A few elementary facts about real square matrices

Let M denote a real, $N \times N$ matrix. Left and right eigenvectors are defined by

$$M\mathbf{u} = \lambda\mathbf{u} \quad , \quad \mathbf{v}^\dagger M = \lambda\mathbf{v}^\dagger, \tag{1}$$

where $\mathbf{u}$ is a column vector, $\mathbf{v}^\dagger$ is a row vector, and the dagger ($\dagger$) denotes a transpose with complex conjugation ($c.c.$).[1] The eigenvalues $\lambda_1, \cdots \lambda_N$ solve the characteristic equation

$$|M - \lambda I| = 0, \tag{2}$$

and there may be degeneracies among these values. Note that there is no distinction between the sets of left and right eigen*values*.

Case 1. If $M = M^T$ then

- all eigenvalues are real,
- the left and right eigenvectors are identical, and can be made to be real, and
- there are N eigenvectors, and they form an orthogonal basis set .

Case 2. If $M \neq M^T$ and there are no degeneracies among the eigenvalues, then

- there are N left eigenvectors $\{\mathbf{v}_1^\dagger, \cdots \mathbf{v}_N^\dagger\}$, forming a complete basis set,
- there are N right eigenvectors, $\{\mathbf{u}_1, \cdots \mathbf{u}_N\}$, forming a complete basis set, and
- with proper choice of normalization, the two sets can be made bi-orthonormal:

$$\mathbf{v}_i^\dagger \cdot \mathbf{u}_j = \delta_{ij} \quad , \tag{3}$$

but in general $\mathbf{u}_i^\dagger \cdot \mathbf{u}_j \neq 0$ and $\mathbf{v}_i^\dagger \cdot \mathbf{v}_j \neq 0$.

[1] Both $\mathbf{v}^\dagger$ and its c.c. are left eigenvectors of M, and both $\mathbf{v}$ and its c.c. are right eigenvectors of M^T. Similar comments apply to the $\mathbf{u}$'s.

In either Case 1 or Case 2, the matrix M is diagonalizable:

$$M = UDV^\dagger = \begin{pmatrix} \uparrow & & \uparrow \\ \mathbf{u}_1 & \cdots & \mathbf{u}_N \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} \begin{pmatrix} \leftarrow & \mathbf{v}_1^\dagger & \rightarrow \\ & \vdots & \\ \leftarrow & \mathbf{v}_N^\dagger & \rightarrow \end{pmatrix} \tag{4}$$

where the notation indicates that the columns of U and the rows of $V^\dagger$ are given by the right and left eigenvectors, respectively. Note that $UV^\dagger = V^\dagger U = I$, and in Case 1 $U = V$.

Case 3. If $M \neq M^T$ and there are degeneracies among the eigenvalues, then

- for each solution λ of Eq. 2 there are k left eigenvectors and k right eigenvectors, where $1 \leq k \leq K$ and K is the degeneracy of the solution. Thus there are $N^* \leq N$ left eigenvectors and an equal number of right eigenvectors.
- If $N^* = N$, the conclusions of Case 2 still apply (in particular M is diagonalizable).
- If $N^* < N$, neither the left nor the right eigenvectors form a complete set (obviously!), and M is not diagonalizable, but can be transformed into Jordan canonical form.

Examples. As you can (and should!) verify, the following matrices illustrate Case 3, with $N^* = N$ (diagonalizable) and $N^* < N$ (non-diagonalizable), respectively:

$$\begin{pmatrix} -3 & 3 & 3 \\ 2 & -4 & 2 \\ 1 & 1 & -5 \end{pmatrix} \quad , \quad \begin{pmatrix} -17 & 19 & 13 \\ 3 & -9 & 9 \\ 14 & -10 & -22 \end{pmatrix} \tag{5}$$