88. On the Adiabatic Theorem for the Hamiltonian System of Differential Equations in the Classical Mechanics. I

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Introduction. In this paper, we shall use the following abbreviations:

$$p=(p_1,\cdots,p_n),\ q=(q_1,\cdots,q_n),\ (p,q)=(p_1,\cdots,p_n,q_1,\cdots,q_n)$$
$$dp/dt=(dp_1/dt,\cdots,dp_n/dt),\ dq/dt=(dq_1/dt,\cdots,dq_n/dt),$$
$$\partial H/\partial p=(\partial H/\partial p_1,\cdots,\partial H/\partial p_n),\ \partial H/\partial q=(\partial H/\partial q_1,\cdots,\partial H/\partial q_n)$$
$$dpdq=dp_1\cdots dp_n\cdot dq_1\cdots dq_n.$$

Let $H(p,q,s)$ be a Hamiltonian containing a parameter s $(1\geqq s\geqq 0)$. Roughly speaking, a quantity $I(p,q,s)$ which is a global integral of the system

$$(1)\qquad dp/dt=-\partial H/\partial q(p,q,s),\ dq/dt=\partial H/\partial p(p,q,s)$$

for every fixed s $(1\geqq s\geqq 0)$, is called an adiabatic invariant of (1), if $I(p,q,t/\lambda)$ is conserved along all (nearly all)[1] trajectories of the following system

$$(2)\qquad dp/dt=-\partial H/\partial q(p,q,t/\lambda),\ dq/dt=\partial H/\partial p(p,q,t/\lambda)$$

in the whole interval of time $0\leqq t\leqq \lambda$, asymptotically for $\lambda\to+\infty$. The fundamental case of the adiabatic theorem in the classical mechanics is the case where (1) has no one-valued Lebesgue measurable global integral other than the functions of the energy integral for almost all s $(1\geqq s\geqq 0)$.[2] In this case, the phase volume

$$\mathfrak{J}(p,q,s)=\int_{I_{E,s}} dpdq$$

where $I_{E,s}$ means the domain in (p,q)-space enclosed by the energy surface $S_{E,s}=\{(p,q)\mid H(p,q,s)=E\}$ of (1) passing through (p,q), is one (and essentially the only one)[3] adiabatic invariant. In the following, we shall call this proposition the adiabatic theorem.

The adiabatic theorem plays an important rôle not only in the statistical mechanics, but also in various other branches of physics. But as far as we know, satisfactory proofs of the theorem exist

1) Here the word "nearly all" is used in a vague sense.

2) The cases where this assumption does not hold, can be reduced to this fundamental case if the reductions as given in T. Levi-civita [7] are possible and the reduced system satisfies an assumption similar to this assumption. Cf. T. Levi-civita [7], also H. Geppert [1].

3) The "only one" part of the adiabatic theorem shall be treated in Part IV of this paper.

only for very special cases where essentially only one pair of canonical variables (p_1, q_1) occurs.[4)5)] In this paper, we shall give a precise formulation of the adiabatic theorem in the general case and prove it by the aid of the operator method in the classical mechanics of J. v. Neumann and B. O. Koopman.[6)]

For convenience sake, we change the scale of time in (2) putting $t'=t/\lambda$ and discuss the asymptotic behaviour for $\lambda\to+\infty$ of the trajectories of the transformed system

(3) $dp/dt'=-\lambda\partial H/\partial q(p, q, t')\ dq/dt'=\lambda\partial H/\partial q(p, q, t')$

in the interval $1\geqq t'\geqq 0$ instead of that of (2) in the interval $\lambda\geqq t\geqq 0$.

In Part I of this paper, we shall determine the region D in R^{2n+1} such that the adiabatic theorem shall be stated for the trajectories of (3) lying in it and shall study the properties of the phase volume function $\tilde{\mathfrak{J}}(p, q, s)$ in D. In Part II, we shall study the flows (Strömungen in the sense of J. v. Neumann) defined by (1) and the infinitesimal generator iA of the unitary group $\{U_t\mid -\infty<t<+\infty\}$ associated with one of the flows.[6)] In Part III, we shall state and prove a form of the adiabatic theorem (in Section 8), under Assumptions 1 and 2 stated in Section 1 and Assumption 3 stated in Section 5. In Part IV, we shall state and prove a more satisfactory form of the adiabatic theorem and some discussions of the results obtained shall follow it. Also in Part IV, we shall prove some lemmas and theorems stated but unproved in Parts I and II.

Notations. If B is a subset of a Euclidian space or a subset of a Hilbert space, we denote by $\bar{B}$, the closure of B in the space and by B^0 the set of all inner points of B in the space. In this paper a function is always complex-valued, if not specially mentioned. We denote by $C^m(B)$, the set of all m times continuously differentiable functions on B if B is a subset of a Euclidian space such that $\bar{B}^0\supset B$ and by $C_0^m(B)$ the set of all m times continuously differentiable functions on B vanishing outside compact sets contained in B, if further B is open in the space. Also we write $C(B), C_0(B)$ for $C^0(B), C_0^0(B)$.

1. We shall call a subset of a subset B of a Euclidian space *relatively open* in B if it can be considered as an intersection of an open set in the Euclidian space with B.

We denote by K the point set $\{(p, q, s)\mid -\infty<p_i<+\infty,\ -\infty<q_i<+\infty\ \ i=1,\cdots,n, a\leqq s\leqq b\}$ in R^{2n+1} where a and b are two fixed

4) For these special cases, cf. H. Kneser [4], A. Lenard [6], Y. Watanabe [10].

5) The arguments in T. Levi-civita [7] only make the holding of the adiabatic theorem for the general case probable. He and other authors proved various generalizations of Theorem 1 of this paper but they did not prove the adiabatic theorem itself for the general case. Cf. T. Levi-civita [7], H. Geppert [1], G. Mattioli [8].

6) Cf. J. v. Neumann [9], B. O. Koopman [5] and E. Hopf [2].

real numbers such that $a<b$.

ASSUMPTION 1. *The point set in R^{2n+1} where the real-valued function $H(p,q,s)$ is defined, contains a subset G of K relatively open in K satisfying the following two conditions*:

i) *H, $\partial H/\partial s$, $\partial H/\partial p_i$, $\partial H/\partial q_i$, $\partial^2 H/\partial p_i \partial q_j$, $\partial^2 H/\partial s \partial p_i$, $\partial^2 H/\partial s \partial q_i$ $(i, j=1, \cdots, n)$ exist and are continuous on G. Also*

$$\sum_{i=1}^{n}\{(\partial H/\partial p_i)^2+(\partial H/\partial q_i)^2\}\neq 0 \text{ on } G.$$

ii) *We denote by G_s the open set $\{(p,q)\mid(p,q,s)\in G\}$ in (p,q)-space and by $S_{E,s}$ the set $\{(p,q)\mid H(p,q,s)=E,\ (p,q)\in G_s\}$ in G_s. Then for each s $(a\leqq s\leqq b)$, there is an open interval $\Gamma_s=\{E\mid E_s<E<E_s'\}$ $(-\infty\leqq E_s<E_s'=+\infty)$ on E-line such that $S_{E,s}$ is non-void and is a closed $(2n-1)$-dimensional C^1-submanifold of R^{2n} enclosing a domain $I_{E,s}$ in R^{2n} for each $E\in\Gamma_s$ and $G_s=\bigcup_{E\in\Gamma_s} S_{E,s}$.*

Under Assumption 1, we can prove after some topological considerations that two alternative cases occur: A) $I_{E_2,s}\supset S_{E_1,s}$ for all E_2, E_1, s such that $a\leqq s\leqq b$, $E_s<E_1<E_2<E_s'$ or B) $I_{E_1,s}\supset S_{E_2,s}$ for all E_1, E_2, s such that $a\leqq s\leqq b$, $E_s<E_1<E_2<E_s'$. We shall assume in the following that the former case A) occurs. The later case B) can be treated just in the same way. Then we can also prove the following two consequences iii), iv) of Assumption 1.

iii) *$I_{E_2,s}\supset\bar{I}_{E_1,s}=I_{E_1,s}\cup S_{E_1,s}$ for all E_1, E_2, s such that $a\leqq s\leqq b$, $E_s<E_1<E_2<E_s'$. We put $I_{E_1,E_2,s}=I_{E_2,s}-\bar{I}_{E_1,s}$ for such E_1, E_2, s. Then $I_{E_1,E_2,s}=\bigcup_{E_1<E<E_2} S_{E,s}$ and*

$$\bar{I}_{E_1,E_2,s}=I_{E_1,E_2,s}\cup S_{E_1,s}\cup S_{E_2,s}$$

and $\bar{I}_{E_1,E_2,s}$ is compact. The point set $\Lambda=\{(E,s)\mid a\leqq s\leqq b,\ E_s<E<E_s'\}$ in (E,s)-plane is relatively open in the point set $\{(E,s)\mid a\leqq s\leqq b,\ -\infty<E<+\infty\}$ in (E,s)-plane.

iv) *If we put*

$$\mathfrak{J}(E,s)=\int_{I_{E,s}} dpdq,$$

then $\mathfrak{J}(E,s)\in C^1(\Lambda)$ and $\partial\mathfrak{J}/\partial E>0$ for $(E,s)\in\Lambda$.

We omit the proof of these results and assume the results since such considerations are not the main purposes of this paper.

By iv), if we put $J_s=\lim_{E\to E_s}\mathfrak{J}(E,s)$ and $J_s'=\lim_{E\to E_s'}\mathfrak{J}(E,s)$, then the point set $\Lambda'=\{(J,s)\mid a\leqq s\leqq b,\ J_s<J<J_s'\}$ is relatively open in the point set $\{(J,s)\mid a\leqq s\leqq b,\ -\infty<J<+\infty\}$ in the (J,s)-plane and if $(J,s)\in\Lambda'$, the equation $J=\mathfrak{J}(E,s)$ can be solved uniquely for E. If we denote the solution by $E=\mathfrak{E}(J,s)$, $\mathfrak{E}(J,s)\in C^1(\Lambda')$ and $\partial\mathfrak{E}/\partial J>0$ for $(J,s)\in\Lambda'$. Also if we put $\tilde{\mathfrak{J}}(p,q,s)=\mathfrak{J}(H(p,q,s),s)$, then we have from i) of Assumption 1

$$\mathfrak{J}(p, q, s) \in C^1(G),\ \sum_{i=1}^{n}\{(\partial\mathfrak{J}/\partial p_i)^2+(\partial\mathfrak{J}/\partial q_i)^2\}\neq 0$$

on G.

ASSUMPTION 2. *There are two numbers J_1^*, J_2^* independent of s such that $J_s'>J_2^*>J_1^*>J_s$ for all s in the interval $a\leqq s\leqq b$.*

In the following, we shall fix such a pair of J_1^* and J_2^* once for all. We put $S(J, s)=S_{\mathfrak{C}(J,s),s}=\{(p, q)\,|\,(p, q)\in G_s,\ \mathfrak{J}(p, q, s)=J\}$, $I(J, s)=I_{\mathfrak{C}(J,s),s}$, $I(J_1, J_2, s)=I_{\mathfrak{C}(J_1,s),\,\mathfrak{C}(J_2,s),s}=\{(p, q)\,|\,(p, q)\in G_s,\ J_2>\mathfrak{J}(p, q, s)>J_1\}$ for J, J_1, J_2, s, such that $a\leqq s\leqq b, J_2^*\geqq J\geqq J_1^*$ and $J_2^*\geqq J_2>J_1\geqq J_1^*$. We also put $I(s)=I(J_1^*, J_2^*, s)=\{(p, q)\,|\,(p, q)\in G_s\ J_1^*<\mathfrak{J}(p, q, s)<J_2^*\}$, $D=\{(p, q, s)\,|\,0\leqq s\leqq b, (p, q)\in I(s)\}=\{(p, q, s)\,|\,(p, q, s)\in G, J_2^*>\mathfrak{J}(p, q, s)>J_1^*\}$, $D(J)=\{(p, q, s)\,|\,a\leqq s\leqq b, (p, q)\in I(J, s)\}$, $W(J)=\{(p, q, s)\,|\,a\leqq s\leqq b, (p, q)\in S(J, s)\}=\{(p, q, s)\,|\,(p, q, s)\in G,\ \mathfrak{J}(p, q, s)=J\}$ for s, J such that $a\leqq s\leqq b$, $J_1^*\leqq J\leqq J_2^*$. Then D is relatively open in K since $\mathfrak{J}(p, q, s)\in C^1(G)$. Also we can prove the following consequence v) of Assumptions 1 and 2.

v) *$D(J)$ is relatively open in K and $\overline{D(J)}, W(J), \bar{D}$ are compact and $\overline{D(J)}=D(J)\cup W(J), \bar{D}=D\cup W(J_2^*)\cup W(J_2^*)$ for $J_1^*\leqq J\leqq J_2^*$.*

We omit the proof of v) and assume v) by the same reason as before.

2. In this paper, a function is always complex-valued if not specially mentioned.

A measure space (X, m) is a pair of a set X and a measure m defined on a Borel field in X. When X is a Lebesgue measurable subset of a Euclidian space R^r and m is the usual Lebesgue measure in R^r defined for all Lebesgue measurable subset of X, we shall often omit the explicit indication of the measure m and for example a function on X is simply called *measurable* or *integrable on X* in the following if it is measurable or integrable on the measure space (X, m).

LEMMA 1. *There is a unique measure $m_{J,s}$ defined for all Borel sets*[7] *on each $S(J, s)(a\leqq s\leqq b, J_2^*>J>J_1^*)$ such that*

$$\frac{d}{dJ}\int_{I(J_1^*,J,s)} f(p, q)dpdq=\int_{S(J,s)} f(p, q)dm_{J,s} \tag{4}$$

for every function $f(p, q)\in C_0[I(s)]$. We denote the completion of $m_{J,s}$ also by $m_{J,s}$. Then for any integrable (or non-negative measurable) function $f(p, q)$ on $I(s)$, $f(p, q)$ is integrable (or non-negative measurable) on the measure space $(S(J, s), m_{J,s})$ for almost all J $(J_2^*>J>J_1^*)$ and

7) A Borel set on $S(J, s)$ is a subset of $S(J, s)$ belonging to the Borel field on $S(J, s)$ generated by all relatively open subsets of $S(J, s)$.

$$\int_{S(J,s)} f(p,q)dm_{J,s}$$

is integrable (or non-negative measurable) with respect to J in the interval $J_2^* > J > J_1^*$ and

$$(5) \qquad \int_{I(s)} f(p,q)dpdq = \int_{J_1^*}^{J_2^*} \Big(\int_{S(J,s)} f(p,q)dm_{J,s} \Big) dJ.$$

Also

$$(6) \qquad \int_{S(J,s)} dm_{J,s} = 1.$$

We shall give a proof of this lemma in Part IV.

Now we give a rigorous proof of the following theorem which is well known in the classical statistical mechanics.

THEOREM 1.

$$\int_{S(J,s)} \frac{\partial \tilde{\mathfrak{J}}}{\partial s}(p,q,s)dm_{J,s} = 0$$

for $J_2^* > J > J_1^*$ and $b \geqq s \geqq a$.

PROOF. We take any function $\varphi(J) \in C_0[(J_1^*, J_2^*)]$ and put

$$G(J) = -\int_J^{J_2^*} \varphi(J')dJ' \quad \text{for} \quad J_2^* > J > J_1^*.$$

Then $G(J) \in C^1[(J_1^*, J_2^*)]$, $dG/dJ = \varphi(J)$ and there exist two numbers J_1 and J_2 such that $J_2^* > J_2 > J_1 > J_1^*$ and

$$G(J) = 0 \quad \text{for} \quad J_2^* > J \geqq J_2$$
$$G(J) = G(J_1) \quad \text{for} \quad J_1 \geqq J > J_1^*.$$

If we define a function $F(p,q,s)$ on K by

$$F(p,q,s) = G(\tilde{\mathfrak{J}}(p,q,s)) \quad \text{for} \quad (p,q,s) \in D$$
$$F(p,q,s) = 0 \quad \text{for} \quad K - \overline{D(J_2^*)}$$
$$F(p,q,s) = G(J_1) \quad \text{for} \quad \overline{D(J_1^*)},$$

then by the consequence v) of Assumptions 1 and 2 in Section 1, $F(p,q,s)$ belongs to $C^1(K)$ and vanishes for all (p,q,s) with sufficiently large $|p|^2 + |q|^2$.

Now by Lemma 1 and the properties of $G(J)$ and the definition of $F(p,q,s)$, we have for $a \leqq s \leqq b$

$$\int_{R^{2n}} F(p,q,s)dpdq = \int_{I(s)} G(\tilde{\mathfrak{J}}(p,q,s))dpdq + \int_{I(J_1^*,s)} G(J_1)dpdq$$
$$= \int_{J_1^*}^{J_2^*} \Big(\int_{S(J,s)} G(\tilde{\mathfrak{J}}(p,q,s))dm_{J,s} \Big) dJ + J_1^* \cdot G(J_1) = \int_{J_1^*}^{J_2^*} G(J)dJ + J_1^* \cdot G(J_1).$$

Hence we have for $a \leqq s \leqq b$

$$(7) \qquad \frac{d}{ds} \int_{R^{2n}} F(p,q,s)dpdq = 0.$$

On the other hand, also by Lemma 1, we have for $a \leqq s \leqq b$

$$
\begin{aligned}
\frac{d}{ds}\int_{R^{2n}} F(p,q,s)dpdq &= \int_{R^{2n}} \frac{\partial}{\partial s}F(p,q,s)dpdq \\
&= \int_{I(s)} \frac{dG}{dJ}(\tilde{\mathfrak{J}}(p,q,s))\frac{\partial\tilde{\mathfrak{J}}}{\partial s}dpdq \\
&= \int_{J_1^*}^{J_2^*}\Big(\int_{S(J,s)} \frac{dG}{dJ}(\tilde{\mathfrak{J}}(p,q,s))\frac{\partial\tilde{\mathfrak{J}}}{\partial s}dm_{J,s}\Big)dJ \\
&= \int_{J_1^*}^{J_2^*}\frac{dG}{dJ}\Big(\int_{S(J,s)} \frac{\partial\tilde{\mathfrak{J}}}{\partial s}dm_{J,s}\Big)dJ.
\end{aligned} \tag{8}
$$

From (7) and (8), we have

$$\int_{J_1^*}^{J_2^*}\varphi(J)\Big(\int_{S(J,s)}\frac{\partial\tilde{\mathfrak{J}}}{\partial s}dm_{J,s}\Big)dJ=0$$

for all s ($a\leqq s\leqq b$) and all $\varphi(J)\in C_0[(J_1^*, J_2^*)]$. From this, by a well-known standard argument we can get the desired result. Q. E. D.

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89. On the Adiabatic Theorem for the Hamiltonian System of Differential Equations in the Classical Mechanics. II

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3. Let (X, m) be a measure space where m is a finite, separable, and complete measure[1] defined on a Borel field in X. A one-parameter group $\{\mathfrak{T}_t \mid -\infty < t < +\infty\}$ of one-to-one mappings $\mathfrak{T}_t$ of X onto X is called a flow on (X, m). A measurable function $f(P)$ on (X, m) is called an invariant function of a flow $\{\mathfrak{T}_t\}$ on (X, m) if

$$f(\mathfrak{T}_t(P)) = f(P)$$

almost everywhere on (X, m) for every fixed t and it is called a strictly invariant function of a flow $\{\mathfrak{T}_t\}$ on (X, m) if it is defined everywhere on X and

$$f(\mathfrak{T}_t(P)) = f(P)$$

for all (P, t) such that $P \in X, -\infty < t < +\infty$. A measure-preserving and measurable flow[2] $\{\mathfrak{T}_t\}$ on (X, m) is ergodic (in the sense of J. v. Neumann) if and only if all its invariant functions are equivalent[3] to constants on (X, m). If a flow $\{\mathfrak{T}_t\}$ on (X, m) is measure-preserving and measurable, then we can associate with it a one-parameter group $\{\mathfrak{U}_t \mid -\infty < t < +\infty\}$ of unitary transformations $\mathfrak{U}_t$ on $L^2(X, m)$ by

$$(\mathfrak{U}_t f)(P) = f(\mathfrak{T}_t(P)) \quad f \in L^2(X, m), \quad P \in X$$

and $\mathfrak{U}_t$ is continuous as a function of t in the strong topology of $\mathfrak{U}_t$.[4]

If X is a Lebesgue measurable subset of a Euclidean space R^r and m is the usual Lebesgue measure in R^r defined for all Lebesgue measurable subsets of X, a flow on (X, m) is simply called a flow on X in the following and we write simply $L^2(X)$ for $L^2(X, m)$.

4. We consider the Hamiltonian system with a parameter s

$$(9) \qquad dp/dt = -\partial H/\partial q(p, q, s) \quad dq/dt = \partial H/\partial p(p, q, s).$$

By Assumption 1, the solution of (9)

$$(10) \qquad p = \mathfrak{p}(t, p^0, q^0, s) \quad q = \mathfrak{q}(t, p^0, q^0, s)$$

in the open set $I(s)$ for a fixed s $(a \leqq s \leqq b)$ with the initial conditions $(p, q) = (p^0, q^0)((p^0, q^0) \in I(s))$ at $t=0$, can be uniquely prolonged for the

1) For the definition of complete or separable measure, cf. P. Halmos [1].

2) For the definition of a measure-preserving, a measurable or an ergodic flow on (X, m), cf. E. Hopf [2, pp. 8-9 and p. 28].

3) Two measurable functions on (X, m) are called equivalent on (X, m) if they coincide almost everywhere on (X, m).

4) For definitions and results concerning flows on a measure space used in this paper, cf. E. Hopf [2].

whole time interval $-\infty<t<+\infty$ and $(\mathfrak{p}(t, p^0, q^0, s), \mathfrak{q}(t, p^0, q^0, s)) \in S(J, s)$ for $-\infty<t<+\infty$ if $(p^0, q^0) \in S(J, s)$, since $H(p, q, s)$ is an integral of (9), $S(J, s)=\{(p, q) \mid H(p, q, s)=\mathfrak{E}(J, s), (p, q) \in G_s\}$ and $S(J, s)$ is compact for $J_2^*>J>J_1^*, b \geqq s \geqq a$.[5] Also $\mathfrak{p}_i(t, p^0, q^0, s), \mathfrak{q}_i(t, p^0, q^0, s) \in C^1[(-\infty, +\infty) \times D]$ $i=1, \cdots, n$.[5] We denote by $T_t^{(s)}$ the one-to-one mapping of $I(s)$ onto $I(s)$

(11) $$(p^0, q^0) \rightarrow (\mathfrak{p}(t, p^0, q^0, s), \mathfrak{q}(t, p^0, q^0, s)).$$

Then $\{T_t^{(s)} \mid -\infty<t<+\infty\}$ constitutes a flow F_s on $I(s)$ for $a \leqq s \leqq b$, since the right sides of (9) do not contain the time t explicitly. The flow F_s on $I(s)$ is measure-preserving and measurable since (9) is a Hamiltonian system[6] (Theorem of Liouville) and $\mathfrak{p}_i(t, p^0, q^0, s)$, $\mathfrak{q}_i(t, p^0, q^0, s)$ $(i=1, \cdots, n)$ has sufficient regularities.

Since $T_t^{(s)}$ transforms $S(J, s)$ onto $S(J, s)$ $T_t^{(s)}$ induces a one-to-one mapping $T_t^{(J,s)}$ of $S(J, s)$ onto $S(J, s)$. $\{T_t^{(J,s)} \mid -\infty<t<+\infty\}$ constitutes a flow $F_{J,s}$ on the measure space $(S(J, s), m_{J,s})$ for $J_2^*>J>J_1^*, b \geqq s \geqq a$.

Lemma 2. *The flow $F_{J,s}$ on $(S(J, s), m_{J,s})$ is measure-preserving and measurable for $J_2^*>J>J_1^*, b \geqq s \geqq a$.*

We shall give a proof of this lemma in Part IV of this paper.

Also we consider the one-to-one mapping T_t of D onto D defined by

(12) $$(p^0, q^0, s) \rightarrow (\mathfrak{p}(t, p^0, q^0, s), \mathfrak{q}(t, p^0, q^0, s), s).$$

$\{T_t \mid -\infty<t<+\infty\}$ constitutes a flow F on D. From the fact that the flow F_s on $I(s)$ is measure-preserving and $\mathfrak{p}(t, p^0, q^0, s), \mathfrak{q}(t, p^0, q^0, s)$ are sufficiently regular, it follows easily that the flow F on D is measure-preserving and measurable. Then we have

Theorem 2. *For any fixed s $(b \geqq s \geqq a)$ the two following conditions* i) *and* ii) *are equivalent:*

i) *Every invariant function $f(p, q)$ of the flow F_s on $I(s)$ is equivalent on $I(s)$ to a function of the form $\varphi(H(p, q, s))$ where $\varphi(E)$ is a measurable function of E for the interval $\mathfrak{E}(J_2^*, s)>E>\mathfrak{E}(J_1^*, s)$.*

ii) *The flow $F_{J,s}$ on $(S(J, s), m_{J,s})$ is ergodic for almost all J in the interval $J_2^*>J>J_1^*$.*

We shall give a proof of this theorem in Part IV. This theorem is not used for the proof of our main theorem (the adiabatic theorem). It is laid here only to clarify the meaning of the following Assumption 3.

5. Now we put a further

Assumption 3. *The condition* i) *in Theorem* 2 (*equivalent to the condition* ii) *in Theorem* 2) *is satisfied at almost all s in the interval $a \leqq s \leqq b$.*

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5) Cf. E. Kamke [3, pp. 135–136 and pp. 161–164].

6) Cf. E. Kamke [3, pp. 155–161].

We can easily prove that Assumption 3 is also equivalent to the following proposition: For almost all s in the interval $a \leqq s \leqq b$, every invariant function $f(p, q)$ of the flow F_s on $I(s)$ is equivalent on $I(s)$ to a function of the form $\psi(\tilde{\mathfrak{J}}(p, q, s))$ where $\psi(J)$ is a measurable function of J for the interval $J_2^* > J > J_1^*$.

LEMMA 3. *If an invariant function $f(p, q, s)$ of the flow F on D belongs to $L^2(D)$, then*

$$\int_D \overline{f(p, q, s)} \frac{\partial \tilde{\mathfrak{J}}}{\partial s}(p, q, s) dp dq ds = 0.$$ [7]

PROOF. We can assume that $f(p, q, s)$ is a strictly invariant function of the flow F on D since for every invariant function of the flow F on D, there is a strictly invariant function of the flow F on D equivalent to it on D.[8] Then for almost all s in the interval $b \geqq s \geqq a$, $f(p, q, s)$ as a function of (p, q) is an invariant function of the flow F_s on $I(s)$ and so by Assumption 3 is equivalent on $I(s)$ to a function of the form $\psi_s(\tilde{\mathfrak{J}}(p, q, s))$ where $\psi_s(J)$ is a measurable function of J on the interval $J_2^* > J > J_1^*$.

Hence

$$\begin{aligned}\int_D \bar{f} \frac{\partial \tilde{\mathfrak{J}}}{\partial s} dp dq ds &= \int_a^b \Big(\int_{I(s)} \bar{f} \frac{\partial \tilde{\mathfrak{J}}}{\partial s} dp dq \Big) ds \\ &= \int_a^b \Big\{ \int_{J_1^*}^{J_2^*} \Big(\int_{S(J,s)} \overline{\psi_s(\tilde{\mathfrak{J}})} \frac{\partial \tilde{\mathfrak{J}}}{\partial s} dm_{J,s} \Big) dJ \Big\} ds \\ &= \int_a^b \Big\{ \int_{J_1^*}^{J_2^*} \overline{\psi_s(J)} \Big(\int_{S(J,s)} \frac{\partial \tilde{\mathfrak{J}}}{\partial s} dm_{J,s} \Big) dJ \Big\} ds = 0\end{aligned}$$

by Fubini's Theorem, Lemma 1, and Theorem 1. Q. E. D.

Now we consider the one-parameter group $\{U_t\}$ of unitary transformations on $L^2(D)$ associated with the flow F on D. We define Af by

$$(13) \qquad \Big\| \frac{U_t f - f}{it} - Af \Big\|_D \overset{9)}{\to} 0 \quad (t \to 0)$$

for all $f \in L^2(D)$ for which such Af exists. Then by a theorem of Stone, A is a self-adjoint operator (in the sense of J. v. Neumann) on $L^2(D)$ and $U_t = e^{iAt}$ in the sense of the operator calculus.[10] We denote the domain and the range of A by $\mathfrak{D}(A)$ and by $\mathfrak{R}(A)$ respectively. For a function $f \in C_0^1(D^0)$, we define $f=0$ on $D-D^0$[11] for

7) Here the bar means the complex conjugate.

8) Cf. E. Hopf [2, pp. 27–28].

9) $\| \ \|_D$ means the norm in $L^2(D)$.

10) Cf. F. Riesz and B. Sz.-Nagy [4, pp. 383–385].

11) If we denote for each s the set $(p, q, s) \mid (p, q) \in I(s)\}$ by $\tilde{I}(s)$, then $D - D^0 = \tilde{I}(a) \cup \tilde{I}(b)$ since D is relatively open in K.

convenience sake in the following. Then $C_0^1(D^0)\subset L^2(D)$ and also $C_0^1(D^0)\subset C^1(D)$. By calculating explicitly the Af in (13) for $f\in C_0^1(D^0)$, we have easily

LEMMA 4. *If* $f\in C_0^1(D^0)$, *then* $f\in\mathfrak{D}(A)$ *and*

$$Af=-i(f,H)=i(H,f).$$ [12]

Now we prove a lemma which is useful for some applications of Stone's Theorem.

LEMMA 5. *Let L be a self-adjoint operator on an abstract complex Hilbert space $\mathfrak{H}$. Let $\{V_t\mid -\infty<t<+\infty\}$ be the strongly continuous one-parameter group of unitary transformations $V_t=e^{iLt}$ on $\mathfrak{H}$. We put $\mathfrak{N}=\{f\mid f\in\mathfrak{D}(L),\ Lf=0\}$. Then $\mathfrak{N}$ is a closed linear subspace of $\mathfrak{H}$. We denote by $\mathfrak{N}^\perp$ the orthogonal complement of $\mathfrak{N}$ in $\mathfrak{H}$. Let $\mathfrak{W}$ be a linear (not necessarily closed) subspace of $\mathfrak{H}$, invariant for the group V_t, that is, such that, $V_t(\mathfrak{W})=\mathfrak{W}$ for all t. Also let $\mathfrak{W}\subset\mathfrak{D}(L)$ and $\overline{\mathfrak{W}}=\mathfrak{H}$. Then we have $\overline{L(\mathfrak{W})}=\mathfrak{N}^\perp$.*[13]

PROOF. $\overline{\mathfrak{R}(L)}=\mathfrak{N}^\perp$ since L is self-adjoint in $\mathfrak{H}$. Hence $\overline{L(\mathfrak{W})}\subset\mathfrak{N}^\perp$. Also $\overline{L(\mathfrak{W})}$ is a closed linear subspace of $\mathfrak{H}$. Let us assume that $\overline{L(\mathfrak{W})}\neq\mathfrak{N}^\perp$. Then there exists an element $f\in\mathfrak{N}^\perp$ such that $f\neq 0$ and $(f,Lg)=0$ for all $g\in\mathfrak{W}$. Now we have for all $g\in\mathfrak{W}$ and for all t

$$\frac{d}{dt}(f,V_tg)=\lim_{\Delta t\to 0}\left(f,\frac{V_{\Delta t}(V_tg)-V_tg}{\Delta t}\right)=(f,iLV_tg)$$

since $V_tg\in\mathfrak{W}\subset\mathfrak{D}(L)$[14] by $g\in\mathfrak{W}$, $V_t(\mathfrak{W})=\mathfrak{W}$ and $\mathfrak{W}\subset\mathfrak{D}(L)$. From this, we have for all $g\in\mathfrak{W}$ and for all t

$$\frac{d}{dt}(f,V_tg)=0$$

since $(f,iLV_tg)=0$ by the assumed properties of f and $V_tg\in\mathfrak{W}$.

Therefore for each $g\in\mathfrak{W}$, (f,V_tg) and so $(V_tf,g)\,(=(f,V_{-t}g))$ are constants for $-\infty<t<+\infty$ so that $(V_tf-f,g)=(V_tf-V_0f,g)=0$ for all $g\in\mathfrak{W}$ and for all t. Hence $V_tf=f$ for all t since $\overline{\mathfrak{W}}=\mathfrak{H}$. From this, it follows easily that $f\in\mathfrak{D}(L)$ and $Lf=0$[14] so that $f\in\mathfrak{N}$. Hence $f=0$ since also $f\in\mathfrak{N}^\perp$ by the assumption. This contradicts the assumption that $f\neq 0$. Q. E. D.

We return to the discussion of the group $\{U_t\}$ of unitary transformations associated with the flow F. We denote by N the set of all invariant functions of the flow F on D belonging to $L^2(D)$. N coincides with the set $\{f\mid f\in L^2(D),\ f=U_tf\ \ -\infty<t<+\infty\}=\{f\mid f\in\mathfrak{D}(A),\ Af=0\}$.[15] Hence N is a closed linear subspace of $L^2(D)$. Also

12) If $f,g\in C^1(D)$, we denote by (f,g) the Poisson bracket $-\sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}-\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}\right)$.

13) $\mathfrak{D}(L)$ and $\mathfrak{R}(L)$ are the domain of definition and range of L respectively. $L(\mathfrak{W})$ and $V_t(\mathfrak{W})$ are the images of $\mathfrak{W}$ by L and V_t respectively.

14) Cf. F. Riesz and B. Sz.-Nagy [4, pp. 383–385].

15) The last identy follows easily from the spectral representations of $\{U_t\}$ and A. Cf. F. Riesz and B. Sz.-Nagy [4, pp. 383–385].

we denote the orthogonal complement of N in $L^2(D)$ by $N^\perp$. Then we have

LEMMA 6. *$A[C_0^1(D^0)]$, the image of $C_0^1(D^0)$ by A, is contained and is dense in $N^\perp$.*

PROOF. $C_0^1(D^0)$ is dense in $L^2(D)$ as is well known. Also by Lemma 4 we have $C_0^1(D^0)\subset\mathfrak{D}(A)$. Further we have $U_t(C_0^1(D^0))=C_0^1(D^0)$ for all $t(-\infty<t<+\infty)$ since $\mathfrak{p}_i(t,p^0,q^0,s)$ and $\mathfrak{q}_i(t,p^0,q^0,s)(i=1,\cdots,n)$ all belong to $C^1[(-\infty,+\infty)\times D]$ and $T_t(D^0)=D^0$ for all t. Therefore we get the desired result by Lemma 5.

Now we prove the most important lemma of this Part II.

LEMMA 7. *For any $\varepsilon>0$, there is a function $f_0(p,q,s)\in C_0^1(D^0)$ such that*

$$\left\| (H,f_0)-\frac{\partial\tilde{\mathfrak{J}}}{\partial s}\right\|_D<\varepsilon.$$

PROOF. By Lemma 3, we have $\partial\tilde{\mathfrak{J}}/\partial s\in N^\perp$, if $\partial\tilde{\mathfrak{J}}/\partial s$ is considered as an element of $L^2(D)$. Hence by Lemma 6, for any $\varepsilon>0$, we have a function $f_0'\in C_0^1(D^0)$ such that

$$\left\| Af_0'-\frac{\partial\tilde{\mathfrak{J}}}{\partial s}\right\|_D<\varepsilon.$$

Therefore if we put $f_0=if_0'$, we get the desired result, since we have

$$Af_0=i(H,f_0)$$

for $f_0\in C_0^1(D^0)$, by Lemma 4. Q. E. D.

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90. On the Adiabatic Theorem for the Hamiltonian System of Differential Equations in the Classical Mechanics. III

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6. We consider the Hamiltonian system containing a parameter $\lambda(>0)$

(14) $$dp/dt=-\lambda\partial H/\partial q(p, q, t),\ dq/dt=\lambda\partial H/\partial p(p, q, t)$$

in D. If $(p^0, q^0, t^0)\in D$ and $\lambda>0$, there is a unique solution of (14) in D passing through $(p^0, q^0, t^0,)$ and prolonged as far as possible to the both directions of the time t, by the regularity of $H(p, q, s)$ in Assumption 1.[1] We denote it by

(15) $$p=\tilde{\mathfrak{p}}(t, p^0, q^0, t^0, \lambda),\ q=\tilde{\mathfrak{q}}(t, p^0, q^0, t^0, \lambda).$$

For a fixed $(p^0, q^0, t^0)\in D$ and a fix $\lambda(>0)$, $\tilde{\mathfrak{p}}(t, p^0, q^0, t^0, \lambda)$, $\tilde{\mathfrak{q}}(t, p^0, q^0, t^0, \lambda)$ are defined on a subinterval of the time interval $a\leqq t\leqq b$ which may be open, closed or half-open according to (p^0, q^0, t^0, λ).[1]

Since $\partial\tilde{\mathfrak{J}}/\partial s$ is continuous on $\bar{D}$ and $\bar{D}$ is compact, there is a number $M(>0)$ such that

(16) $$|\partial\tilde{\mathfrak{J}}/\partial s|\leqq M \qquad \text{on } D.$$

THEOREM 3. *Let a' and b' be two numbers such that $a\leqq a'<b'\leqq b$ and $(b'-a')<(J_2^*-J_1^*)/(2M)$ and let us put $J_2=J_2^*-M(b'-a')$, $J_1=J_1^*+M(b'-a')$. Then the solution of* (14) *passing through (p^0, q^0, a') where $(p^0, q^0)\in I(J_1, J_2, a')$ can be prolonged in D to the time interval $a'\leqq t\leqq b'$ for every $\lambda(>0)$.*

PROOF. Let β be the least upper bound of β' such that the solution in D of (14), $p=\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda)$, $q=\tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda)$ for a fixed $(p^0, q^0)\in I(J_1, J_2, a')$ and a fixed $\lambda>0$, can be defined for the time interval $a'\leqq t<\beta'$ and such that $a'<\beta'\leqq b'$. Then $a'<\beta\leqq b'$ and this solution in D can be defined on the the time interval $a'\leqq t<\beta$. Since $\partial H/\partial p$, $\partial H/\partial q$ are bounded on D by their continuity on the compact set $\bar{D}$, the functions $\tilde{\mathfrak{p}}_i(t, p^0, q^0, a', \lambda)$ $\tilde{\mathfrak{q}}_i(t, p^0, q^0, a', \lambda)$ $(i=1,\cdots, n)$ of t representing a solution of (14) in D, are uniformly continuous on the interval $a'\leqq t<\beta$. Hence the limits

$$\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda)\to p'(t\to\beta-0)$$
$$\tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda)\to q'(t\to\beta-0)$$

exist and $(p', q', \beta)\in\bar{D}$.

We shall sometimes abbreviate $\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda)$ and $\tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda)$

1) Cf. E. Kamke [1, pp. 135-136 and pp. 137-142].

as $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{q}}$ in this proof of Theorem 3.

Now for $a' \leqq t < \beta$, we have[2)]

$$\begin{aligned}(17)\qquad &\frac{d}{dt}\tilde{\mathfrak{F}}\{\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda), t\}\\ &=\frac{d}{dt}\mathfrak{F}\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\}\\ &=\frac{\partial \mathfrak{F}}{\partial E}\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\}\frac{d}{dt}H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)+\frac{\partial \mathfrak{F}}{\partial s}\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\}\\ &=\frac{\partial \mathfrak{F}}{\partial E}\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\}\frac{\partial H}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)+\frac{\partial \mathfrak{F}}{\partial s}\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\}\\ &=\frac{\partial \tilde{\mathfrak{F}}}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)\end{aligned}$$

since we can easily verify that

$$\frac{d}{dt}H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)=\frac{\partial H}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)$$

for the solution $(p=\tilde{\mathfrak{p}}, q=\tilde{\mathfrak{q}})$ of (14). Hence by (16) we have

$$\begin{aligned}|\tilde{\mathfrak{F}}(p', q', \beta)-\tilde{\mathfrak{F}}(p^0, q^0, a')| &=\left|\int_{a'}^{\beta}\frac{d}{dt}\tilde{\mathfrak{F}}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)dt\right|\\ &=\left|\int_{a'}^{\beta}\frac{\partial \tilde{\mathfrak{F}}}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)dt\right| \leqq M(\beta-a') \leqq M(b'-a').\end{aligned}$$

Hence we have

$$J_2^*=J_2+M(b'-a')>\tilde{\mathfrak{F}}(p', q', \beta)>J_1-M(b'-a')=J_1^*$$

since $J_2>\tilde{\mathfrak{F}}(p^0, q^0, a')>J_1$ by $(p^0, q^0)\in I(J_1, J_2, a')$. Thus we have proved that $(p', q', \beta)\in D$ and that the solution $p=\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda)$, $q=\tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda)$ of (14) in D can be defined on the interval $a'\leqq t\leqq \beta$.[1)] If $a'\leqq \beta < b'$, then $(p', q', \beta)\in D^0$[3)] and the solution can be continued beyond $t=\beta$.[1)] This contradicts the definition of β. Hence $\beta=b'$. This completes the proof of Theorem 2.

7. When $a\leqq a'<b'\leqq b$, $(p^0, q^0)\in I(a')$ and $\lambda>0$, we define $\Delta(a', b', p^0, q^0, \lambda)$ as follows. We put

$$\Delta(a', b', p^0, q^0, \lambda)=\max_{a'\leqq t\leqq b'}|\tilde{\mathfrak{F}}\{\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(t, p^0, q^0\, a', \lambda), t\}-\tilde{\mathfrak{F}}(p^0, q^0, a')|$$

if the solution $p=\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda)$, $q=\tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda)$ of (14) can be continued to $t=b'$ in D and we put

$$\Delta(a', b', p^0, q^0, \lambda)=+\infty,$$

if the above solution of (14) can not be continued to $t=b'$ in D. When $a\leqq a'<b'\leqq b$, $\lambda>0$ and $\delta>0$, we denote the subset of $I(a')$, $\{(p^0, q^0)\,|\,(p^0, q^0)\in I(a'), \Delta(a', b', p^0, q^0, \lambda)<\delta\}$ by $L(a', b', \lambda, \delta)$. We can easily

2) $\partial\mathfrak{F}/\partial E\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\}$, $\partial\mathfrak{F}/\partial s\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\}$ are the values of $\partial\mathfrak{F}/\partial E(E, s)$, $\partial\mathfrak{F}/\partial s(E, s)$ for $E=H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)$, $s=t$ and $\partial H/\partial s(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)$, $\partial\tilde{\mathfrak{F}}/\partial s(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)$ are the values of $\partial H/\partial s(p, q, s)$, $\partial\tilde{\mathfrak{F}}/\partial s(p, q, s)$ for $p=\tilde{\mathfrak{p}}$, $q=\tilde{\mathfrak{q}}$, $s=t$.

3) Cf. footnote 11) of Part II.

prove that $L(a', b', \lambda, \delta)$ is an open set in R^{2n} by the continuity of $\tilde{\mathfrak{J}}$ on D and by the theorems[4] on the dependence of the solutions of (14) in D on the initial conditions.

In the following, we denote the m-dimensional Lebesgue measure of a measurable set in R^m by $\mu_m[\quad]$.

LEMMA 8. *Let a', b', J_1 and J_2 be the same with those in Theorem 3. Then for any fixed $\delta(>0)$, $\mu_{2n}[I(J_1, J_2, a')-L(a', b', \lambda, \delta)]\to 0$ $(\lambda\to +\infty)$.*

PROOF. By Theorem 3, the solution $p=\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda)$, $q=\tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda)$ of (14) can be continued in D to $t=b'$ if $(p^0, q^0)\in I(J_1, J_2, a')$.

Now we take any positive number ε. Then by Lemma 7, we can take a function $f_0(p, q, s)\in C_0^1(D^0)$ such that

$$\left(\int_D\left|(H,f_0)-\frac{\partial\tilde{\mathfrak{J}}}{\partial s}\right|^2 dpdqds\right)^{\frac{1}{2}}<\frac{\varepsilon}{2}[\mu_{2n+1}(D)]^{-\frac{1}{2}}.$$

Hence if we put $g_0(p, q, s)=\partial\tilde{\mathfrak{J}}/\partial s-(H,f_0)$ on D, then we have by Schwartz inequality

$$\int_D |g_0(p, q, s)|\, dpdqds<\frac{\varepsilon}{2}. \tag{18}$$

We have in the same way as in (17)

$$\begin{aligned}&\frac{d}{dt}\tilde{\mathfrak{J}}\{\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda), t\}\\ &=\frac{\partial\tilde{\mathfrak{J}}}{\partial s}\{\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda), t\}\end{aligned} \tag{19}$$

for $a'\leqq t\leqq b'$ and $(p^0, q^0)\in I(J_1, J_2, a')$. We shall sometimes abbreviate $\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda)$ and $\tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda)$ as $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{q}}$ in this proof of Lemma 8. By (19) and the definition of $g_0(p, q, s)$, we get

$$\begin{aligned}&|\tilde{\mathfrak{J}}\{\tilde{\mathfrak{p}}(t', p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(t', p^0, q^0, a', \lambda), t'\}-\tilde{\mathfrak{J}}(p^0, q^0, a')|\\ &=\left|\int_{a'}^{t'}\frac{d}{dt}\tilde{\mathfrak{J}}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)dt\right|=\left|\int_{a'}^{t'}\frac{\partial\tilde{\mathfrak{J}}}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)dt\right|\leqq\left|\int_{a'}^{t'}g_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)dt\right|\\ &+\left|\int_{a'}^{t'}(H,f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)dt\right|^{5)}.\end{aligned}$$

Hence by the definition of $\Delta(a', b', p^0, q^0, \lambda)$, we have

$$\Delta(a', b', p^0, q^0, \lambda)\leqq\int_{a'}^{b'}|g_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)|\,dt+\max_{a'\leqq t'\leqq b'}\left|\int_{a'}^{t'}(H,f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)dt\right|. \tag{20}$$

Now for any fixed $s(a'\leqq s\leqq b')$ and for any fixed $\lambda(>0)$, the one-to-one mapping of $I(J_1, J_2, a')$ onto an open set $V(s, \lambda)$ in $I(s)$

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4) Cf. E. Kamke [1, pp. 149–153].

5) $(H,f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)$ is the value of the Poisson bracket $(H,f_0)(p, q, s)$ for $p=\tilde{\mathfrak{p}}$, $q=\tilde{\mathfrak{q}}$, $s=t$.

$$(p^0, q^0) \to \{\tilde{\mathfrak{p}}(s, p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(s, p^0, q^0, a', \lambda)\}$$

is measure-preserving, since (14) is a Hamiltonian system (Theorem of Liouville).[6] Hence

$$\text{(21)}\quad \begin{aligned} &\int_{I(J_1,J_2,a')} \left(\int_{a'}^{b'} | g_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) | \, dt\right) dp^0 dq^0 \\ &= \int_{a'}^{b'} \left(\int_{I(J_1,J_2,a')} | g_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) | \, dp^0 dq^0\right) dt \\ &= \int_{a'}^{b'} \left(\int_{V(s,\lambda)} | g_0(p, q, s) | \, dp dq\right) ds \\ &\leqq \int_D | g_0(p, q, s) | \, dp dq ds, \end{aligned}$$

since $V(s, \lambda) \subset I(s)$ and $D = \{(p, q, s) \mid (p, q) \in I(s), a \leqq s \leqq b\}$.

On the other hand[7]

$$\begin{aligned} (H, f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) &= \frac{1}{\lambda} \frac{\partial f_0}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) - \frac{1}{\lambda}\left\{\frac{\partial f_0}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) - \lambda (H, f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)\right\} \\ &= \frac{1}{\lambda} \frac{\partial f_0}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) - \frac{1}{\lambda} \frac{d}{dt} f_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) \end{aligned}$$

since $\dfrac{d}{dt} f_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) = -\lambda (H, f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) + \dfrac{\partial f_0}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)$ for the solution $p = \tilde{\mathfrak{p}}$, $q = \tilde{\mathfrak{q}}$ of (14) in D as can be easily verified. Hence for $a' \leqq t' \leqq b'$

$$\begin{aligned} &\left| \int_{a'}^{t'} (H, f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) dt \right| \leqq \frac{1}{\lambda} \left| \int_{a'}^{t'} \frac{\partial f_0}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) dt \right| \\ &\quad + \frac{1}{\lambda} \left| \int_{a'}^{t'} \frac{d}{dt} f_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) dt \right| \leqq \frac{1}{\lambda} \int_{a'}^{b'} \left| \frac{\partial f_0}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) \right| dt \\ &\quad + \frac{1}{\lambda} \left| f_0\{\tilde{\mathfrak{p}}(t', p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(t', p^0, q^0, a', \lambda), t'\} - f_0(p^0, q^0, a') \right|. \end{aligned}$$

Now there is an M' such that $|f_0|, |\partial f_0/\partial s| \leqq M'$ on D since $f_0 \in C_0^1(D^0)$. Therefore we have

$$\text{(22)}\quad \max_{a' \leqq t' \leqq b'} \left| \int_{a'}^{t'} (H, f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) dt \right| \leqq \frac{1}{\lambda}((b' - a') + 2) M'.$$

By (18), (20), (21) and (22), we get

$$\int_{I(J_1,J_2,a')} \Delta(a', b', p^0, q^0, \lambda) dp^0 dq^0$$

$\leqq \dfrac{\varepsilon}{2} + \dfrac{1}{\lambda}\{(b' - a') + 2\} M' \mu_{2n}[I(J_1, J_2, a')] \leqq \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} \leqq \varepsilon$ if $\lambda \geqq 2\{(b' - a') + 2\} \times M' \mu_{2n}[I(J_1, J_2, a')]/\varepsilon$. Thus we have proved that

$$\int_{I(J_1,J_2,a')} \Delta(a', b', p^0, q^0, \lambda) dp^0 dq^0 \to 0 \quad (\lambda \to +\infty).$$

From this we can easily deduce the desired results. Q. E. D.

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6) Cf. E. Kamke [1, pp. 155–161].

7) Cf. footnote 2) and 5).

8. Now we state and prove a form of the adiabatic theorem.

THEOREM 4. *Under Assumptions* 1, 2 *and* 3,

$$\mu_{2n}[I(J_1, J_2, a)-L(a, b, \lambda, \delta)]\to 0 \ (\lambda\to+\infty)$$

for any fixed J_1, J_2, δ *such that* $J_2^*>J_2>J_1>J_1^*$ *and* $\delta>0$.

PROOF. We fix any J_1, J_2 such that $J_2^*>J_2>J_1>J_1^*$, then by Lemma 8, if $b\geqq\beta'>a$ and β' is sufficiently close to a,

$$\mu_{2n}[I(J_1, J_2, a)-L(a, \beta', \lambda, \delta)]\to 0 \ (\lambda\to+\infty) \tag{23}$$

for all $\delta>0$. We denote by β the least upper bound of β' such that $b\geqq\beta'>a$ and (23) holds for all $\delta>0$. Also we fix a $\delta_0>0$ such that $J_2^*>J_2+\delta_0>J_1-\delta_0>J_1^*$. Then if $b\geqq\beta''\geqq\beta>\alpha''>a$ and $\beta''-\alpha''$ is sufficiently small, we have by Lemma 8

$$\mu_{2n}[I(J_1-\delta_0, J_2+\delta_0, \alpha'')-L(\alpha'', \beta'', \lambda, \delta/2)]\to 0 \ (\lambda\to+\infty) \tag{24}$$

for all $\delta>0$. Also by the definition of β, there is an α'' arbitrarily close to β such that $\beta>\alpha''>a$ and

$$\mu_{2n}[I(J_1, J_2, a)-L(a, \alpha'', \lambda, \delta/2)]\to 0 \ (\lambda\to+\infty) \tag{25}$$

for all $\delta>0$. Hence we can take for any β'' such that $b\geqq\beta''\geqq\beta$ and $\beta''-\beta$ is sufficiently small or zero, an α'' such that $\beta>\alpha''>a$ and (24), (25) are satisfied for all $\delta>0$. We take such α'' and β'' in the following.

Now if $(p^0, q^0)\in L(a, \alpha'', \lambda, \delta/2)$, then the solution $p=\tilde{\mathfrak{p}}(t, p^0, q^0, a, \lambda)$, $q=\tilde{\mathfrak{q}}(t, p^0, q^0, a, \lambda)$ of (14) can be continued in D to $t=\alpha''$, by the definition of $L(a, \alpha'', \lambda, \delta/2)$. We denote by $\mathfrak{A}_{\lambda,\delta}$ the one-to-one mapping of $L(a, \alpha'', \lambda, \delta/2)$ into $I(\alpha'')$

$$(p^0, q^0)\to\{\tilde{\mathfrak{p}}(\alpha'', p^0, q^0, a, \lambda),\ \tilde{\mathfrak{q}}(\alpha'', p^0, q^0, a, \lambda)\}$$

and also by $R(\lambda, \delta)$ the set $\mathfrak{A}_{\lambda,\delta}[I(J_1, J_2, a)\cap L(a, \alpha'', \lambda, \delta/2)]$. Then by the definition of $L(a, \alpha'', \lambda, \delta/2)$ we have for $0<\delta<2\delta_0$, $\lambda>0$

$$R(\lambda, \delta)\subset I(J_1-\delta_0, J_2+\delta_0, \alpha''). \tag{26}$$

Also the mapping $\mathfrak{A}_{\lambda,\delta}$ is measure-preserving since (14) is a Hamiltonian system (Theorem of Liouville).[6] Hence for $\delta>0, \lambda>0$

$$\mu_{2n}[R(\lambda, \delta)]=\mu_{2n}[I(J_1, J_2, a)\cap L(a, \alpha'', \lambda, \delta/2)]. \tag{27}$$

By (25) and (27), we have for all $\delta>0$

$$\mu_{2n}[R(\lambda, \delta)]\to\mu_{2n}[I(J_1, J_2, a)] \ (\lambda\to+\infty). \tag{28}$$

From (28), (24) and (26), we have for $0<\delta<2\delta_0$

$$\mu_{2n}[R(\lambda, \delta)\cap L(\alpha'', \beta'', \lambda, \delta/2)]\to\mu_{2n}[I(J_1, J_2, a)] \ (\lambda\to+\infty).$$

Therefore we have for $0<\delta<2\delta_0$,

$$\mu_{2n}\{\mathfrak{A}_{\lambda,\delta}^{-1}[R(\lambda, \delta)\cap L(\alpha'', \beta'', \lambda, \delta/2)]\}\to\mu_{2n}[I(J_1, J_2, a)] \ (\lambda\to+\infty)$$

since $\mathfrak{A}_{\lambda,\delta}$ and so $\mathfrak{A}_{\lambda,\delta}^{-1}$ is measure-preserving. Hence we have for $0<\delta<2\delta_0$,

$$\mu_{2n}[I(J_1, J_2, a)\cap L(a, \beta'', \lambda, \delta)]\to\mu_{2n}[I(J_1, J_2, a)] \ (\lambda\to+\infty)$$

since we can easily see that $I(J_1, J_2, a)\cap L(a, \beta'', \lambda, \delta)\supset\mathfrak{A}_{\lambda,\delta}^{-1}[R(\lambda, \delta)\cap L(\alpha'', \beta'', \lambda, \delta/2)]$ by the definitions of $L(a', b', \lambda, \delta)$ and $R(\lambda, \delta)$. Thus we get

$$\mu_{2n}[I(J_1, J_2, a)-L(a, \beta'', \lambda, \delta)]\to 0 \ (\lambda\to+\infty) \tag{29}$$

for $0<\delta<2\delta_0$ and so for all $\delta>0$, since $L(a, \beta'', \lambda, \delta_2)\supset L(a, \beta'', \lambda, \delta_1)$ if $\delta_2>\delta_1$.

If $\beta<b$, then we can take the above β'' in such a manner that $b\geqq\beta''>\beta$. But then (29) contradicts the definition of β. Hence $\beta=b$. Then if we take $\beta''=\beta=b$ in the above argument, we have from (29) the desired results. Q. E. D.

Reference

[1] Kamke, E.,: Differentialgleichungen reeller Funktionen, Leipzig (1930).