

Modified SIR Model and Nondimensionalization

Brian Hunt

University of Maryland

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Modified SIR Model

- A standard extension to the SIR model adds terms representing births and deaths that are proportional to the overall population.
- If we're modeling an adult subpopulation that is either infected or “at risk”, it may be more appropriate to add a net influx that is independent of the current susceptible/infectious/removed populations:

$$dS/dt = q - pSI$$

$$dI/dt = pSI - rI$$

$$dR/dt = rI$$

- The cumulative number of people infected is $I + R$, and the rate of new infections is pSI .

Units of Variables and Parameters

- The variables S , I and R have units of “population”. A unit of population could be one person, but sometimes other units are used; e.g., census data is often tabulated in units of thousands of people.
- The derivatives dS/dt , dI/dt , dR/dt have units of population/time. Therefore:
 - p has units of $1/(\text{population} \cdot \text{time})$.
 - q has units of population/time.
 - r has units of $1/\text{time}$.

Change of Parameters

- In the SI model, we developed new parameters N , λ , and δ that had units of population, 1/time, and time, respectively.
- This separated a parameter N that controlled the size of the outbreak from parameters λ and δ that controlled the speed and timing of the outbreak.
- Let's do something similar for our modified SIR model.
- If we let $N = S + I + R$, then N changes over time (if $q \neq 0$). To make N a constant, let's let $N = S(0) + I(0) + R(0)$.

Normalized/Nondimensional Parameters

- Let's make N be the only parameter whose units involve population.
- As before, let $\lambda = \rho N$; then λ has units of 1/time.
- Let's make the other parameters be “dimensionless” – i.e., be independent of the units of time and population.
- Let $\mu = q/(\lambda N)$; then μ is dimensionless.
- Let $\nu = r/\lambda$; then ν is dimensionless.

Model with New Parameters

- The modified SIR model with the new parameters is

$$dS/dt = \lambda(\mu N - SI/N)$$

$$dI/dt = \lambda(SI/N - \nu I)$$

$$dR/dt = \lambda\nu I$$

- This is a bit messier than before, but it allows us to isolate better the effects of changing a parameter.
- If we multiply N by a constant a , then multiplying S , I , R by the same constant yields a solution to the model with the same values of λ, μ, ν .
- If we change λ , this only changes the rate of the outbreak (not the size).

Interpretation of New Parameters

- N controls the size of the outbreak. However, it does not represent the total number who will eventually be infected.
- λ controls the rate of the outbreak.
- μ and ν are rates of the renewal (new susceptibles) and recovery/removal processes **relative** to the infection process.
- There are two more parameters that determine the relative sizes of $S(0)$, $I(0)$, and $R(0)$, whose sum we have called N .
- One of these parameters could be replaced by a parameter like δ , which for the SI model was the time at which $S(t) = I(t)$.

Questions to Consider

- What happens to $S(t)$, $I(t)$, $R(t)$, and the rate of new infections as $t \rightarrow \infty$?
- How much do the new processes (renewal, governed by q and μ , and recovery/removal, governed by r and ν) affect the model's ability to fit a data set? (It may help to consider separately the cases $q = \mu = 0$ and $r = \nu = 0$.)