Rate matrices, two examples.

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathcal{R}\mathbf{p} \qquad , \qquad \mathcal{R} = U\Lambda V^\dagger \quad , \quad \mathcal{T}(t) \equiv e^{\mathcal{R}t} = Ue^{\Lambda t}V^\dagger \tag{1}$$

Example 1.

$$\mathcal{R} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -z & -z^* \\ 1 & -z^* & -z \end{pmatrix}\begin{pmatrix} 0 & & \\ & -1-z^* & \\ & & -1-z \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -z^* & -z \\ 1 & -z & -z^* \end{pmatrix} \tag{2}$$

where

$$z = e^{i\phi} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \qquad , \qquad z^* = e^{-i\phi} = \frac{1}{2} - i\frac{\sqrt{3}}{2} \qquad , \qquad \phi = \frac{\pi}{3} \quad . \tag{3}$$

Using this decomposition we get

$$\mathcal{T}(t) \equiv e^{\mathcal{R}t} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{2}{3}e^{-3t/2}\begin{pmatrix} \cos(\alpha t) & \cos(\alpha t - 4\phi) & \cos(\alpha t - 2\phi) \\ \cos(\alpha t - 2\phi) & \cos(\alpha t) & \cos(\alpha t - 4\phi) \\ \cos(\alpha t - 4\phi) & \cos(\alpha t - 2\phi) & \cos(\alpha t) \end{pmatrix} \tag{4}$$

where $\alpha = \sqrt{3}/2$. Note that $\mathcal{T}(0) = I$.

Example 2.

$$\mathcal{R} = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix} = \begin{pmatrix} \sqrt{1/3} & \sqrt{1/2} & \sqrt{1/6} \\ \sqrt{1/3} & 0 & -\sqrt{2/3} \\ \sqrt{1/3} & -\sqrt{1/2} & \sqrt{1/6} \end{pmatrix}\begin{pmatrix} 0 & & \\ & -\frac{3}{2} & \\ & & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \\ \sqrt{1/2} & 0 & -\sqrt{1/2} \\ \sqrt{1/6} & -\sqrt{2/3} & \sqrt{1/6} \end{pmatrix} \tag{5}$$

Using this decomposition we get

$$\mathcal{T}(t) \equiv e^{\mathcal{R}t} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{3}e^{-3t/2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \quad . \tag{6}$$

Again, $\mathcal{T}(0) = I$.