THE EFFECT OF PROJECTIONS ON FRACTAL SETS AND MEASURES IN BANACH SPACES

WILLIAM OTT, BRIAN HUNT, AND VADIM KALOSHIN

Abstract. We study the extent to which the Hausdorff dimension and the dimension spectrum of a fractal measure supported on a compact subset of a Banach space are affected by a typical mapping into a finite-dimensional Euclidean space. Let X be a compact subset of a Banach space B with thickness exponent $\tau(X)$ and Hausdorff dimension $\dim_H(X)$. Let M be any subspace of the Borel measurable functions from B to $\mathbb{R}^m$ that contains the space of linear functions and is contained in the space of locally Lipschitz functions. We prove that for almost every (in the sense of prevalence) function $f \in M$, one has $\dim_H(f(X)) \geqslant \min\{m, \dim_H(X)/(1+\tau(X))\}$. We also prove an analogous result for a certain part of the dimension spectra of Borel probability measures supported on X. The factor $1/(1+\tau(X))$ can be improved to $1/(1+\tau(X)/2)$ if B is a Hilbert space. Since dimension cannot increase under a locally Lipschitz function, these theorems become dimension preservation results when $\tau(X) = 0$. We conjecture that many of the attractors associated with the evolution equations of mathematical physics have zero thickness. The sharpness of our results in the case $\tau(X) \neq 0$ is discussed.

1. Introduction

Many infinite-dimensional dynamical systems have been shown to have compact finite-dimensional attractors [3, 4, 24, 27, 28]. Such attractors exist for a variety of the evolution equations of mathematical physics, including the Navier-Stokes system, various classes of reaction-diffusion systems, nonlinear dissipative wave equations, and complex Ginzburg-Landau equations. When an attractor is measured experimentally, one observes a 'projection' of the attractor into finite-dimensional Euclidean space. This technique of observation via projection leads to a natural and fundamental question. How accurately does the image of the attractor reflect the attractor itself? We address this question from a dimension-theoretic perspective and we consider the following problem. For an attractor of an infinite-dimensional dynamical system, how is its dimension affected by a typical projection into a finite-dimensional Euclidean space?

One may define the dimension of an attractor in many different ways. Setting aside dynamics, the attractor may be viewed as a compact set of points in a metric space. Viewing the attractor in this light, the dimension of the attractor may be defined as the box-counting dimension or the Hausdorff dimension of the attracting set. Measure-dependent notions of attractor dimension take into account the distribution of points induced by the dynamics and are thought to be more accurately measured from numerical or experimental data. One often analyzes the 'natural

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measure,' the probability measure induced by the statistics of a typical trajectory that approaches the attractor. A natural measure is not known to exist for arbitrary systems, but it does exist for Axiom A attractors and for certain classes of systems satisfying conditions weaker than uniform hyperbolicity. See [14, 29] for expository discussions of systems that are known to have natural measures.

The dimension spectrum (D_q spectrum) characterizes the multifractal structure of an attractor. Given a Borel measure μ with compact support X in some metric space, for $q \geqslant 0$ and $q \neq 1$ let

$$D_q(\mu) = \lim_{\epsilon \to 0} \frac{\log \int_X \left[\mu(B(x,\epsilon))\right]^{q-1} d\mu(x)}{(q-1)\log \epsilon}, \tag{1.1}$$

provided the limit exists, where $B(x,\epsilon)$ is the ball of radius ϵ centered at x. (If the limit does not exist, define $D_q^+(\mu)$ and $D_q^-(\mu)$ to be the lim sup and lim inf, respectively.) Let

$$D_1(\mu) = \lim_{q \to 1} D_q(\mu),$$

again provided the limit exists. This spectrum includes the box-counting dimension (D_0), the information dimension (D_1), and the correlation dimension (D_2). In particular, when $q = 0$ the dimension depends only on the support X of μ and we write $D_0(X) = D_0(\mu)$. See Section 2 for a discussion of this definition and its relationship to other definitions of D_q in the literature.

The goal of this paper is to extend the following theorems, as much as possible, to infinite-dimensional Banach spaces. In all of the results in this paper, 'almost every' is in the sense of *prevalence*, a generalization of 'Lebesgue almost every' to infinite-dimensional spaces. See Section 2 and [10, 11] for details.

Theorem 1.1 ([26]). *Let $X \subset \mathbb{R}^n$ be a compact set. For almost every function $f \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, one has*

$$\dim_H(f(X)) = \min\{m, \dim_H(X)\}$$

where $\dim_H(\cdot)$ is the Hausdorff dimension.

Theorem 1.2 ([12]). *Let μ be a Borel probability measure on $\mathbb{R}^n$ with compact support and let q satisfy $1 < q \leqslant 2$. Assume that $D_q(\mu)$ exists. Then for almost every function $f \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, $D_q(f(\mu))$ exists and is given by*

$$D_q(f(\mu)) = \min\{m, D_q(\mu)\}.$$

For each result, the space C^1 can be replaced by any space that contains the linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ and is contained in the locally Lipschitz functions. Theorem 1.1 extends to smooth functions a result of Mattila [18] (generalizing earlier results of Marstrand [17] and Kaufmann [16]) that makes the same conclusion for almost every linear function from $\mathbb{R}^n$ into $\mathbb{R}^m$, in the sense of Lebesgue measure on the space of m-by-n matrices. Strictly speaking, Marstrand, Kaufmann, and Mattila considered orthogonal projections, but the analogous results for general linear projections follow immediately. Sauer and Yorke [26] prove Theorem 1.2 for the correlation dimension (D_2) and recover (1.1) by invoking a variational principle for Hausdorff dimension [5]. Theorem 1.1 and its predecessors follow from a potential-theoretic characterization of the dimensions involved. Roughly speaking, the dimension is the largest exponent for which a certain singular integral converges. Theorem 1.2 follows from a similar characterization of D_q for $q > 1$ [12].

The potential-theoretic approach only leads to a dimension preservation result for D_q if $1 < q \leqslant 2$. For $0 \leqslant q < 1$ and $q > 2$, [12] gives examples for which D_q is not preserved by any linear transformation into $\mathbb{R}^m$. For $0 \leqslant q < 1$, the construction is based on the discovery by Kan [25, 26] of a class of examples for which the box dimension is not preserved by any C^1 function.

When the ambient space is not finite-dimensional, one does not expect a dimension preservation result analogous to Theorem 1.1 or Theorem 1.2 to hold. We use the thickness exponent to study the extent to which the dimension spectrum is affected by projection from a Banach space to $\mathbb{R}^m$. This exponent, defined precisely in Section 2 and denoted $\tau(X)$, measures how well a compact subset X of a Banach space B can be approximated by finite-dimensional subspaces of B, with smaller values of the thickness exponent indicating better approximability. In general one has $\tau(X) \leqslant D_0^+(X)$, the upper box-counting dimension of X, and equality is possible. We expect that the thickness exponent can be shown to be significantly smaller than the box-counting dimension for many attractors of infinite-dimensional systems. Studying the Hölder regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces, [13] establishes a bound on the amount the dimension may drop for a typical projection.

Theorem 1.3 ([13]). *Let B denote a Banach space. Let $X \subset B$ be a compact set with box-counting dimension d and thickness exponent $\tau(X)$. Let $m > 2d$ be an integer, and let α be a real number with*

$$0 < \alpha < \frac{m - 2d}{m(1 + \tau(X))}.$$

Then for almost every (in the sense of prevalence) bounded linear function (or C^1 function, or Lipschitz function) $f : B \to \mathbb{R}^m$ there exists $C > 0$ such that for all x, $y \in X$,

$$C|f(x) - f(y)|^\alpha \geqslant |x - y|. \tag{1.2}$$

For such a function f, one has

$$\frac{m - 2d}{m(1 + \tau(X))} \dim(X) \leqslant \dim(f(X)) \leqslant \dim(X)$$

where $\dim(X)$ represents either the box-counting dimension or Hausdorff dimension.

This theorem generalizes earlier results in [2] and [6].

For a function f satisfying (1.2), the factor by which the dimension may drop is the product of two terms, $(m-2d)/m$ and $1/(1+\tau(X))$. The first term depends on the embedding dimension m and converges to one as $m \to \infty$ while the second term depends intrinsically on X via its thickness. We prove that the Hausdorff dimension is preserved by a typical projection up to a factor of $1/(1+\tau(X))$. In particular, the factor $(m-2d)/m$ has been removed. We now state the main theorem for compact subsets of Banach spaces. Because of the possibility of dimension drop, the existence of $D_q(\mu)$ does not imply the existence of $D_q(f(\mu))$ for functions f satisfying the conclusion of the theorem. We therefore formulate the result in terms of the lower dimension D_q^-.

Banach Space Theorem. *Let B be a Banach space, and let M be any subspace of the Borel measurable functions from B to $\mathbb{R}^m$ that contains the space of linear* *functions and is contained in the space of locally Lipschitz functions. Let $X \subset B$ be a compact set with thickness exponent $\tau(X)$. Let μ be a Borel probability measure supported on X. For almost every $f \in M$, one has*

$$\dim_H(f(X)) \geqslant \min\left\{m, \frac{\dim_H(X)}{1+\tau(X)}\right\},$$

and, for $1 < q \leqslant 2$,

$$D_q^-(f(\mu)) \geqslant \min\left\{m, \frac{D_q^-(\mu)}{1+\tau(X)}\right\}. \tag{1.3}$$

Notice that for sets with thickness zero, the Banach space theorem is a dimension preservation result. Every compact set $X \subset \mathbb{R}^n$ has thickness zero. Thus, the Banach space theorem generalizes Theorems 1.1 and 1.2. Furthermore, it strengthens Theorem 1.2, because for a prevalent set of functions, (1.3) holds simultaneously for all $1 < q \leqslant 2$. On the other hand, suppose $\tau(X) > 0$. The Hausdorff dimension of X may be *noncomputable* in the sense that for any positive integer m and any subspace M of the Borel measurable functions from B to $\mathbb{R}^m$, $\dim_H(f(X)) < \dim_H(X)$ for all $f \in M$. In other words, the Hausdorff dimension of X cannot be ascertained from any finite-dimensional representation of X.

The proof of the Banach space theorem uses only the most general information about the structure of the dual space B'. In specific situations, additional knowledge about the structure of the dual space may yield improved theorems. We show that this does indeed happen in the Hilbert space setting.

Hilbert Space Theorem. *Let H be a Hilbert space, and let M be any subspace of the Borel measurable functions from H to $\mathbb{R}^m$ that contains the space of linear functions and is contained in the space of locally Lipschitz functions. Let $X \subset H$ be a compact set with thickness exponent $\tau(X)$. Let μ be a Borel probability measure supported on X. For almost every $f \in M$, one has*

$$\dim_H(f(X)) \geqslant \min\left\{m, \frac{\dim_H(X)}{1+\tau(X)/2}\right\},$$

and, for $1 < q \leqslant 2$,

$$D_q^-(f(\mu)) \geqslant \min\left\{m, \frac{D_q^-(\mu)}{1+\tau(X)/2}\right\}. \tag{1.4}$$

As we have mentioned, examples in [12] preclude similar results for $0 \leqslant q < 1$ and $q > 2$. The case $q = 1$ is of interest because it corresponds to the commonly used notion of information dimension, in the following sense. In general, the limit (1.1) need not exist. However, $D_q^-(\mu)$ is a nonincreasing function of q and is continuous for $q \neq 1$ [1]. From this it follows that (1.3) and (1.4) hold for $q = 1$ if we define

$$D_1^-(\mu) = \lim_{q \to 1^+} D_q^-(\mu).$$

Next, we consider the sharpness of the Banach and Hilbert space theorems. In [13], the authors give an example of a compact subset X of Hausdorff dimension d in ℓ^p for $1 \leqslant p < \infty$ such that for all bounded linear functions $\pi : \ell^p \to \mathbb{R}^m$,

$$\dim_H(\pi(X)) \leqslant \frac{d}{1+d/q}$$

where $q = p/(p-1)$. In these cases, $\tau(X) = d$. Thus, the Hausdorff dimension parts of the Banach and Hilbert space theorems are sharp, in the sense that there is no better bound in terms of $\tau(X)$ that holds for all such spaces (notice that $q = 2$ for the separable Hilbert space ℓ^2 and $q \to 1$ as $p \to \infty$).

On the other hand, when $p = 1$, q is infinite, and the example in [13] does not rule out the possibility of a dimension preservation result for subsets of ℓ^1 of arbitrary thickness. We demonstrate that such a result is not possible by constructing a compact subset X of Hausdorff dimension d in ℓ^1 such that for all bounded linear functions $\pi : \ell^1 \to \mathbb{R}$,

$$\dim_H(\pi(X)) \leqslant \frac{d}{1+d/2}.$$

In light of this example, we are somewhat pessimistic regarding the existence of infinite-dimensional spaces for which a general dimension preservation theorem holds. It is thus natural to consider the following fundamental question. Suppose X represents the global attractor of a flow on a function space generated by an evolution equation. Under what hypotheses on the flow does one have $\tau(X) = 0$? If one assumes that the flow is sufficiently dissipative and smoothing, then X will have finite box dimension. We conjecture that similar *dynamical* hypotheses imply that $\tau(X) = 0$. Friz and Robinson [7] obtain a result of this type. They prove that if an attractor is uniformly bounded in the Sobolev space H^s on an appropriate bounded domain in $\mathbb{R}^m$, then its thickness is at most m/s. This result implies that certain attractors of the Navier-Stokes equations have thickness exponent zero. Roughly speaking, thickness is inversely proportional to smoothness.

Section 2 reviews prevalence, the dimension spectrum, and the thickness exponent. The main two theorems are presented and proved in Section 3. In Section 4 we describe the counterexample to the dimension preservation conjecture for subsets of ℓ^1 of arbitrary thickness.

2. Preliminaries

We discuss prevalence, the dimension spectrum, and the thickness exponent.

2.1. **Prevalence.** Mathematicians often use topological notions of genericity when formulating theorems in dynamical systems and topology. In topological terms, ‘generic’ refers to an open and dense subset of mappings, or to a countable intersection of such sets (a ‘residual’ subset). In finite-dimensional spaces, there exists considerable discord between the topological notion of genericity and the measure-theoretic notion of the size of a set (see [10, 21] for examples). Prevalence is intended to be a better analogue to “probability one” on function spaces where no Lebesgue or Haar measure exists.

To motivate the definition of prevalence on a Banach space B, consider how the notion of ‘Lebesgue almost every’ on $\mathbb{R}^n$ can be formulated in terms of the same notion on lower-dimensional spaces. Foliate $\mathbb{R}^n$ by k-dimensional planes, which by an appropriate choice of coordinates we think of as translations of $\mathbb{R}^k \subset \mathbb{R}^n$ by elements of $\mathbb{R}^{n-k}$. If ‘Lebesgue almost every’ translation of $\mathbb{R}^k$ intersects a Borel set $S \subset \mathbb{R}^n$ in full k-dimensional Lebesgue measure, then S has full n-dimensional Lebesgue measure by the Fubini theorem. If $\mathbb{R}^n$ is replaced by an infinite-dimensional space B, we cannot formulate the same condition because the space of translations of a k-dimensional subspace is infinite-dimensional. However, we can impose the stronger condition that *every* translation of the subspace intersects S in a set of full Lebesgue measure. A preliminary notion of prevalence is obtained by declaring that a Borel set $S \subset B$ is prevalent if there exists some finite k and some k-dimensional subspace V such that every translation of V intersects S in a set of full k-dimensional Lebesgue measure. In order to ensure that a countable intersection of prevalent sets is prevalent, we must enlarge the space of measures under consideration beyond Lebesgue measure supported on finite-dimensional subspaces.

Definition 2.1. A Borel set $S \subset B$ is said to be prevalent if there exists a measure μ on B such that

(1) $0 < \mu(C) < \infty$ for some compact subset C of B, and
(2) the set $S - x$ has full μ-measure (that is, the complement of $S - x$ has measure 0) for all $x \in B$.

A non-Borel set that contains a prevalent Borel set is also prevalent.

The measure μ may be a Lebesgue measure on a finite-dimensional subspace of B. More generally, one may think of μ as describing a family of perturbations in B. In this sense, S is prevalent if for all $x \in B$, choosing a perturbation at random with respect to μ and adding it to x yields a point in S with probability one. Prevalent sets share several of the desirable properties of residual sets. A prevalent subset of B is dense and the countable intersection of prevalent sets is prevalent. See [10] for details. One may formulate a notion of prevalence appropriate for spaces without a linear structure [15]. This notion applies to the space of diffeomorphisms of a compact smooth manifold.

2.2. **The Dimension Spectrum.** Let μ be a Borel probability measure on a metric space X. For $q \geqslant 0$ and $\epsilon > 0$ define

$$C_q(\mu, \epsilon) = \int_X [\mu(B(x, \epsilon))]^{q-1} \, d\mu(x)$$

where $B(x, \epsilon)$ is the open ball of radius ϵ centered at x.

Definition 2.2. For $q \geqslant 0$, $q \neq 1$, the lower and upper q-dimensions of μ are

$$D_q^-(\mu) = \liminf_{\epsilon \to 0} \frac{\log C_q(\mu, \epsilon)}{(q-1)\log(\epsilon)},$$
$$D_q^+(\mu) = \limsup_{\epsilon \to 0} \frac{\log C_q(\mu, \epsilon)}{(q-1)\log(\epsilon)}.$$

If $D_q^-(\mu) = D_q^+(\mu)$, their common value is denoted $D_q(\mu)$ and is called the q-dimension of μ.

For a measure μ such that $D_q(\mu)$ exists, the function $q \to D_q(\mu)$ is called the dimension spectrum of μ. For measures on $\mathbb{R}^n$, one encounters the following alternative definition of the dimension spectrum [8, 9, 22]. For $\epsilon > 0$, cover the support of μ with a grid of cubes with edge length ϵ. Let $N(\epsilon)$ be the number of cubes that intersect the support of μ, and let the measure of these cubes be $p_1, p_2, \ldots, p_{N(\epsilon)}$. Write

$$D_q^-(\mu) = \liminf_{\epsilon \to 0} \frac{\sum_{i=1}^{N(\epsilon)} p_i^q}{(q-1)\log(\epsilon)},$$
$$D_q^+(\mu) = \limsup_{\epsilon \to 0} \frac{\sum_{i=1}^{N(\epsilon)} p_i^q}{(q-1)\log(\epsilon)}.$$

For $q \geqslant 0$, $q \neq 1$, these limits are independent of the choice of ϵ-grids, and give the same values as Definition 2.2. See [23] for a proof of this equivalence for $q > 1$. The grid definition of the dimension spectrum is not appropriate for measures on general metric spaces. We therefore adopt Definition 2.2 as the natural notion in the general case.

A potential-theoretic definition of the lower q-dimension $D_q^-(\mu)$ for $q > 1$ is introduced in [12]. For $s \geqslant 0$ the s-potential of the measure μ at the point x is given by

$$\varphi_s(\mu, x) = \int_X |x-y|^{-s}\, d\mu(y).$$

Definition 2.3. The (s,q)-energy of μ, denoted $I_{s,q}(\mu)$, is given by

$$I_{s,q}(\mu) = \int_X [\varphi_s(\mu,x)]^{q-1}\, d\mu(x) = \int_X \left(\int_X \frac{d\mu(y)}{|x-y|^s} \right)^{q-1} d\mu(x).$$

For $q = 2$, the (s,q)-energy of μ reduces to the more standard notion of the s-energy of μ, written

$$I_s(\mu) = \int_X \varphi_s(\mu,x)\, d\mu(x) = \int_X \int_X \frac{d\mu(x) d\mu(y)}{|x-y|^s}.$$

Sauer and Yorke [26] show that the lower correlation dimension $D_2^-(\mu)$ can be expressed as

$$D_2^-(\mu) = \sup\{s : I_s(\mu) < \infty\}. \tag{2.1}$$

This characterization of $D_2^-(\mu)$ is used to establish the preservation of correlation dimension. The following proposition generalizes (2.1) to the lower-dimension spectrum for $q > 1$.

Proposition 2.4 ([12])**.** *If $q > 1$ and μ is a Borel probability measure, then*

$$D_q^-(\mu) = \sup\{s \geqslant 0 : I_{s,q}(\mu) < \infty\}.$$

2.3. **The Thickness Exponent.** Let B denote a Banach space.

Definition 2.5. The thickness exponent $\tau(X)$ of a compact set $X \subset B$ is defined as follows. Let $d(X,\epsilon)$ be the minimum dimension of all finite-dimensional subspaces $V \subset B$ such that every point of X lies within ϵ of V; if no such V exists, then $d(X,\epsilon) = \infty$. Let

$$\tau(X) = \limsup_{\epsilon \to 0} \frac{\log d(X,\epsilon)}{\log(1/\epsilon)}.$$

There is no general relationship between the thickness exponent and the Hausdorff dimension. A definitive statement may be made concerning the box-counting dimension D_0.

Lemma 2.6 ([13])**.** *Let $X \subset B$ be a compact set. Then $\tau(X) \leqslant D_0^+(X)$.*

3. Main Results

We begin with the main results for general Banach spaces.

Theorem 3.1. *Let B be a Banach space, and let M be any subspace of the Borel measurable functions from B to $\mathbb{R}^m$ that contains the bounded linear functions. Let $X \subset B$ be a compact set with thickness exponent $\tau(X)$, and let μ be a Borel probability measure supported on X. For almost every function $f \in M$,*

$$D_q^-(f(\mu)) \geqslant \min\left\{m, \frac{D_q^-(\mu)}{1+\tau(X)}\right\}$$

for all $q \in (1, 2]$.

Corollary 3.2. *Assume in addition that M is contained in the space of locally Lipschitz functions, that $\tau(X) = 0$, and that $D_q(\mu)$ exists ($D_q^-(\mu) = D_q^+(\mu)$) for all $q \in (1, 2]$. Then for almost every function $f \in M$, $D_q(f(\mu))$ exists and equals $\min\{m, D_q(\mu)\}$ for all $q \in (1, 2]$.*

Remark 3.3. For $r \geqslant 1$, the space $M = C^r(B, \mathbb{R}^m)$ satisfies the hypotheses of Theorem 3.1 and Corollary 3.2.

The corollary follows immediately from Theorem 3.1 and the fact that for all μ and all locally Lipschitz f, $D_q^+(f(\mu)) \leqslant \min\{m, D_q^+(\mu)\}$.

Corollary 3.4. *Let B be a Banach space. Let $X \subset B$ be a compact set with thickness exponent $\tau(X)$. For almost every function $f \in M$,*

$$\dim_H(f(X)) \geqslant \min\left\{m, \frac{\dim_H(X)}{1+\tau(X)}\right\}. \tag{3.1}$$

Proof. Let $\mathcal{M}(X)$ denote the set of Borel probability measures on X. The Hausdorff dimension of X may be expressed in terms of the lower correlation dimension of measures supported on X via the variational principle [5]

$$\dim_H(X) = \sup_{\mu \in \mathcal{M}(X)} D_2^-(\mu).$$

For each $i \in \mathbb{N}$, there exists $\mu_i \in \mathcal{M}(X)$ such that $D_2^-(\mu_i) > \dim_H(X) - 1/i$. Applying Theorem 3.1, there exists a prevalent set $P_i \subset M$ of functions such that for $f \in P_i$,

$$D_2^-(f(\mu_i)) \geqslant \min\left\{m, \frac{D_2^-(\mu_i)}{1+\tau(X)}\right\}.$$

The set $\bigcap_{i=1}^\infty P_i$ is prevalent. For $f \in \bigcap_{i=1}^\infty P_i$, the bound (3.1) follows from the variational principle. $\square$

Proof of Theorem 3.1. Fix $1 < q \leqslant 2$. Let $L \subset M$ denote the space of bounded linear functions from B into $\mathbb{R}^m$. We construct a 'Banach brick' $Q \subset L$ of perturbations and a probability measure λ on Q. For $f \in M$ and $\pi \in Q$, write $f_\pi = f + \pi$. Utilizing the potential-theoretic description of $D_q^-(\mu)$ for $1 < q \leqslant 2$, we must show that for any $f \in M$, $t > 0$, and $0 \leqslant s < \min\{m, t/(1+\tau(X))\}$,

$$I_{t,q}(\mu) < \infty \Rightarrow I_{s,q}(f_\pi(\mu)) < \infty \tag{3.2}$$

for λ-almost every $\pi \in Q$. The result follows because we can choose t arbitrarily close to $D_q^-(\mu)$.

We define the Banach brick Q as follows. For $j \in \mathbb{N}$, let $d_j = d(X, 2^{-j})$ and let $V_j \subset B$ be a subspace of dimension d_j such that every point of X lies within 2^{-j} of V_j. Fix $\sigma > \tau(X)$. By Definition 2.5 of $\tau(X)$, there exists $C_1 > 0$, depending only on X and σ, such that $d_j \leqslant C_1 2^{j\sigma}$. Let S_j be the closed unit ball in the dual space V_j' of V_j. There is no natural embedding of V_j' into B', but it follows from the Hahn-Banach theorem that there exists an isometric embedding of V_j' into B'. As such, we can think of S_j as a subset of B'. On the other hand, V_j' is linearly isomorphic to $\mathbb{R}^{d_j}$, and S_j corresponds to a convex set $U_j \subset \mathbb{R}^{d_j}$. The uniform (Lebesgue) probability measure on U_j induces a measure λ_j on S_j. Define the Banach brick Q by

$$Q = \left\{ \pi = (\pi_1, \dots, \pi_m) : \pi_i = \sum_{j=1}^{\infty} j^{-2} \phi_{ij} \text{ with } \phi_{ij} \in S_j \ \forall j \right\}.$$

Since each $S_j \subset B'$ is compact, $Q \subset L$ is compact. Let λ be the probability measure on Q that results from choosing the elements ϕ_{ij} randomly and independently with respect to the measures λ_j on the sets S_j. (While the term "brick" suggests that Q is the product of compact sets $j^{-2}S_j$ that are all transverse to each other, these sets may have nontrivial intersection, in which case Q and λ are still well-defined.)

Choose $\rho > \sigma > \tau(X)$. We will show that for $0 \leqslant s < m$,

$$I_{s(1+\rho),q}(\mu) < \infty \Rightarrow I_{s,q}(f_\pi(\mu)) < \infty$$

for λ-almost every $\pi \in Q$. Since ρ and σ can be arbitrarily close to $\tau(X)$, this implies (3.2). Computing the (s,q)-energy of $f_\pi(\mu)$, we have

$$\begin{aligned} I_{s,q}(f_\pi(\mu)) &= \int_{\mathbb{R}^m} \left[\int_{\mathbb{R}^m} \frac{df_\pi(\mu)(v)}{|u-v|^s} \right]^{q-1} df_\pi(\mu)(u) \\ &= \int_B \left[\int_B \frac{d\mu(y)}{|f_\pi(x) - f_\pi(y)|^s} \right]^{q-1} d\mu(x). \end{aligned}$$

Integrating the energy over Q and using the Fubini/Tonelli theorem and the fact that $0 < q-1 \leqslant 1$, we have

$$\begin{aligned} \int_Q I_{s,q}(f_\pi(\mu))\, d\lambda(\pi) &= \int_Q \int_B \left[\int_B \frac{d\mu(y)}{|f_\pi(x) - f_\pi(y)|^s} \right]^{q-1} d\mu(x) d\lambda(\pi) \\ &= \int_B \int_Q \left[\int_B \frac{d\mu(y)}{|f_\pi(x) - f_\pi(y)|^s} \right]^{q-1} d\lambda(\pi) d\mu(x) \\ &\leqslant \int_B \left[\int_Q \int_B \frac{d\mu(y)}{|f_\pi(x) - f_\pi(y)|^s}\, d\lambda(\pi) \right]^{q-1} d\mu(x) \\ &= \int_B \left[\int_B \left(\int_Q \frac{d\lambda(\pi)}{|f_\pi(x) - f_\pi(y)|^s} \right) d\mu(y) \right]^{q-1} d\mu(x). \end{aligned}$$

We now estimate the interior integral.

Lemma 3.5 (Perturbation Lemma). *If $s < m$, there exists a constant C_2 depending only on s, σ, and ρ, such that for all x, $y \in X$,*

$$\int_Q \frac{d\lambda(\pi)}{|f_\pi(x) - f_\pi(y)|^s} \leqslant \frac{C_2}{\min\{|x-y|, 1\}^{s(1+\rho)}}.$$

Proof. Set $\zeta = \min\{|x-y|, 1\}$. Choose $j \in \mathbb{N}$ such that $2 - \log_2 \zeta \leqslant j \leqslant 3 - \log_2 \zeta$. There exist points $\gamma_j(x)$ and $\gamma_j(y)$ in V_j satisfying $|x - \gamma_j(x)| \leqslant 2^{-j}$ and $|y - \gamma_j(y)| \leqslant 2^{-j}$. Estimating the distance between $\gamma_j(x)$ and $\gamma_j(y)$, we have

$$|\gamma_j(x) - \gamma_j(y)| \geqslant |x-y| - 2^{-j+1} \geqslant |x-y| - \frac{\zeta}{2} \geqslant \frac{|x-y|}{2}.$$

For $\pi \in Q$, write $\pi = \xi_j + j^{-2}\phi_j$ where $\phi_j = (\phi_{1j}, \ldots, \phi_{mj}) \in S_j^m$ and $\xi_j = (\xi_{1j}, \ldots, \xi_{mj})$ with

$$\xi_{ij} = \sum_{\substack{k \in \mathbb{N} \\ k \neq j}} k^{-2} \phi_{ik}$$

for each i. We fix ξ_j and integrate over $\phi_j \in S_j^m$. We have

$$\begin{aligned}
&\int_{S_j^m} \frac{d\lambda_j^m(\phi_j)}{|f_{\xi_j + j^{-2}\phi_j}(x) - f_{\xi_j + j^{-2}\phi_j}(y)|^s} \\
&= \int_{S_j^m} \frac{d\lambda_j^m(\phi_j)}{|f_{\xi_j}(x) - f_{\xi_j}(y) + j^{-2}\phi_j(x-y)|^s} \\
&\leqslant \int_{S_j^m} \frac{d\lambda_j^m(\phi_j)}{|j^{-2}\phi_j(x-y)|^s} \\
&= j^{2s} \int_{S_j^m} \frac{d\lambda_j^m(\phi_j)}{|\phi_j(x-y)|^s}.
\end{aligned}$$

Let $P \subset B'$ be the annihilator of $x - y$. By the Hahn-Banach theorem, there exists $\psi \in B'$ such that $\psi(x-y) = |x-y|$ and $\|\psi\|_{B'} = 1$. By restricting P and ψ to V_j, we may think of them as belonging to V_j', and hence also to $\mathbb{R}^{d_j}$. Notice that

$$\frac{|\psi(\gamma_j(x) - \gamma_j(y))|}{|\gamma_j(x) - \gamma_j(y)|} \geqslant \frac{|x-y| - \zeta/2}{|x-y| + \zeta/2} \geqslant \frac{|x-y|/2}{3|x-y|/2} = \frac{1}{3},$$

so $\|\psi\|_{V_j'} \geqslant \frac{1}{3}$. Let b be such that $\|b\psi\|_{V_j'} = 1$ and set $\widetilde{\psi} = b\psi$. By convexity, S_j contains the cones with base $P \cap S_j$ and vertices $\widetilde{\psi}$ and $-\widetilde{\psi}$. Let C_j be the union of this pair of cones and let $\widetilde{\lambda}_j$ denote the restriction of λ_j to C_j. We have

$$j^{2s} \int_{S_j^m} \frac{d\lambda_j^m(\phi_j)}{|\phi_j(x-y)|^s} \leqslant j^{2s} \left(\frac{\int_{C_j^m} \frac{d\widetilde{\lambda}_j^m(\phi_j)}{|\phi_j(x-y)|^s}}{\int_{C_j^m} d\widetilde{\lambda}_j^m(\phi_j)} \right). \tag{3.3}$$

Let W_j be the right side of (3.3). In order to estimate W_j, we use the $(P, \widetilde{\psi})$ foliation given by

$$C_{j,i} = \{C_{j,i} \cap (P + \alpha_i \widetilde{\psi}) : \alpha_i \in [-1, 1]\}$$

for each $i = 1, \ldots, m$.

Lemma 3.6 (Integral Asymptotics)**.** *Let $m \in \mathbb{N}$ and $s < m$. There exists a constant K, independent of $n \in \mathbb{N}$, such that*

$$\frac{\int_0^1 \cdots \int_0^1 \frac{(1-\alpha_1)^{n-1} \cdots (1-\alpha_m)^{n-1}}{|\alpha|^s} \, d\alpha_1 \cdots d\alpha_m}{\int_0^1 \cdots \int_0^1 (1-\alpha_1)^{n-1} \cdots (1-\alpha_m)^{n-1} \, d\alpha_1 \cdots d\alpha_m} \leqslant K n^s, \tag{3.4}$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$.

Proof. Since $e^{-z} \geqslant 1 - z$ for all real z, and the denominator of (3.4) is n^{-m}, the ratio of integrals in (3.4) is bounded above by

$$n^m \int_0^\infty \cdots \int_0^\infty \frac{\exp\left(-\sum_{i=1}^m \alpha_i (n-1)\right)}{|\alpha|^s} \, d\alpha_1 \cdots d\alpha_m.$$

Setting $u_i = \alpha_i(n-1)$, this becomes

$$n^m(n-1)^{s-m} \int_0^\infty \cdots \int_0^\infty \frac{\exp\left(-\sum_{i=1}^m u_i\right)}{|u|^s}\, du_1 \cdots du_m.$$

Since $|u|^{-s}$ is integrable in a neighborhood of 0 for $s < m$, the lemma is established. □

We are now in position to complete the proof of Lemma 3.5. Estimating the ratio of integrals in W_j using the $(P, \widetilde{\psi})$ foliation, it follows from Lemma 3.6 with $n = d_j \leqslant C_1 2^{j\sigma}$ that there exists K, independent of j, such that

$$\begin{aligned} W_j &\leqslant j^{2s} K |x-y|^{-s} \left(C_1 2^{j\sigma}\right)^s \\ &\leqslant K C_1^s j^{2s} |x-y|^{-s} \left(2^j\right)^{\sigma s} \\ &\leqslant K C_1^s j^{2s} |x-y|^{-s} \left(8\zeta^{-1}\right)^{\sigma s} \\ &\leqslant 8^{\sigma s} K C_1^s j^{2s} \zeta^{-s(1+\sigma)} \\ &\leqslant 8^{\sigma s} K C_1^s \left(3 - \log_2 \zeta\right)^{2s} \zeta^{-s(1+\sigma)}. \end{aligned}$$

Thus, since $\rho > 0$, there exists C_2 such that

$$W_j \leqslant \frac{C_2}{\zeta^{s(1+\rho)}}.$$

We have established that

$$\int_{S_j^m} \frac{d\lambda_j^m(\phi_j)}{|f_{\xi_j + j^{-2}\phi_j}(x) - f_{\xi_j + j^{-2}\phi_j}(y)|^s} \leqslant \frac{C_2}{\zeta^{s(1+\rho)}}$$

for all ξ_j, and hence by integrating over ξ_j that

$$\int_Q \frac{d\lambda(\pi)}{|f_\pi(x) - f_\pi(y)|^s} \leqslant \frac{C_2}{\zeta^{s(1+\rho)}}.$$

The proof of the perturbation lemma is complete. □

Returning to the proof of Theorem 3.1, recall that $0 \leqslant s < \min\{m, t/(1+\tau(X))\}$ and $\rho > \sigma > \tau(X)$ have been fixed. Applying the perturbation lemma, we have

$$\begin{aligned} \int_Q I_{s,q}(f_\pi(\mu))\, d\lambda(\pi) &\leqslant \int_B \left[\int_B \left(\int_Q \frac{d\lambda(\pi)}{|f_\pi(x) - f_\pi(y)|^s}\right) d\mu(y)\right]^{q-1} d\mu(x) \\ &\leqslant \int_B \left[\int_B \frac{C_2}{\min\{|x-y|, 1\}^{s(1+\rho)}}\, d\mu(y)\right]^{q-1} d\mu(x). \end{aligned}$$

Therefore,

$$I_{s(1+\rho),q}(\mu) < \infty \Rightarrow I_{s,q}(f_\pi(\mu)) < \infty$$

for λ-almost every $\pi \in Q$. Since ρ and σ can be arbitrarily close to $\tau(X)$, this implies (3.2) for fixed t. Because we can choose t arbitrarily close to $D_q^-(\mu)$, there exists a prevalent set $P_q \subset M$ such that for $f \in P_q$,

$$D_q^-(f(\mu)) \geqslant \min\left\{m, \frac{D_q^-(\mu)}{1+\tau(X)}\right\}.$$

Let $\{q_i\}$ be a countable dense subset of $(1,2]$. The set $\bigcap_{i=1}^{\infty} P_{q_i}$ is prevalent. For $f \in \bigcap_{i=1}^{\infty} P_{q_i}$, the continuity of D_q^- on $(1,2]$ implies that

$$D_q^-(f(\mu)) \geqslant \min\left\{m, \frac{D_q^-(\mu)}{1+\tau(X)}\right\}$$

for all $1 < q \leqslant 2$. $\square$

The proof of the perturbation lemma uses only the convexity of S_j. In specific cases, additional information about the structure of the dual space may lead to an improved perturbation lemma and hence to an improvement of the factor $1/(1+\tau(X))$. We establish such an improvement for Hilbert spaces.

Theorem 3.7. *Let H be a Hilbert space, and let M be any subspace of the Borel measurable functions from H to $\mathbb{R}^m$ that contains the bounded linear functions. Let $X \subset H$ be a compact set with thickness exponent $\tau(X)$, and let μ be a Borel probability measure supported on X. For almost every function $f \in M$,*

$$D_q^-(f(\mu)) \geqslant \min\left\{m, \frac{D_q^-(\mu)}{1+\tau(X)/2}\right\}$$

for all $q \in (1,2]$.

Corollary 3.8. *Let H be a Hilbert space. Let $X \subset H$ be a compact set with thickness exponent $\tau(X)$. For almost every function $f \in M$,*

$$\dim_H(f(X)) \geqslant \min\left\{m, \frac{\dim_H(X)}{1+\tau(X)/2}\right\}.$$

Remark 3.9. For the example from [13] discussed in the introduction, this Hausdorff dimension estimate is sharp.

Proof of Theorem 3.7. Let $L \subset M$ denote the space of bounded linear functions from H into $\mathbb{R}^m$. We must show that for any $f \in M$ and $0 \leqslant s < \min\{m, t/(1+\tau(X)/2)\}$,

$$I_{t,q}(\mu) < \infty \Rightarrow I_{s,q}(f_\pi(\mu)) < \infty$$

for λ-almost every $\pi \in Q$. The construction of the Hilbert brick Q follows that of the Banach brick. Notice that each S_j is isometric to a Euclidean ball. The dual space V_j' embeds canonically into $H' = H$: an element of V_j' acts on an element of H by composition with the orthogonal projection onto V_j. Let $\rho > \sigma > \tau(X)$. We will show that for $0 \leqslant s < m$,

$$I_{s(1+\rho),q}(\mu) < \infty \Rightarrow I_{s,q}(f_\pi(\mu)) < \infty$$

for λ-almost every $\pi \in Q$. The proof of this implication follows the argument given in the proof of Theorem 3.1. We only need to apply the following improved perturbation lemma. $\square$

Lemma 3.10 (Perturbation Lemma)**.** *If $s < m$, there exists a constant C_3, depending only on s, σ, and ρ, such that for all x, $y \in X$,*

$$\int_Q \frac{d\lambda(\pi)}{|f_\pi(x) - f_\pi(y)|^s} \leqslant \frac{C_3}{\min\{|x-y|, 1\}^{s(1+\rho/2)}}.$$

Proof. Set $\zeta = \min\{|x-y|, 1\}$. Select j as before and note that

$$\int_{S_j^m} \frac{d\lambda_j^m(\phi_j)}{|f_{\xi_j+j^{-2}\phi_j}(x) - f_{\xi_j+j^{-2}\phi_j}(y)|^s} \leqslant j^{2s} \int_{S_j^m} \frac{d\lambda_j^m(\phi_j)}{|\phi_j(\gamma_j(x) - \gamma_j(y))|^s}.$$

Lemma 3.11 (Integral Asymptotics)**.** *There exists $K > 0$, independent of $n \in \mathbb{N}$, such that for $s < m$,*

$$\frac{\int_0^1 \cdots \int_0^1 \frac{(1-\alpha_1^2)^{\frac{n-1}{2}} \cdots (1-\alpha_m^2)^{\frac{n-1}{2}}\, d\alpha_1 \cdots d\alpha_m}{|\alpha|^s}}{\int_0^1 \cdots \int_0^1 (1-\alpha_1^2)^{\frac{n-1}{2}} \cdots (1-\alpha_m^2)^{\frac{n-1}{2}}\, d\alpha_1 \cdots d\alpha_m} \leqslant K n^{\frac{s}{2}}.$$

Proof. The proof is similar to that of Lemma 3.6 and is left to the reader. □

Let P be the annihilator of $\gamma_j(x) - \gamma_j(y)$ in V_j'. Foliating S_j into leaves parallel to P and using Lemma 3.11 with $n = d_j \leqslant C_1 2^{j\sigma}$, we have

$$\begin{aligned}
& j^{2s} \int_{S_j^m} \frac{d\lambda_j^m(\phi_j)}{|\phi_j(\gamma_j(x) - \gamma_j(y))|^s} \\
& \leqslant K j^{2s} |\gamma_j(x) - \gamma_j(y)|^{-s} \left(C_1 2^{j\sigma}\right)^{s/2} \\
& \leqslant 2^s K j^{2s} |x-y|^{-s} \left(C_1 2^{j\sigma}\right)^{s/2} \\
& \leqslant 2^s K C_1^{s/2} j^{2s} |x-y|^{-s} \left(2^j\right)^{\sigma s/2} \\
& \leqslant 2^s K C_1^{s/2} j^{2s} |x-y|^{-s} \left(8\zeta^{-1}\right)^{\sigma s/2} \\
& \leqslant 2^s 8^{\sigma s/2} K C_1^{s/2} j^{2s} \zeta^{-s(1+\sigma/2)} \\
& \leqslant C_3 \zeta^{-s(1+\rho/2)}
\end{aligned}$$

for some $C_3 > 0$. We have established that

$$\int_{S_j^m} \frac{d\lambda_j^m(\phi_j)}{|f_{\xi_j+j^{-2}\phi_j}(x) - f_{\xi_j+j^{-2}\phi_j}(y)|^s} \leqslant \frac{C_3}{\zeta^{s(1+\rho/2)}}$$

for all ξ_j, and hence by integrating over ξ_j that

$$\int_Q \frac{d\lambda(\pi)}{|f_\pi(x) - f_\pi(y)|^s} \leqslant \frac{C_3}{\zeta^{s(1+\rho/2)}}.$$

The proof of the perturbation lemma is complete. □

4. Nonpreservation of Hausdorff Dimension

Theorems 3.1 and 3.7 are sharp in the following sense. Given $d > 0$, $1 \leqslant p \leqslant \infty$, and a positive integer m, there is a compact subset X of Hausdorff dimension d in ℓ^p such that for all bounded linear functions $\pi : \ell^p \to \mathbb{R}^m$,

$$\dim_H(\pi(X)) \leqslant \frac{d}{1 + d/q},$$

where $q = p/(p-1)$ [13]. The cases $p = \infty$ and $p = 2$ show respectively that Theorems 3.1 and 3.7 are sharp for bounded linear functions on these particular Banach spaces. On the other hand, this class of examples does not rule out a dimension preservation result in ℓ^1.

Here we construct a compact subset X of Hausdorff dimension d in ℓ^1 such that for all bounded linear functions $\pi : \ell^1 \to \mathbb{R}$,

$$\dim_H(\pi(X)) \leqslant \frac{d}{1+d/2}.$$

Let $\{e_i\}$ be the standard basis of ℓ^1, and let $\lambda = 2^{-1/d}$. Consider the inductively constructed sets X_k, defined as follows. Let $X_0 = \{0\}$ and $X_1 = \{\pm p\}$, where

$$p = \frac{1}{2}(e_1 - e_2).$$

For the next step, construct the two points

$$p_0 = \frac{\lambda}{4}(e_3 - e_4 + e_5 - e_6), \text{ and}$$
$$p_1 = \frac{\lambda}{4}(e_3 + e_4 - e_5 - e_6).$$

Attach these points to the nodes of X_1, forming the set

$$X_2 = \{p \pm p_0,\ -p \pm p_1\}.$$

We now describe the construction of X_{k+1} given X_k. Let

$$\alpha_k = 1 + \sum_{i=0}^{k-1} 2^{2^i}.$$

Define the collection of 2^k points

$$\left\{p_{\beta_1\beta_2\cdots\beta_k} : \beta_1, \beta_2, \ldots, \beta_k \in \{0,1\}\right\}$$

by setting

$$p_{\beta_1\beta_2\cdots\beta_k} = \frac{\lambda^k}{2^{2^k}} \sum_{i=0}^{2^{2^k}-1} (-1)^{\left[\frac{i}{2^{\gamma_{\beta_1\cdots\beta_k}}}\right]} e_{\alpha_k+i},$$

where $\gamma_{\beta_1\cdots\beta_k}$ is the integer in $[0, 2^k)$ whose binary representation is $\beta_1 \cdots \beta_k$; that is,

$$\gamma_{\beta_1\cdots\beta_k} = \beta_1 2^{k-1} + \beta_2 2^{k-2} + \cdots + \beta_k.$$

Notice that $\|p_{\beta_1\cdots\beta_k}\|_{\ell^1} = \lambda^k$. Attach these points to the nodes of X_k, forming

$$X_{k+1} = \left\{(-1)^{\beta_1} p + (-1)^{\beta_2} p_{\beta_1} + \cdots + (-1)^{\beta_{k+1}} p_{\beta_1\cdots\beta_k} : \beta_1, \ldots, \beta_{k+1} \in \{0,1\}\right\}.$$

Figure 1 illustrates the third step in the construction. Let X be the set of all limit points of

$$\bigcup_{k=0}^{\infty} X_k.$$

Equivalently,

$$X = \left\{(-1)^{\beta_1} p + (-1)^{\beta_2} p_{\beta_1} + (-1)^{\beta_3} p_{\beta_1\beta_2} + \cdots : \beta_1, \beta_2, \beta_3, \ldots \in \{0,1\}\right\}.$$

Proposition 4.1. *For the set $X \subset \ell^1$ constructed above,*

$$\dim_H(X) = D_0^+(X) = \frac{\log 2}{\log(1/\lambda)} = d.$$

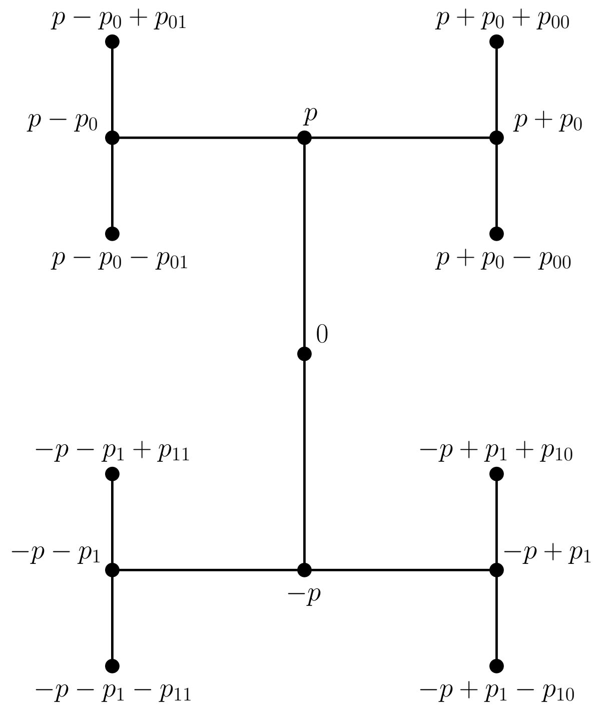

FIGURE 1. The sets X_0, X_1, X_2, and X_3 consist of the nodes of the binary tree above.

Proof. The set X can be covered by 2^k balls of radius $\lambda^k/(1-\lambda)$ centered at the points of X_k, so $\dim_H(X) \leqslant D_0^+(X) \leqslant d$. To show that $\dim_H(X) \geqslant d$, we apply Frostman's lemma [5, 20]. The binary tree X may be identified with the set of binary strings $S = \{\beta = \beta_1\beta_2\beta_3 \cdots : \beta_1, \beta_2, \beta_3, \ldots \in \{0,1\}\}$. Consider the measure μ on X induced by the uniform probability measure on S. Since every two points in X corresponding to different initial strings $\beta_1 \cdots \beta_k \beta_{k+1}$ and $\beta_1 \cdots \beta_k \beta'_{k+1}$ must lie at least $2\lambda^k$ apart, the measure of a ball of radius less than λ^k is at most the measure of all strings in S starting with a given $\beta_1 \cdots \beta_{k+1}$, which is $2^{-(k+1)} = (\lambda^k)^d/2$. By Frostman's lemma, $\dim_H(X) \geqslant d$. $\square$

Proposition 4.2. *For every bounded linear map $\pi : \ell^1 \to \mathbb{R}$,*

$$\dim_H(\pi(X)) \leqslant \frac{d}{1 + d/2}.$$

Proof. Let $s = d/(1 + d/2) = (1/d + 1/2)^{-1}$. Let $\pi \in \ell^\infty$ and assume $\|\pi\|_{\ell^\infty} = 1$. We will show for each $k \geqslant 0$ that $\pi(X)$ can be covered by a collection of 2^k intervals $\mathcal{C}_k = \{I_0, I_1, \ldots, I_{2^k-1}\}$ such that

$$\lim_{k\to\infty} \max_{I \in \mathcal{C}_k} \operatorname{diam}(I) = 0$$

and

$$\sum_{j=0}^{2^k-1} \operatorname{diam}(I_j)^s$$

remains bounded as $k \to \infty$. It then follows that the s-dimensional Hausdorff measure of $\pi(X)$ is finite, and therefore that the Hausdorff dimension of $\pi(X)$ is at most s, as desired. The proposition is trivially true if $s \geqslant 1$, so assume henceforth that $s < 1$. Then by convexity,

$$2^{-k} \sum_{j=0}^{2^k-1} \operatorname{diam}(I_j)^s \leqslant \left(2^{-k} \sum_{j=0}^{2^k-1} \operatorname{diam}(I_j) \right)^s,$$

so it suffices to show that

$$\begin{aligned} 2^{-k} \sum_{j=0}^{2^k-1} \operatorname{diam}(I_j) &\leqslant C_4 2^{-k/s} \\ &= C_4 2^{-k/2-k/d} \\ &= C_4 2^{-k/2} \lambda^k \end{aligned}$$

for some constant C_4 independent of k.

Each interval I_j will be the convex hull of the image under π of the part P_j of X corresponding to point j in X_k. As in the proof of Proposition 4.1, P_j is contained in a ball of radius $\lambda^k/(1-\lambda)$. Thus in effect, we want to show that on average (over j), π contracts P_j by a factor proportional to $2^{-k/2}$. Recall that

$$N_k = \left\{ (-1)^{\beta_{k+1}} p_{\beta_1 \cdots \beta_k} : \beta_1, \ldots, \beta_{k+1} \in \{0,1\} \right\}$$

is the set of points used to perturb the 2^k points of X_k to form the 2^{k+1} points of X_{k+1}. We seek an asymptotic bound on the quantity

$$Z_k = \sup_{\|\pi\|_{\ell^\infty}=1} \frac{1}{2^{k+1}} \sum_{s \in N_k} \frac{|\pi(s)|}{\|s\|_{\ell^1}} = \sup_{\|\pi\|_{\ell^\infty}=1} \frac{1}{2^{k+1}} \sum_{s \in N_k} \frac{|\pi(s)|}{\lambda^k}.$$

Lemma 4.3. *There exists $C_5 > 0$ such that $Z_k \leqslant C_5 2^{-k/2}$.*

Proof. For each $\beta_1 \cdots \beta_k \in \{0,1\}^k$, N_k contains $p_{\beta_1 \cdots \beta_k}$ and $-p_{\beta_1 \cdots \beta_k}$. Define

$$N_k^+ = \left\{ p_{\beta_1 \cdots \beta_k} : \beta_1, \ldots, \beta_k \in \{0,1\} \right\}.$$

We reindex the elements of N_k^+ by $\gamma_{\beta_1 \cdots \beta_k}$, obtaining $N_k^+ = \{p_i : i = 0, \ldots, 2^k - 1\}$. For each $\pi = (\pi_i) \in \ell^\infty$, there exists a permutation σ such that $\pi_\sigma = (\pi_{\sigma(i)})$ satisfies the positivity condition

$$\pi_\sigma(p_i) \geqslant 0$$

for all $i = 0, \ldots, 2^k - 1$. Therefore, we express Z_k in terms of N_k^+, yielding

$$Z_k = \sup_{\|\pi\|_{\ell^\infty}=1} \frac{1}{2^k} \sum_{i=0}^{2^k-1} \frac{\pi(p_i)}{\lambda^k}.$$

Think of the points of N_k^+ as the rows of a $2^k \times 2^{2^k}$ matrix. The entry in row i, column j of this matrix (starting the numbering at $i = 0$ and $j = 0$) is

$$p_{ij} = \frac{\lambda^k}{2^{2^k}} (-1)^{\left[\frac{j}{2^i}\right]} e_{\alpha_k + j}.$$

Let (s_{ij}) be the associated matrix of signs, defined by

$$s_{ij} = (-1)^{\left[\frac{j}{2^i}\right]}.$$

The set of columns of (s_{ij}) maps bijectively onto the set of vectors

$$\big\{\big((-1)^{\rho_1},\dots,(-1)^{\rho_k}\big) : \rho_1,\dots,\rho_k \in \{0,1\}\big\}. \tag{4.1}$$

We construct an element $\pi^* \in \ell^\infty$ as follows. For $0 \leqslant j < 2^{2^k}$, set

$$\pi^*_{\alpha_k+j} = \begin{cases} 1, & \text{if } \sum_{i=0}^{2^k-1} p_{ij} \geqslant 0; \\ -1, & \text{if } \sum_{i=0}^{2^k-1} p_{ij} < 0, \end{cases}$$

and set $\pi^*_l = 0$ for $l < \alpha_k$ and $l \geqslant \alpha_k + 2^{2^k}$. Writing

$$r_{ij} = s_{ij} e_{\alpha_k+j} \text{ and } r_i = \sum_{j=0}^{2^{2^k}-1} r_{ij},$$

we have

$$Z_k = \frac{1}{2^k}\sum_{i=0}^{2^k-1}\frac{\pi^*(p_i)}{\lambda^k} = \frac{1}{2^k 2^{2^k}}\sum_{i=0}^{2^k-1}\pi^*(r_i).$$

Since the columns of (s_{ij}) correspond bijectively to (4.1), Z_k may be related to the expected value of a binomially distributed random variable. Let Y be a binomial random variable such that for $0 \leqslant m \leqslant 2^k$, the probability that $Y = m$ is given by

$$\binom{2^k}{m}\left(\frac{1}{2}\right)^{2^k}.$$

Summing over m, we have

$$\begin{aligned} Z_k &= \frac{1}{2^k 2^{2^k}}\sum_{i=0}^{2^k-1}\pi^*(r_i) \\ &= \frac{1}{2^k 2^{2^k}}\sum_{m=0}^{2^k}\binom{2^k}{m}|2^k - 2m| \\ &= \frac{1}{2^{2^k}}\sum_{m=0}^{2^k}\binom{2^k}{m}|1 - 2m/2^k| \\ &= E[|1 - 2Y/2^k|], \end{aligned}$$

where $E[\cdot]$ denotes the expectation. By the central limit theorem, there exists $C_5 > 0$ such that

$$E[|1 - 2Y/2^k|] \leqslant C_5 2^{-k/2}.$$

The proof of Lemma 4.3 is complete. □

Returning to the proof of Proposition 4.2, we show that for each $k \geqslant 0$, $\pi(X)$ can be covered by 2^k intervals $I_0, \dots, I_{2^k-1}$ such that

$$2^{-k}\sum_{j=0}^{2^k-1}\operatorname{diam}(I_j) \leqslant C_4 2^{-k/2}\lambda^k$$

for some constant C_4 independent of k. Fix $k \geqslant 0$. For each string $\beta_1 \cdots \beta_k$, the subtree

$$\begin{aligned} X^{\beta_1 \cdots \beta_k} = \big\{ &(-1)^{\beta_1} p + (-1)^{\beta_2} p_{\beta_1} + \cdots + (-1)^{\beta_k} p_{\beta_1 \cdots \beta_{k-1}} + (-1)^{\beta_{k+1}} p_{\beta_1 \cdots \beta_k} \\ &+ (-1)^{\beta_{k+2}} p_{\beta_1 \cdots \beta_{k+1}} + \cdots : \beta_{k+1}, \beta_{k+2}, \ldots \in \{0,1\} \big\} \end{aligned}$$

can be covered by an interval $I_j = I_{\gamma_{\beta_1 \cdots \beta_k}}$ containing

$$\pi \left((-1)^{\beta_1} p + (-1)^{\beta_2} p_{\beta_1} + \cdots + (-1)^{\beta_k} p_{\beta_1 \cdots \beta_{k-1}} \right)$$

of length

$$\sum_{i=1}^{\infty} \sum_{\beta_{k+1} \cdots \beta_{k+i}} \left| \pi \left((-1)^{\beta_{k+i}} p_{\beta_1 \cdots \beta_{k+i-1}} \right) \right| .$$

Applying Lemma 4.3, we have

$$\begin{aligned} 2^{-k} \sum_{j=0}^{2^k - 1} \operatorname{diam}(I_j) &= 2^{-k} \sum_{j=0}^{2^k-1} \sum_{i=1}^{\infty} \sum_{\beta_{k+1} \cdots \beta_{k+i}} \left| \pi \left((-1)^{\beta_{k+i}} p_{\beta_1 \cdots \beta_{k+i-1}} \right) \right| \\ &= \sum_{i=1}^{\infty} 2^{-k} \sum_{j=0}^{2^k-1} \sum_{\beta_{k+1} \cdots \beta_{k+i}} \left| \pi \left((-1)^{\beta_{k+i}} p_{\beta_1 \cdots \beta_{k+i-1}} \right) \right| \\ &\leqslant \sum_{n=0}^{\infty} 2^{n+1} \lambda^{k+n} \cdot C_5 \cdot 2^{-(k+n)/2} \\ &= 2 C_5 \lambda^k 2^{-k/2} \sum_{n=0}^{\infty} \left(\sqrt{2} \lambda \right)^n . \end{aligned}$$

The assumption that $s < 1$ implies that $\lambda < 1/\sqrt{2}$. Setting

$$C_4 = 2 C_5 \sum_{n=0}^{\infty} \left(\sqrt{2} \lambda \right)^n = \frac{2 C_5}{1 - \sqrt{2} \lambda},$$

we have

$$2^{-k} \sum_{j=0}^{2^k-1} \operatorname{diam}(I_j) \leqslant C_4 2^{-k/2} \lambda^k. \tag{4.2}$$

Finally, (4.2) implies that $\operatorname{diam}(I_j) \leqslant C_4 2^{k(1-1/s)}$ for each $j = 0, \ldots, 2^k - 1$. $\square$

References

[1] C. Beck, *Upper and Lower Bounds on the Renyi Dimensions and the Uniformity of Multifractals*, Physica D **41** (1990), 67-78.

[2] A. Ben-Artzi, A. Eden, C. Foias and B. Nicolaenko, *Hölder Continuity for the Inverse of Mañé's Projection*, J. Math. Anal. Appl. **178** (1993), no. 1, 22-29.

[3] V. Chepyzhov and M. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society, 2002.

[4] A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, John Wiley & Sons, 1994.

[5] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, New York, 1990.

[6] C. Foias and E. Olson, *Finite Fractal Dimension and Hölder-Lipschitz Parametrization*, Indiana Univ. Math. J. **45** (1996), no. 3, 603-616.

[7] P. Friz and J. Robinson, *Smooth Attractors have Zero "Thickness,"* J. Math. Anal. Appl. **240** (1999), no. 1, 37-46.

[8] P. Grassberger, *Generalized Dimension of Strange Attractors*, Phys. Lett. **97A** (1983), 227-231.

[9] H.G. Hentschel and I. Procaccia, *The Infinite Number of Generalized Dimensions of Fractals and Strange Attractors*, Physica D **8** (1983), 435-444.

[10] B. Hunt, T. Sauer and J. Yorke, *Prevalence: A Translation-Invariant "Almost Every" on Infinite-Dimensional Spaces*, Bulletin of the AMS **27** (1992), 217-238.

[11] B. Hunt, T. Sauer and J. Yorke, *Prevalence: An Addendum*, Bulletin of the AMS **28** (1993), no. 2, 306-307.

[12] B. Hunt and V. Kaloshin, *How Projections Affect the Dimension Spectrum of Fractal Measures*, Nonlinearity **10** (1997), 1031-1046.

[13] B. Hunt and V. Kaloshin, *Regularity of Embeddings of Infinite-Dimensional Fractal Sets into Finite-Dimensional Spaces*, Nonlinearity **12** (1999), 1263-1275.

[14] B. Hunt, J. Kennedy, T.Y. Li, and H. Nusse, *SLYRB Measures: Natural Invariant Measures for Chaotic Systems*, Phys. D **170** (2002), no. 1, 50-71.

[15] V. Yu. Kaloshin, *Some Prevalent Properties of Smooth Dynamical Systems*, Proc. Steklov Inst. Math. **213** (1996), no. 2, 115-140.

[16] R. Kaufmann, *On Hausdorff Dimension of Projections*, Mathematika **15** (1968), 153-155.

[17] J.M. Marstrand, *Some Fundamental Geometrical Properties of Plane Sets of Fractional Dimension*, Proc. London Math. Soc. **4** (1954), 257-302.

[18] P. Mattila, *Hausdorff Dimension, Orthogonal Projections and Intersections with Planes*, Ann. Acad. Sci. Fennicae A **1** (1975), 227-244.

[19] P. Mattila, *Integralgeometric Properties of Capacities*, Trans. Am. Math. Soc. **266** (1981), 539-554.

[20] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge, 1995.

[21] J.C. Oxtoby, *Measure and Category*, Springer, New York, 1971.

[22] A. Rényi, *Probability Theory*, North-Holland, Amsterdam, 1970.

[23] R.H. Riedi, *An Improved Multifractal Formalism and Self-Affine Measures*, J. Math. Anal. Appl. **189** (1995), 462-490.

[24] J. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge University Press, 2001.

[25] T. Sauer, J.A. Yorke, and M. Casdagli, *Embedology*, Journal of Statistical Physics **65** (1991), nos. 3/4, 579-616.

[26] T. Sauer and J. Yorke, *Are the Dimensions of a Set and its Image Equal Under Typical Smooth Functions?*, Ergodic Theory and Dynamical Systems **17** (1997), no. 4, 941-956.

[27] G. Sell and Y. You, *Dynamics of Evolutionary Equations*, Springer, 2002.

[28] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer, 1997.

[29] L.-S. Young, *What are SRB measures, and which dynamical systems have them? Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.*, Journal of Statistical Physics **108** (2002), no. 5-6, 733-754.

Department of Mathematics, University of Maryland, College Park, MD 20742
E-mail address, William Ott: ott@math.umd.edu
URL, William Ott: http://www.math.umd.edu/~ott

Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742
E-mail address, Brian Hunt: bhunt@ipst.umd.edu
URL, Brian Hunt: http://www.ipst.umd.edu/~bhunt

Department of Mathematics, California Institute of Technology, Pasadena, CA 91125
E-mail address, Vadim Kaloshin: kaloshin@its.caltech.edu