eigenvalue 1 of the operator U_1 be precisely 1; in other words, the only L^2 functions invariant under the flow are the constant functions. This property may thus be viewed as an alternate definition of ergodicity.

We now proceed to prove Von Neumann's theorem by first restricting our attention to the discrete time case. In passing to the continuous case, it suffices to consider the operator U in the proof below as successive applications of the flow T_t with $t = 1$. The statement is as follows:

Von Neumann's Ergodic Theorem (discrete form):

Let $\mathcal{H}$ be a Hilbert space, U a unitary operator (the evolution operator), V the eigenspace corresponding to the eigenvalue 1 of the operator U, and P the orthogonal projection onto this eigenspace. For each vector φ in $\mathcal{H}$, the time average

$$<\varphi>_N = 1/N \sum_{0\leqslant n\leqslant N} U^n(\varphi)$$

converges in norm to the projection $P\varphi$.

Since U is unitary, $\mathcal{H}$ may be decomposed as :

(3) $\mathcal{H} = \mathrm{Ker}(I-U^*) \oplus \mathrm{clos}(\mathrm{Im}(I-U))$,

where clos(A) denotes the closure of the set A. If φ belongs to $\mathrm{Ker}(I - U^*)$, then $U\varphi = \varphi$, $P\varphi = \varphi$, and the theorem holds.

Suppose now that φ belongs to the image $\mathrm{Im}(I - U)$ of $I - U$, and set $\varphi = \psi - U\psi$. Since

(4) $1/N \sum_{0\leqslant n<N} U^n(\varphi) = (\psi - U^N \psi)/N$,

we have the bound:

(5) $| <\varphi>_N | \leqslant 2 \| \psi \|/N$,

so that tends to zero as N approaches infinity. Since $P\varphi = 0$, the theorem holds for φ in $\mathrm{Im}(I - U)$.

This latter result extends to the case where $\varphi \in \mathrm{clos}(\mathrm{Im}(I - U))$, which proves the theorem. ∎

It is interesting how disarmingly simple the above proof is, compared to the somewhat tricky proof of Birkhoff's theorem, thus illustrating the power of Hilbert space techniques.

We now give a third definition of ergodicity, which may be compared with the definitions of strong and weak mixing given below. In fact, using Von Neumann's theorem, it is easy to show that T_t is ergodic if and only if for two arbitrary measurable sets A and B, as T approaches infinity the time average $1/T \int_0^T \mu[T_s A \cap B]\, ds$ converges to the product $\mu(A).\mu(B)$ of the measures of A and B; we thus have convergence of $\mu[T_t A \cap B]$ to $\mu(A).\mu(B)$ in the Césaro sense.

There is a hierarchy of statistical properties stronger than ergodicity: weak mixing (which implies ergodicity), mixing (which implies weak mixing), the properties of

Bernouilli, Lebesgue spectra, K and C systems, and so on. Here we limit ourselves to recalling the definitions of mixing properties which further illustrate the relation between dynamical and spectral properties.

We saw previously that preservation of measure under the flow is equivalent to the unitarity of the evolution operators U_t and that ergodicity is equivalent to 1 being an eigenvalue of multiplicity 1 of the evolution operators U_t . We now give similar definitions of the mixing properties; proof of their equivalence is a simple exercise.

Weak Mixing:

1) The flow T_t is weakly mixing if for two arbitrary measurable sets A and B:

$$(6) \qquad \lim_{T\to\infty} 1/T \int_0^T | \mu(T_t(A)\cap B) - \mu(A).\mu(B) | \, dt = 0.$$

2) The flow T_t is weakly mixing if and only if 1 is the only eigenvalue of U_1 and it occurs with multiplicity 1.

Mixing:

1) The flow T_t is mixing if and only if for any two measurable sets A and B:

$$(7) \qquad \lim_{t\to\infty} \mu(T_t(A) \cap B) = \mu(A).\mu(B).$$

2) The flow T_t is mixing if and only if it is weakly mixing and the spectrum of the restriction of U_1 to the orthogonal complement of the eigenspace associated to the eigenvalue 1 is purely absolutely continuous.

To end this very brief discussion of ergodicity, it may be useful to simply state the underlying classification problem. Let (X, μ, T) and (X', μ', T') be two dynamical systems; here T and T' are measure preserving applications from X (X') to itself. When are these systems equivalent, in the sense that there exists an almost everywhere bijective transformation $S: X \to X'$ which is measure preserving and such that $T = S^{-1}T'S$? This problem is not completely solved, but spectral theory furnishes invariants defined by means of the associated unitary operators on $L^2(X, \mu)$ and $L^2(X', \mu')$. The Kolmogorov-Sinaï entropy is also an important, *nonspectral* invariant, but any detailed discussion of it would of course take us too far afield.