SIR Model and Nonlinear Least Squares

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SI Model

- The SI model we discussed before is often written

$$dS/dt = -pS\mathcal{I}$$
$$d\mathcal{I}/dt = pS\mathcal{I}$$

  where S is the “susceptible” population – those at risk to become infected at a given time – and $\mathcal{I}$ is the infectious population. For this model the sum $S+\mathcal{I}$ remains constant over time; we called the sum N and substituted $S = N - \mathcal{I}$ in the second equation.

- The resulting solution was

$$\mathcal{I}(t) = \frac{N\mathcal{I}(0)}{\mathcal{I}(0) + [N - \mathcal{I}(0)]e^{-pNt}}$$

SIR Model

- The SIR model (Kermack & McKendrick, 1927) is

$$dS/dt = -pS\mathcal{I}$$
$$d\mathcal{I}/dt = pS\mathcal{I} - r\mathcal{I}$$
$$dR/dt = r\mathcal{I}$$

  where R (for “recovered” or “removed”) is the number of people who were infected but are no longer infectuous. In this case, $\mathcal{I} + R$ is the cumulative number of people infected.
- One can add a term to the first equation representing new arrivals to the susceptible population.
- There is no formula for the solutions.

Properties of Solution Families

- Each model's family of solutions has some properties that are useful for fitting parameters to data.
- A time-shifted solution is also a solution: If $\mathcal{I}(t)$ is a solution, then $\mathcal{I}(t + c)$ is also a solution (with a different initial condition). This is because the model is "autonomous" – no explicit t dependence.
- A rescaled solution is also a solution: If $\mathcal{I}(t)$ is a solution, then $a\mathcal{I}(bt)$ is a solution of the same model with different parameters.
- Given a data set and the graph of a solution $\mathcal{I}(t)$, we can try to shift and rescale the graph to fit the data.

Change of parameters for SI model solution

- We can rewrite

$$\mathcal{I}(t) = \frac{N}{1 + [N/\mathcal{I}(0) - 1]e^{-pNt}}$$
$$= \frac{N}{1 + e^{-\lambda(t-\delta)}} = Ng(\lambda(t - \delta))$$

where

$$\lambda = pN$$
$$\delta = \log[N/\mathcal{I}(0) - 1]/\lambda$$
$$g(x) = 1/(1 + e^{-x}).$$

Interpretation of new parameters

- If we find parameters N, λ, δ that fit the data, we can solve for the original parameters p and $\mathcal{I}(0)$. However, the new parameters may be more interesting in their own right.
- N is the total number of people who will be infected over the outbreak, according to the model.
- δ is the time at which $N/2$ people have been infected, and at which $d\mathcal{I}/dt$ peaks; it is more relevant than $\mathcal{I}(0)$ to the data and to the intepretation of the model.
- λ is the rate at which the outbreak unfolds; it represents the rate per unit time a single person is infecting others early in the outbreak.

Data Fitting Problems

- Given data points $[t_j, \mathcal{I}_j]$, where $\mathcal{I}_j$ is an estimate of the cumulative number of people infected at time t_j, we can try to minimize the sum of squared residuals

$$E_{\mathcal{I}}(N, \lambda, \delta) = \sum_{j=1}^{J} [\mathcal{I}_j - Ng(\lambda(t_j - \delta))]^2.$$

- If the data is $[t_j, y_j]$ where $t_j = j$ and y_j is the number of new diagnoses per unit time, then we can fit $d\mathcal{I}/dt$ to the data by minimizing

$$E_y(N, \lambda, \delta) = \sum_{j=1}^{J} [y_j - N\lambda g'(\lambda(t_j - \delta))]^2.$$

Partial Solution

- We have posed nonlinear least squares problems.
- Numerical methods for optimization can yield approximate minimizers N, λ, δ.
- We can make some progress algebraically, since E is a quadratic function of N. Minimizing E_I over N yields

$$N_{\lambda,\delta} = \sum_{j=1}^{J} \mathcal{I}_j g(\lambda(t_j - \delta)) \Bigg/ \sum_{j=1}^{J} [g(\lambda(t_j - \delta))]^2.$$

- Substituting and simplifying yields

$$E_{\mathcal{I}}(N_{\lambda,\delta}, \lambda, \delta) = \sum_{j=1}^{J} \mathcal{I}_j^2 - N_{\lambda,\delta} \sum_{j=1}^{J} \mathcal{I}_j g(\lambda(t_j - \delta)).$$

Simple Approaches to Minimizing E

- Fix one parameter (say δ) and compute $E(N_{\lambda,\delta}, \lambda, \delta)$ for various λ; look for the value of λ that minimizes E for the chosen value of δ. Then fix λ and adjust δ to make E as small as you can. Then go back and see if you can make E smaller by adjusting λ, etc.
- Make a contour plot of $E(N_{\lambda,\delta}, \lambda, \delta)$ over a range of plausible λ and δ values. Zoom in near the apparent minimum and make another contour plot, etc.
- These approaches can be automated, and of course there are more sophisticated approaches; the latter become important when there are more parameters and/or when the function to be minimized takes a very long time to compute.