Modeling Portfolios that Contain Risky Assets

Portfolio Models II: Long Portfolios

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Portfolio Models II: Long Portfolios

Long Portfolio Constraints. *Because the value of any portfolio with short positions has the potential to go negative, many investors will not hold a short position in any risky asset.* For these investors we consider only portfolios that hold either a long or neutral position in each risky asset. These so-called *long portfolios* satisfy the inequality constraints $\mathbf{f} \geq \mathbf{0}$.

Let Ω be the set of all long portfolios and Ω_μ be the set of all long portfolios with return rate mean μ. These sets are given by

$$\Omega = \left\{ \mathbf{f} \in \mathbb{R}^N \; : \; \mathbf{f} \geq \mathbf{0} \, , \; \mathbf{1}^{\mathsf{T}}\mathbf{f} = 1 \right\} ,$$
$$\Omega_\mu = \left\{ \mathbf{f} \in \Omega \; : \; \mathbf{m}^{\mathsf{T}}\mathbf{f} = \mu \right\} .$$

Clearly $\Omega_\mu \subset \Omega$ for every $\mu \in \mathbb{R}$.

We first consider the set Ω of all long portfolios. Let e_i denote the vector whose i^{th} entry is 1 while every other entry is 0. For every $\mathbf{f} \in \Omega$ we have

$$\mathbf{f} = \sum_{i=1}^{N} f_i \mathbf{e}_i \,,$$

where $f_i \geq 0$ for every $i = 1, \cdots, N$ and

$$\sum_{i=1}^{N} f_i = \mathbf{1}^{\mathsf{T}} \mathbf{f} = 1 \,.$$

This shows that Ω is simply all convex combinations of the vectors $\{\mathrm{e}_i\}_{i=1}^{N}$. Moreover, Ω is closed. Indeed, for any $\mathbf{f}$ in the closure of Ω there exists a sequence $\{\mathbf{f}_n\}_{n \in \mathbb{N}} \subset \Omega$ such that $\mathbf{f}_n \to \mathbf{f}$. Because $\mathbf{f}_n \geq \mathbf{0}$ and $\mathbf{1}^{\mathsf{T}} \mathbf{f}_n = 1$ for every $n \in \mathbb{N}$, we see that $\mathbf{f} \in \Omega$ because

$$\mathbf{f} = \lim_{n \to \infty} \mathbf{f}_n \geq \mathbf{0} \,, \qquad \mathbf{1}^{\mathsf{T}} \mathbf{f} = \lim_{n \to \infty} \mathbf{1}^{\mathsf{T}} \mathbf{f}_n = 1 \,.$$

Therefore Ω is a nonempty, closed, bounded, convex set.

We can visualize Ω when N is small. When $N = 2$ it is the line segment that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

When $N = 3$ it is the triangle that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

When $N = 4$ it is the tetrahedron that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

For general N it is the simplex that connects the unit vectors $\{\mathbf{e}_i\}_{i=1}^N$.

Remark. When $N = 4$ it is easy to check that the tetrahedron $\Omega \subset \mathbb{R}^4$ is the image of the tetrahedron $\Psi \subset \mathbb{R}^3$ given by

$$\Psi = \left\{ \mathbf{x} \in \mathbb{R}^3 \ : \ \mathbf{w}_k^\top \mathbf{x} \leq 1 \ \text{ for } k = 1, 2, 3, 4 \right\} ,$$

where

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \quad \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} , \quad \mathbf{w}_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} , \quad \mathbf{w}_4 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} ,$$

under the one-to-one affine mapping $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$\mathbf{f}(\mathbf{x}) = \frac{1}{4} \begin{pmatrix} 1 - \mathbf{w}_1^\top \mathbf{x} \\ 1 - \mathbf{w}_2^\top \mathbf{x} \\ 1 - \mathbf{w}_3^\top \mathbf{x} \\ 1 - \mathbf{w}_4^\top \mathbf{x} \end{pmatrix} .$$

Therefore the set Ω in $\mathbb{R}^4$ can be visualized in $\mathbb{R}^3$ as the tetrahedron Ψ.

We saw that the simplex Ω is a nonempty, closed, bounded, convex set. For every $\mu \in \mathbb{R}$ the set Ω_μ is the intersection of the simplex Ω with the hyperplane $\{\mathbf{f} \in \mathbb{R}^N : \mathbf{m}^\mathsf{T}\mathbf{f} = \mu\}$. *This intersection might be empty.*

We now derive a condition that μ must satisfy for Ω_μ to be nonempty. Let

$$\mu_{\mathsf{mn}} = \mathsf{min}\{m_i \,:\, i = 1, \cdots, N\},$$
$$\mu_{\mathsf{mx}} = \mathsf{max}\{m_i \,:\, i = 1, \cdots, N\}.$$

Because $\mathbf{f} \geq \mathbf{0}$ and $\mathbf{1}^\mathsf{T}\mathbf{f} = 1$, for every $\mathbf{f} \in \Omega_\mu$ we have the inequalities

$$\mu_{\mathsf{mn}} = \mu_{\mathsf{mn}}\mathbf{1}^\mathsf{T}\mathbf{f} = \mu_{\mathsf{mn}}\sum_{i=1}^N f_i \leq \sum_{i=1}^N m_i f_i = \mathbf{m}^\mathsf{T}\mathbf{f} = \mu\,,$$
$$\mu = \mathbf{m}^\mathsf{T}\mathbf{f} = \sum_{i=1}^N m_i f_i \leq \mu_{\mathsf{mx}}\sum_{i=1}^N f_i = \mu_{\mathsf{mx}}\mathbf{1}^\mathsf{T}\mathbf{f} = \mu_{\mathsf{mx}}\,.$$

Therefore if Ω_μ is nonempty then $\mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$.

Conversely, let $\mu \in [\mu_{\text{mn}}, \mu_{\text{mx}}]$ and set

$$\mathbf{f} = \frac{\mu_{\text{mx}}-\mu}{\mu_{\text{mx}}-\mu_{\text{mn}}}\,\mathbf{e}_{\text{mn}} + \frac{\mu-\mu_{\text{mn}}}{\mu_{\text{mx}}-\mu_{\text{mn}}}\,\mathbf{e}_{\text{mx}}\,.$$

where

$$\mathbf{e}_{\text{mn}} = \mathbf{e}_i \quad \text{for any } i \text{ that satisfies } m_i = \mu_{\text{mn}}\,,$$
$$\mathbf{e}_{\text{mx}} = \mathbf{e}_j \quad \text{for any } j \text{ that satisfies } m_j = \mu_{\text{mx}}\,.$$

Clearly $\mathbf{f} \geq \mathbf{0}$. Because $\mathbf{1}^{\mathsf{T}}\mathbf{e}_{\text{mn}} = \mathbf{1}^{\mathsf{T}}\mathbf{e}_{\text{mx}} = 1$, $\mathbf{m}^{\mathsf{T}}\mathbf{e}_{\text{mn}} = \mu_{\text{mn}}$, and $\mathbf{m}^{\mathsf{T}}\mathbf{e}_{\text{mx}} = \mu_{\text{mx}}$, we see that

$$\begin{aligned}
\mathbf{1}^{\mathsf{T}}\mathbf{f} &= \frac{\mu_{\text{mx}}-\mu}{\mu_{\text{mx}}-\mu_{\text{mn}}}\,\mathbf{1}^{\mathsf{T}}\mathbf{e}_{\text{mn}} + \frac{\mu-\mu_{\text{mn}}}{\mu_{\text{mx}}-\mu_{\text{mn}}}\,\mathbf{1}^{\mathsf{T}}\mathbf{e}_{\text{mx}} \\
&= \frac{\mu_{\text{mx}}-\mu}{\mu_{\text{mx}}-\mu_{\text{mn}}} + \frac{\mu-\mu_{\text{mn}}}{\mu_{\text{mx}}-\mu_{\text{mn}}} = 1\,, \\
\mathbf{m}^{\mathsf{T}}\mathbf{f} &= \frac{\mu_{\text{mx}}-\mu}{\mu_{\text{mx}}-\mu_{\text{mn}}}\,\mathbf{m}^{\mathsf{T}}\mathbf{e}_{\text{mn}} + \frac{\mu-\mu_{\text{mn}}}{\mu_{\text{mx}}-\mu_{\text{mn}}}\,\mathbf{m}^{\mathsf{T}}\mathbf{e}_{\text{mx}} \\
&= \frac{\mu_{\text{mx}}-\mu}{\mu_{\text{mx}}-\mu_{\text{mn}}}\,\mu_{\text{mn}} + \frac{\mu-\mu_{\text{mn}}}{\mu_{\text{mx}}-\mu_{\text{mn}}}\,\mu_{\text{mx}} = \mu\,.
\end{aligned}$$

Hence, $\mathbf{f} \in \Omega_\mu$. *Therefore if $\mu \in [\mu_{\text{mn}}, \mu_{\text{mx}}]$ then Ω_μ is nonempty.*

Remark. Recall from our last remark that when $N = 4$ the set $\Omega \subset \mathbb{R}^4$ is the image of the tetrahedron $\Psi \subset \mathbb{R}^3$ under the one-to-one affine mapping $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ given there. Hence, the set $\Omega_\mu \subset \mathbb{R}^4$ is the image under $\mathbf{f}$ of the intersection of the tetrahedron Ψ with the hyperplane H_μ given by

$$H_\mu = \left\{ \mathbf{x} \in \mathbb{R}^3 \; ; \; \mathbf{m}^\mathsf{T} \mathbf{f}(\mathbf{x}) = \mu \right\} .$$

Hence, the set Ω_μ in $\mathbb{R}^4$ can be visualized in $\mathbb{R}^3$ as the set $\Psi_\mu = \Psi \cap H_\mu$. Because $\mathbf{m}$ is arbitrary, H_μ can be any hyperplane in $\mathbb{R}^3$. Therefore Ψ_μ can be the intersection of the tetrahedron Ψ with any hyperplane in $\mathbb{R}^3$. When such an intersection is nonempty it can be either

1. a *point* that is a vertex of Ψ ,
2. a *line segment* that is an edge of Ψ ,
3. a *triangle* with vertices on edges of Ψ ,
4. a *convex quadrilateral* with vertices on edges of Ψ .

In general Ω_μ will be a *convex polytope* of dimension at most $N - 2$.

Long Frontiers. The set Ω in $\mathbb{R}^N$ of all long portfolios is associated with the set Σ in the $\sigma\mu$-plane of volatilities and return rate means given by

$$\Sigma = \left\{ (\sigma, \mu) \in \mathbb{R}^2 \ : \ \sigma = \sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}\,,\ \mu = \mathbf{m}^{\mathsf{T}}\mathbf{f}\,,\ \mathbf{f} \in \Omega \right\}.$$

The set Σ is the image in $\mathbb{R}^2$ of the simplex Ω in $\mathbb{R}^N$ under the mapping $\mathbf{f} \mapsto (\sigma, \mu)$. Because the set Ω is compact (closed and bounded) and the mapping $\mathbf{f} \mapsto (\sigma, \mu)$ is continuous, the set Σ is compact.

We have seen that the set Ω_μ of all long portfolios with return rate mean μ is nonempty if and only if $\mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$. Hence, Σ can be expressed as

$$\Sigma = \left\{ \left(\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}\,,\ \mu\right) \ : \ \mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]\,,\ \mathbf{f} \in \Omega_\mu \right\}.$$

The points on the boundary of Σ that correspond to those long portfolios that have less volatility than every other long portfolio with the same return rate mean is called the *long frontier.*

The point of the long frontier associated with $\mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ is $(\sigma_{\mathsf{lf}}(\mu), \mu)$ where $\sigma_{\mathsf{lf}}(\mu)$ is obtained by solving the constrainted minimization problem

$$\sigma_{\mathsf{lf}}(\mu)^2 = \min\left\{ \sigma^2 \;:\; (\sigma, \mu) \in \Sigma \right\} = \min\left\{ \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \Omega_\mu \right\}.$$

This problem can not be solved by Lagrange multipliers because of the inequality constraints $\mathbf{f} \geq \mathbf{0}$ associated with the set of long portfolios Ω_μ.

Because the function $\mathbf{f} \mapsto \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}$ is continuous over the compact set Ω_μ, *a minimizer exists.* Because $\mathbf{V}$ is positive definite, the function $\mathbf{f} \mapsto \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}$ is strictly convex over the convex set Ω_μ. Because the function $\mathbf{f} \mapsto \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}$ is strictly convex over the convex set Ω_μ, *the minimizer is unique.* If this unique minimizer is denoted by $\mathbf{f}_{\mathsf{lf}}(\mu)$ then the long frontier is given by the equation $\sigma = \sigma_{\mathsf{lf}}(\mu)$ over $\mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ where $\sigma_{\mathsf{lf}}(\mu)$ is given by

$$\sigma_{\mathsf{lf}}(\mu) = \sqrt{\mathbf{f}_{\mathsf{lf}}(\mu)^{\mathsf{T}}\mathbf{V}\mathbf{f}_{\mathsf{lf}}(\mu)}\,.$$

For every $\mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ the portfolio $\mathbf{f}_{\mathsf{lf}}(\mu)$ can be expressed as

$$\mathbf{f}_{\mathsf{lf}}(\mu) = \arg\min\Big\{ \tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \geq \mathbf{0}\,,\; \mathbf{1}^{\mathsf{T}}\mathbf{f} = 1\,,\; \mathbf{m}^{\mathsf{T}}\mathbf{f} = \mu \Big\}\,.$$

Here $\arg\min$ is read *“the argument that minimizes”.* It means that $\mathbf{f}_{\mathsf{lf}}(\mu)$ is the minimizer of the function $\mathbf{f} \mapsto \frac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}$ subject to the given constraints. Because the function being minimized is quadratic in $\mathbf{f}$ while the constraints are linear in $\mathbf{f}$, this is called a *quadratic programming problem*.

This problem can be solved for a particular $\mathbf{V}$, $\mathbf{m}$, and μ by using the Matlab command “quadprog”. In general quadprog$(\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d}, \mathbf{C}_{\mathsf{eq}}, \mathbf{d}_{\mathsf{eq}})$ returns the solution of the quadratic programming problem given by

$$\arg\min\Big\{ \tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{A}\mathbf{f} + \mathbf{b}^{\mathsf{T}}\mathbf{f} \;:\; \mathbf{C}\mathbf{f} \leq \mathbf{d}\,,\; \mathbf{C}_{\mathsf{eq}}\mathbf{f} = \mathbf{d}_{\mathsf{eq}} \Big\}\,,$$

where $\mathbf{A} \in \mathbb{R}^{N\times N}$ is Hermitian positive, $\mathbf{b} \in \mathbb{R}^{N}$, $\mathbf{C} \in \mathbb{R}^{M\times N}$, $\mathbf{d} \in \mathbb{R}^{M}$, $\mathbf{C}_{\mathsf{eq}} \in \mathbb{R}^{M_{\mathsf{eq}}\times N}$, and $\mathbf{d}_{\mathsf{eq}} \in \mathbb{R}^{M_{\mathsf{eq}}}$. Here M and M_{eq} are the number of inequality and equality constraints respectively.

By comparing this general quadratic programming problem with the one above it that yields $\mathbf{f}_{\mathsf{lf}}(\mu)$ for a given $\mathbf{V}$, $\mathbf{m}$, and μ, we see that

$$\mathbf{A} = \mathbf{V}\,, \quad \mathbf{b} = \mathbf{0}\,, \quad \mathbf{C} = -\mathbf{I}\,, \quad \mathbf{d} = \mathbf{0}\,, \quad \mathbf{C}_{\mathsf{eq}} = \begin{pmatrix} \mathbf{1}^{\mathsf{T}} \\ \mathbf{m}^{\mathsf{T}} \end{pmatrix}, \quad \mathbf{d}_{\mathsf{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix},$$

where $\mathbf{I}$ is the $N\times N$ identity. Here $M = N$ because $\mathbf{f} \geq \mathbf{0}$ gives N inequality constraints while $M_{\mathsf{eq}} = 2$ because $\mathbf{1}^{\mathsf{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathsf{T}}\mathbf{f} = \mu$ are two equality constraints. There are other ways to use quadprog to obtain $\mathbf{f}_{\mathsf{lf}}(\mu)$. Documentation for this command is easy to find on the web.

In practice $\mathbf{f}_{\mathsf{lf}}(\mu)$ can be obtained as the output f of a quadprog command that is formated as

$$\mathsf{f} = \mathsf{quadprog}(\mathsf{V}, \mathsf{z}, -\mathsf{I}, \mathsf{z}, \mathsf{C}, \mathsf{d})\,,$$

where the matrices V, I, and C, and vectors z and d are given by

$$\mathsf{V} = \mathbf{V}\,, \quad \mathsf{z} = \mathbf{0}\,, \quad \mathsf{I} = \mathbf{I}\,, \quad \mathsf{C} = \begin{pmatrix} \mathbf{1}^{\mathsf{T}} \\ \mathbf{m}^{\mathsf{T}} \end{pmatrix}, \quad \mathsf{d} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}.$$

The long frontier can be computed numerically with the Matlab command quadprog. First, partition the interval $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ as

$$\mu_{\mathsf{mn}} = \mu_0 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = \mu_{\mathsf{mx}} \,.$$

For example, set $\mu_k = \mu_{\mathsf{mn}} + k(\mu_{\mathsf{mx}} - \mu_{\mathsf{mn}})/n$ for a uniform partition. Second, compute σ_0 and σ_n from the minimization problems

$$\sigma_0 = \min\{\sqrt{v_{ii}} \,:\, m_i = \mu_0\} \;, \qquad \sigma_n = \min\{\sqrt{v_{ii}} \,:\, m_i = \mu_n\} \;.$$

Typically there is just one asset to consider in each of these problems. Third, for every $k = 1, \cdots, n-1$ use quadprog to compute $\mathbf{f}_{\mathsf{lf}}(\mu_k)$ and compute σ_k by

$$\sigma_k = \sigma_{\mathsf{lf}}(\mu_k) = \sqrt{\mathbf{f}_{\mathsf{lf}}(\mu_k)^{\mathsf{T}} \mathbf{V} \mathbf{f}_{\mathsf{lf}}(\mu_k)} \,.$$

Finally, “connect the dots” between the points $\{(\sigma_k, \mu_k)\}_{k=0}^{n}$ to build an approximation to the long frontier in the $\sigma\mu$-plane. Here n should be large enough to resolve the features of the long frontier.

When computing a long frontier, it helps to know some general properties of the function $\sigma_{\mathsf{lf}}(\mu)$. These include:

- $\sigma_{\mathsf{lf}}(\mu)$ is *continuous* over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$;
- $\sigma_{\mathsf{lf}}(\mu)$ is *strictly convex* over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$;
- $\sigma_{\mathsf{lf}}(\mu)$ is *piecewise hyperbolic* over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$.

This means that $\sigma_{\mathsf{lf}}(\mu)$ is built up from segments of hyperbolas that are connected at a finite number of *nodes* within $(\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}})$. At each of these nodes $\sigma_{\mathsf{lf}}(\mu)$ has either a jump discontinuity in its first derivative or a jump discontinuity in its second derivative.

General Portfolio with Two Risky Assets. Recall the portfolio of two risky assets with mean vector $\mathbf{m}$ and covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}.$$

Here we will assume that $m_1 < m_2$, so that $\mu_{\mathsf{mn}} = m_1$ and $\mu_{\mathsf{mx}} = m_2$. The frontier portfolios are

$$\mathbf{f}_{\mathsf{f}}(\mu) = \frac{1}{m_2 - m_1} \begin{pmatrix} m_2 - \mu \\ \mu - m_1 \end{pmatrix} \qquad \text{for } \mu \in \mathbb{R}.$$

Clearly $\mathbf{f}_{\mathsf{f}}(\mu) \geq \mathbf{0}$ if and only if $\mu \in [m_1, m_2] = [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$. Therefore

$$\mathbf{f}_{\mathsf{lf}}(\mu) = \mathbf{f}_{\mathsf{f}}(\mu) \qquad \text{for } \mu \in [m_1, m_2],$$

and the long frontier is determined by

$$\sigma_{\mathsf{lf}}(\mu) = \sigma_{\mathsf{f}}(\mu) = \sqrt{\mathbf{f}_{\mathsf{f}}(\mu)^{\mathsf{T}} \mathbf{V}\, \mathbf{f}_{\mathsf{f}}(\mu)} \qquad \text{for } \mu \in [m_1, m_2].$$

In this case $\sigma_{\mathsf{lf}}(\mu)$ is a segment of a hyperbola with no nodes.

Nested Approaches to Long Frontiers. The basis for these approaches is the fact that the long frontier lies on the left-hand side of the boundary of the set Σ, which is the image of the simplex Ω under the mapping $\mathbf{f} \mapsto (\sigma, \mu)$. *These approaches build up the long frontier from the long frontiers of successively larger subportfolios.*

Given any subportfolio consisting of n assets $P = \{i_1, \cdots, i_n\}$, let $\mathbf{m}_P$ and $\mathbf{V}_P$ be the n-vector and $n{\times}n$-matrix of the associated return rate means and covariances. The associated frontier portfolios are given by

$$\mathbf{f}_{\mathsf{f}}^{P}(\mu) = (\sigma_{\mathsf{mv}}^{P})^2 \mathbf{V}_P^{-1} \mathbf{1}_n + \frac{\mu - \mu_{\mathsf{mv}}^{P}}{(\nu_{\mathsf{as}}^{P})^2} \mathbf{V}_P^{-1} \left(\mathbf{m}_P - \mu_{\mathsf{mv}}^{P} \mathbf{1}_n \right) ,$$

where σ_{mv}^{P}, μ_{mv}^{P}, and ν_{as}^{P} are the frontier parameters associated with the subportfolio P and $\mathbf{1}_n$ is the n-vector with every entry equal to 1. Because $\mathbf{1}_n^{\mathsf{T}} \mathbf{V}_P^{-1} (\mathbf{m}_P - \mu_{\mathsf{mv}}^{P} \mathbf{1}_n) = 0$, and because $\mathbf{m}_P$ and $\mathbf{1}_n$ are not co-linear, the second term above has both positive and negative entries when $\mu \neq \mu_{\mathsf{mv}}^{P}$.

We can determine the values of μ (if any) for which these portfolios are long. Let $f_k(\mu)$ be the k^{th} entry of $\mathbf{f}_{\mathsf{f}}^P(\mu)$. By the foregoing discussion each $f_k(\mu)$ is an affine (linear) function of μ. Moreover, some of them will be decreasing and some of them will be increasing.

If $f_k(\mu)$ is decreasing then $f_k(\mu) \geq 0$ over $(-\infty, \overline{\eta}_k]$ where $f_k(\overline{\eta}_k) = 0$.

If $f_k(\mu)$ is increasing then $f_k(\mu) \geq 0$ over $[\underline{\eta}_k, \infty)$ where $f_k(\underline{\eta}_k) = 0$.

If $f_k(\mu)$ is a nonnegative constant then $f_k(\mu) \geq 0$ over $(-\infty, \infty)$.

If $f_k(\mu)$ is a negative constant then $f_k(\mu) \geq 0$ nowhere.

If the fourth case does not arise and $\underline{\eta}^P = \max\{\underline{\eta}_k\} \leq \min\{\overline{\eta}_k\} = \overline{\eta}^P$ *then the portfolios* $\mathbf{f}_{\mathsf{f}}^P(\mu)$ *are long if and only if* $\mu \in [\underline{\eta}^P, \overline{\eta}^P]$*.*

Remark. For every two asset subportfolio P the frontier portfolios $\mathbf{f}_{\mathsf{f}}^P(\mu)$ are long over the interval $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ — i.e. $\underline{\eta}^P = \mu_{\mathsf{mn}}$ and $\overline{\eta}^P = \mu_{\mathsf{mx}}$.

Consider every subportfolio P of N risky assets.

If $\underline{\eta}^P > \overline{\eta}^P$ or the fourth case arises then there is nothing to plot.

If $\underline{\eta}^P \leq \overline{\eta}^P$ then plot the frontier $\sigma = \sigma_{\mathsf{f}}^P(\mu)$ for $\mu \in [\underline{\eta}^P, \overline{\eta}^P]$. When there are three or more assets in P then at each of the endpoints $\mu = \underline{\eta}^P$ and $\mu = \overline{\eta}^P$ there are assets for which $f_k^P(\mu)$ vanishes at the endpoint. Then the frontier $\sigma = \sigma_{\mathsf{f}}^P(\mu)$ will be tangent and to the left of the frontier portfolio for the subportfolio of P with the vanishing assets removed.

The long frontier for the N risky assets is the left-most curves of all those that are plotted. There are $2^N - 1$ subportfolios to be considered, so this approach is only pratical if N is not too large.

Remark. For N risky assets there are $N!/(n!(N-n)!)$ subportfolios with n risky assets. For $N = 4$ there are 4 subportfolios with 3 assets and 6 subportfolios with 2 assets. For $N = 7$ there are 7 subportfolios with 6 assets, 21 subportfolios with 5 assets, 35 subportfolios with 4 assets, 35 subportfolios with 3 assets, and 21 subportfolios with 2 assets.

Remark. It is more efficient to begin by computing the frontier portfolios $\mathbf{f}_{\mathsf{f}}(\mu)$ for all N risky assets. If some of these frontier portfolios are long — i.e. if $\mathbf{f}_{\mathsf{f}}(\mu) \geq \mathbf{0}$ for some μ — then the long frontier can be constructed by the method in the next section. If there are no long portfolios on this frontier then we consider all subportfolios with $N-1$ risky assets — there are only N of them. If for any of these subportfolios P some of the frontier portfolios $\mathbf{f}_{\mathsf{f}}^{P}(\mu)$ are long then we can construct the long frontier associated with P by the method in the next section. Moreover, we no longer have to consider the smaller subportfolios of P. We then consider all subportfolios with $N-2$ risky assets that have not been eliminated from consideration and repeat the procedure.

Long Frontiers that Intersect Frontiers. The long frontier is given by $\sigma = \sigma_{\mathsf{lf}}(\mu)$ where the function $\sigma_{\mathsf{lf}}(\mu)$ is only defined over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$. This function is given by a finite list of formulas that can be obtain when N is not too large. Here we show how this can be done when the long frontier intersects the frontier.

Suppose that $\mathbf{f}_{\mathsf{f}}(\mu_0) \geq \mathbf{0}$ for some μ_0. The frontier portfolio distribution $\mathbf{f}_{\mathsf{f}}(\mu)$ can be expressed as

$$\mathbf{f}_{\mathsf{f}}(\mu) = \mathbf{f}_{\mathsf{f}}(\mu_0) + \frac{\mu - \mu_0}{\nu_{\mathsf{as}}^2} \mathbf{V}^{-1}\Big(\mathbf{m} - \mu_{\mathsf{mv}}\mathbf{1}\Big) \,.$$

Because $\mathbf{1}^{\mathsf{T}}\mathbf{V}^{-1}\Big(\mathbf{m} - \mu_{\mathsf{mv}}\mathbf{1}\Big) = b - \mu_{\mathsf{mv}} a = 0$, and because $\mathbf{1}$ and $\mathbf{m}$ are not co-linear, we see that the second term above has both positive and negative entries whenever $\mu \neq \mu_0$. Because $\mathbf{f}_{\mathsf{f}}(\mu_0) \geq \mathbf{0}$, the set of μ for which $\mathbf{f}_{\mathsf{f}}(\mu) \geq \mathbf{0}$ is satisfied must be a closed interval containing μ_0.

Remark. We might expect that $\mathbf{f}_{\text{mv}} \geq \mathbf{0}$ because the least risky position in a market should not require any assets to be held short. If that were the case it would be natural to take $\mu_0 = \mu_{\text{mv}}$. However, it is false that $\mathbf{f}_{\text{mv}} \geq \mathbf{0}$ for every positive definite $\mathbf{V}$. Indeed, for the case $N = 2$ we have

$$\mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}, \qquad \mathbf{V}^{-1} = \frac{1}{v_{11}v_{22} - v_{12}^2} \begin{pmatrix} v_{22} & -v_{12} \\ -v_{12} & v_{11} \end{pmatrix},$$

while $\mathbf{f}_{\text{mv}}$ is a positive multiple of

$$\mathbf{V}^{-1}\mathbf{1} = \frac{1}{v_{11}v_{22} - v_{12}^2} \begin{pmatrix} v_{22} - v_{12} \\ v_{11} - v_{12} \end{pmatrix}.$$

On one hand, $\mathbf{V}$ is positive definite if and only if $v_{11} > 0$, $v_{22} > 0$, and $v_{11}v_{22} > v_{12}^2$. On the other hand, if $\mathbf{V}$ is positive definite then $\mathbf{V}^{-1}\mathbf{1}$ has nonnegative entries if and only if $v_{11} \geq v_{12}$ and $v_{22} \geq v_{12}$. We can find many positive definite matrices for which these conditions do not hold.

Because $\mathbf{f}_{\mathsf{f}}(\mu_0) \geq \mathbf{0}$, the distributions of those frontier portfolios that are long portfolios are given by $\mathbf{f}_{\mathsf{f}}(\mu)$ where $\mu \in [\underline{\mu}_1, \overline{\mu}_1]$ with

$$\overline{\mu}_1 = \max\left\{\mu \in \mathbb{R} \,:\, \mathbf{f}_{\mathsf{f}}(\mu) \geq \mathbf{0} \quad \text{entrywise}\right\},$$
$$\underline{\mu}_1 = \min\left\{\mu \in \mathbb{R} \,:\, \mathbf{f}_{\mathsf{f}}(\mu) \geq \mathbf{0} \quad \text{entrywise}\right\}.$$

The long frontier coincides with the frontier for $\mu \in [\underline{\mu}_1, \overline{\mu}_1]$. That is to say,

$$\sigma_{\mathsf{lf}}(\mu) = \sigma_{\mathsf{f}}(\mu) \qquad \text{for } \mu \in [\underline{\mu}_1, \overline{\mu}_1] .$$

We can extend $\sigma_{\mathsf{lf}}(\mu)$ to the interval $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ by an iterative process. We will show how to do this for the right endpoint. The steps are analogous for the left endpoint. We initialize the iteration by setting

$$\overline{\mathbf{m}}_0 = \mathbf{m}\,, \qquad \overline{\mathbf{V}}_0 = \mathbf{V}\,, \qquad \sigma_{\overline{\mathsf{f}}_0}(\mu) = \sigma_{\mathsf{f}}(\mu)\,, \qquad \mathbf{f}_{\overline{\mathsf{f}}_0}(\mu) = \mathbf{f}_{\mathsf{f}}(\mu)\,.$$

Suppose we have extended $\sigma_{\mathsf{lf}}(\mu)$ to an interval with right endpoint $\overline{\mu}_k$. If $\overline{\mu}_k = \mu_{\mathsf{mx}}$ then we are done. Otherwise, let the vector $\overline{\mathbf{m}}_k$ and matrix $\overline{\mathbf{V}}_k$ be obtained from $\overline{\mathbf{m}}_{k-1}$ and $\overline{\mathbf{V}}_{k-1}$ by removing every entry with an index corresponding to an entry of $\mathbf{f}_{\overline{\mathsf{f}}_{k-1}}(\overline{\mu}_k)$ that is zero. In other words, let $\overline{\mathbf{m}}_k$ be the return rate mean vector and $\overline{\mathbf{V}}_k$ be the return rate covariance matrix after we drop from consideration every asset corresponding to an entry of $\mathbf{f}_{\overline{\mathsf{f}}_{k-1}}(\overline{\mu}_k)$ that is zero. (Typically only one asset will be dropped each time.)

Let $\sigma = \sigma_{\overline{\mathsf{f}}_k}(\mu)$ be the frontier of this reduced portfolio. The dimension of the associated frontier distribution $\mathbf{f}_{\overline{\mathsf{f}}_k}(\mu)$ is less than that of $\mathbf{f}_{\overline{\mathsf{f}}_{k-1}}(\mu)$ by the number of zero entries of $\mathbf{f}_{\overline{\mathsf{f}}_{k-1}}(\overline{\mu}_k)$. The entries of $\mathbf{f}_{\overline{\mathsf{f}}_k}(\overline{\mu}_k)$ are exactly the positive entries of $\mathbf{f}_{\overline{\mathsf{f}}_{k-1}}(\overline{\mu}_k)$. Therefore $\sigma_{\overline{\mathsf{f}}_k}(\mu)$ satisfies

$$\begin{aligned}\sigma_{\overline{\mathsf{f}}_k}(\overline{\mu}_k) &= \mathbf{f}_{\overline{\mathsf{f}}_k}(\overline{\mu}_k)^{\mathsf{T}}\overline{\mathbf{V}}_k\mathbf{f}_{\overline{\mathsf{f}}_k}(\overline{\mu}_k) \\ &= \mathbf{f}_{\overline{\mathsf{f}}_{k-1}}(\overline{\mu}_k)^{\mathsf{T}}\overline{\mathbf{V}}_{k-1}\mathbf{f}_{\overline{\mathsf{f}}_{k-1}}(\overline{\mu}_k) = \sigma_{\overline{\mathsf{f}}_{k-1}}(\overline{\mu}_k)\,.\end{aligned}$$

Because $\sigma_{\bar{\mathsf{f}}_k}(\mu)$ is associated with fewer assets, we also know that

$$\sigma_{\bar{\mathsf{f}}_k}(\mu) \geq \sigma_{\bar{\mathsf{f}}_{k-1}}(\mu) \qquad \text{for every } \mu\,.$$

Because these functions are equal at $\mu = \overline{\mu}_k$, we conclude that moreover

$$\sigma'_{\bar{\mathsf{f}}_k}(\overline{\mu}_k) = \sigma'_{\bar{\mathsf{f}}_{k-1}}(\overline{\mu}_k)\,.$$

Now let

$$\overline{\mu}_{k+1} = \max\Big\{\mu \in \mathbb{R} \,:\, \mathbf{f}_{\bar{\mathsf{f}}_k}(\mu) \geq \mathbf{0} \quad \text{entrywise}\Big\}\,.$$

Because $\mathbf{f}_{\bar{\mathsf{f}}_k}(\overline{\mu}_k) > \mathbf{0}$, it is clear that $\overline{\mu}_{k+1} > \overline{\mu}_k$. Finally, set

$$\sigma_{\mathsf{lf}}(\mu) = \sigma_{\bar{\mathsf{f}}_k}(\mu) \qquad \text{for } \mu \in [\overline{\mu}_k, \overline{\mu}_{k+1}]\,.$$

We have thereby extended $\sigma_{\mathsf{lf}}(\mu)$ to an interval with right endpoint $\overline{\mu}_{k+1}$, whereby we can return to the beginning of the iteration.

After applying the analogous iterative process to extend the left endpoint, you find that $\sigma_{\mathsf{lf}}(\mu)$ is expressed over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ as the list function

$$\sigma_{\mathsf{lf}}(\mu) = \begin{cases} \sigma_{\underline{\mathsf{f}}_k}(\mu) & \text{for } \mu \in [\underline{\mu}_{k+1}, \underline{\mu}_k]\,, \\ \sigma_{\mathsf{f}}(\mu) & \text{for } \mu \in [\underline{\mu}_1, \overline{\mu}_1]\,, \\ \sigma_{\overline{\mathsf{f}}_k}(\mu) & \text{for } \mu \in [\overline{\mu}_k, \overline{\mu}_{k+1}]\,. \end{cases}$$

This is strictly convex and continuously differentiable over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$. Its second derivative will have a jump discontinuity at each $\underline{\mu}_k$ and $\overline{\mu}_k$ that lies in $(\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}})$.

Remark. Here we will not show why the above algorithm for computing $\sigma_{\mathsf{lf}}(\mu)$ works. The proof is far more complicated than others in this course. The algorithm is straightforward to implement when N is not too large. When either N is large or no μ_0 exists then $\sigma_{\mathsf{lf}}(\mu)$ can be approximated numerically using a *primal-dual interior algorithm* for *convex optimization*. Such algorithms are taught in some graduate courses on optimization.

Long Frontier Portfolios. Associated with each of the distributions $\mathbf{f}_{\underline{\mathsf{f}}_k}(\mu)$ and $\mathbf{f}_{\overline{\mathsf{f}}_k}(\mu)$ of the reduced portfolios in the above construction we define the distributions $\underline{\mathbf{f}}_{\underline{\mathsf{f}}_k}(\mu)$ and $\overline{\mathbf{f}}_{\overline{\mathsf{f}}_k}(\mu)$ to be the N-vectors obtained by adding zero entries corresponding to assets that are not held by the respective reduced portfolios.

The distibutions associated with the long frontier portfolios are then given over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ by the list function

$$\mathbf{f}_{\mathsf{lf}}(\mu) = \begin{cases} \underline{\mathbf{f}}_{\underline{\mathsf{f}}_k}(\mu) & \text{for } \mu \in [\underline{\mu}_{k+1}, \underline{\mu}_k]\,, \\ \mathbf{f}_{\mathsf{f}}(\mu) & \text{for } \mu \in [\underline{\mu}_1, \overline{\mu}_1]\,, \\ \overline{\mathbf{f}}_{\overline{\mathsf{f}}_k}(\mu) & \text{for } \mu \in [\overline{\mu}_k, \overline{\mu}_{k+1}]\,, \end{cases}$$

This is continuous and piecewise linear over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$. Its first derivative will have a jump discontinuity at each $\underline{\mu}_k$ and $\overline{\mu}_k$ that lies in $(\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}})$.

Because $\mathbf{f}_{\mathsf{lf}}(\mu)$ is continuous and piecewise linear over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ with nodes $\underline{\mu}_k$ and $\overline{\mu}_k$ in $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$, it can be expressed in terms of the so-called *nodal portfolio distributions* given by

$$\underline{\mathbf{f}}_k = \underline{\mathbf{f}}_{\underline{\mathsf{f}}_k}(\underline{\mu}_k)\,, \qquad \overline{\mathbf{f}}_k = \overline{\mathbf{f}}_{\overline{\mathsf{f}}_k}(\overline{\mu}_k)\,.$$

Because

$$\underline{\mathbf{f}}_{k+1} = \underline{\mathbf{f}}_{\underline{\mathsf{f}}_k}(\underline{\mu}_{k+1})\,, \qquad \overline{\mathbf{f}}_{k+1} = \overline{\mathbf{f}}_{\overline{\mathsf{f}}_k}(\overline{\mu}_{k+1})\,,$$

by the two mutual fund property we have

$$\begin{aligned}
\underline{\mathbf{f}}_{\underline{\mathsf{f}}_k}(\mu) &= \frac{\underline{\mu}_{k+1} - \mu}{\underline{\mu}_{k+1} - \underline{\mu}_k}\,\underline{\mathbf{f}}_k + \frac{\mu - \underline{\mu}_k}{\underline{\mu}_{k+1} - \underline{\mu}_k}\,\underline{\mathbf{f}}_{k+1}\,, \\
\mathbf{f}_{\mathsf{f}}(\mu) &= \frac{\overline{\mu}_1 - \mu}{\overline{\mu}_1 - \underline{\mu}_1}\,\underline{\mathbf{f}}_1 + \frac{\mu - \underline{\mu}_1}{\overline{\mu}_1 - \underline{\mu}_1}\,\overline{\mathbf{f}}_1\,, \\
\overline{\mathbf{f}}_{\overline{\mathsf{f}}_k}(\mu) &= \frac{\overline{\mu}_{k+1} - \mu}{\overline{\mu}_{k+1} - \overline{\mu}_k}\,\overline{\mathbf{f}}_k + \frac{\mu - \overline{\mu}_k}{\overline{\mu}_{k+1} - \overline{\mu}_k}\,\overline{\mathbf{f}}_{k+1}\,.
\end{aligned}$$

Simple Portfolio with Three Risky Assets. Recall the portfolio of three risky assets with mean vector $\mathbf{m}$ and covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m - d \\ m \\ m + d \end{pmatrix}, \qquad \mathbf{V} = s^2 \begin{pmatrix} 1 & r & r \\ r & 1 & r \\ r & r & 1 \end{pmatrix}.$$

Here $m \in \mathbb{R}$, $d, s \in \mathbb{R}_+$, and $r \in (-\frac{1}{2}, 1)$, where the last condition is equivalent to the condition that $\mathbf{V}$ is positive definite given $s > 0$. Its frontier parameters are

$$\sigma_{\text{mv}} = \sqrt{\frac{1}{a}} = s\sqrt{\frac{1+2r}{3}}, \qquad \mu_{\text{mv}} = \frac{b}{a} = m,$$

$$\nu_{\text{as}} = \sqrt{c - \frac{b^2}{a}} = \frac{d}{s}\sqrt{\frac{2}{1-r}}.$$

Its minimum volatility portfolio is $\mathbf{f}_{\text{mv}} = \frac{1}{3}\mathbf{1}$, whereby we can take $\mu_0 = m$. Clearly $[\mu_{\text{mn}}, \mu_{\text{mx}}] = [m - d, m + d]$.

Its frontier is determined by

$$\sigma_{\mathsf{f}}(\mu) = s\sqrt{\frac{1+2r}{3} + \frac{1-r}{2}\left(\frac{\mu-m}{d}\right)^2} \qquad \text{for } \mu \in (-\infty, \infty)\,,$$

while the distribution of the frontier portfolio with return rate mean μ is

$$\mathbf{f}_{\mathsf{f}}(\mu) = \begin{pmatrix} \frac{1}{3} - \frac{\mu-m}{2d} \\ \frac{1}{3} \\ \frac{1}{3} + \frac{\mu-m}{2d} \end{pmatrix} = \begin{pmatrix} \frac{m+\frac{2}{3}d-\mu}{2d} \\ \frac{1}{3} \\ \frac{\mu-m+\frac{2}{3}d}{2d} \end{pmatrix}\,.$$

The frontier portfolio holds long postitions when $\mu \in (m - \frac{2}{3}d, m + \frac{2}{3}d)$. Therefore $[\underline{\mu}_1, \overline{\mu}_1] = [m - \frac{2}{3}d, m + \frac{2}{3}d]$ and the long frontier satisfies

$$\sigma_{\mathsf{lf}}(\mu) = \sigma_{\mathsf{f}}(\mu) \qquad \text{for } \mu \in [m - \tfrac{2}{3}d, m + \tfrac{2}{3}d]\,.$$

The distribution weight of first asset vanishes at the right endpoint while that of the third vanishes at the left endpoint.

In order to extend the long frontier beyond the right endpoint $\bar{\mu}_1 = m + \frac{2}{3}d$ to $\mu_{\mathrm{mx}} = m + d$ we reduce the portfolio by removing the first asset and set

$$\overline{\mathbf{m}}_1 = \begin{pmatrix} m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m \\ m + d \end{pmatrix}, \qquad \overline{\mathbf{V}}_1 = s^2 \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}.$$

Then

$$\overline{\mathbf{V}}_1^{-1} = \frac{1}{s^2(1-r^2)} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}, \qquad \overline{\mathbf{V}}_1^{-1}\mathbf{1} = \frac{1}{s^2(1+r)}\mathbf{1},$$

whereby

$$\bar{a}_1 = \mathbf{1}^{\mathsf{T}}\overline{\mathbf{V}}_1^{-1}\mathbf{1} = \frac{2}{s^2(1+r)}, \qquad \bar{b}_1 = \mathbf{1}^{\mathsf{T}}\overline{\mathbf{V}}_1^{-1}\overline{\mathbf{m}}_1 = \frac{2m+d}{s^2(1+r)},$$

$$\bar{c}_1 = \overline{\mathbf{m}}_1^{\mathsf{T}}\overline{\mathbf{V}}_1^{-1}\overline{\mathbf{m}}_1 = \frac{2m(m+d)}{s^2(1+r)} + \frac{d^2}{s^2(1-r^2)}.$$

The associated frontier parameters are

$$\sigma_{\mathsf{mv}_1} = \sqrt{\frac{1}{\bar{a}_1}} = s\sqrt{\frac{1+r}{2}}\,, \qquad \mu_{\mathsf{mv}_1} = \frac{\bar{b}_1}{\bar{a}_1} = m + \tfrac{1}{2}d\,,$$

$$\nu_{\mathsf{as}_1} = \sqrt{\bar{c}_1 - \frac{\bar{b}_1^{\,2}}{\bar{a}_1}} = \frac{d}{2s}\sqrt{\frac{2}{1-r}}\,,$$

whereby the frontier of the reduced portfolio is given by

$$\sigma_{\bar{\mathsf{f}}_1}(\mu) = s\sqrt{\frac{1+r}{2} + \frac{1-r}{2}\left(\frac{\mu - m - \frac{1}{2}d}{\frac{1}{2}d}\right)^2}\,.$$

Similarly, to extend beyond the left endpoint we remove the third asset and find that the frontier of the reduced portfolio is given by

$$\sigma_{\underline{\mathsf{f}}_1}(\mu) = s\sqrt{\frac{1+r}{2} + \frac{1-r}{2}\left(\frac{\mu - m + \frac{1}{2}d}{\frac{1}{2}d}\right)^2}\,.$$

By putting these pieces together we see that the long frontier is given by

$$\sigma_{\mathsf{lf}}(\mu) = \begin{cases} s\sqrt{\dfrac{1+r}{2} + \dfrac{1-r}{2}\left(\frac{\mu-m+\frac{1}{2}d}{\frac{1}{2}d}\right)^2} & \text{for } \mu \in [m-d, m-\tfrac{2}{3}d]\,, \\ s\sqrt{\dfrac{1+2r}{3} + \dfrac{1-r}{2}\left(\frac{\mu-m}{d}\right)^2} & \text{for } \mu \in [m-\tfrac{2}{3}d, m+\tfrac{2}{3}d]\,, \\ s\sqrt{\dfrac{1+r}{2} + \dfrac{1-r}{2}\left(\frac{\mu-m-\frac{1}{2}d}{\frac{1}{2}d}\right)^2} & \text{for } \mu \in [m+\tfrac{2}{3}d, m+d]\,. \end{cases}$$

This is strictly convex and continuously differentiable over $[m-d, m+d]$. Its second derivative is defined and positive everywhere in $[m-d, m+d]$ except at the nodes $\mu = m \pm \frac{2}{3}d$ where it has jump discontinuities. We have

$$\sigma_{\mathsf{lf}}(m \pm \tfrac{2}{3}d) = s\sqrt{\frac{5+4r}{9}}\,, \qquad \sigma_{\mathsf{lf}}(m \pm d) = s\,.$$

Finally, the long frontier distributions are given by

$$\mathrm{f}_{\mathsf{lf}}(\mu) = \begin{cases} \begin{pmatrix} \frac{m-\mu}{d} \\ \frac{\mu-m+d}{d} \\ 0 \end{pmatrix} & \text{for } \mu \in [m-d, m-\tfrac{2}{3}d]\,, \\ \begin{pmatrix} \frac{1}{3} - \frac{\mu-m}{2d} \\ \frac{1}{3} \\ \frac{1}{3} + \frac{\mu-m}{2d} \end{pmatrix} & \text{for } \mu \in [m-\tfrac{2}{3}d, m+\tfrac{2}{3}d]\,, \\ \begin{pmatrix} 0 \\ \frac{m+d-\mu}{d} \\ \frac{\mu-m}{d} \end{pmatrix} & \text{for } \mu \in [m+\tfrac{2}{3}d, m+d]\,. \end{cases}$$

Notice that the distribution weights do not depend on either s or r. They are continuous and piecewise linear over $[m-d, m+d]$. Their first derivatives are defined everywhere in $[m-d, m+d]$ except at the nodes $\mu = m \pm \frac{2}{3}d$ where they have jump discontinuities.

Exercise. Find a 2×2 positive definite matrix $\mathbf{V}$ such that the vector $\mathbf{V}^{-1}\mathbf{1}$ has a negative entry.

Exercise. Consider the following groups of assets:

(a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2009;

(b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2007;

(c) S&P 500 and Russell 1000 and 2000 index funds in 2009;

(d) S&P 500 and Russell 1000 and 2000 index funds in 2007.

For group (a), group (c), and groups (a) and (c) combined, determine if $\mathbf{f}_{\mathsf{f}}(\mu_0) \geq \mathbf{0}$ for some μ_0. If so, add plots of the associated long frontiers to the graph you produced for these assets in the last exercise of the last section. (Use daily data.) Do the same thing for groups (b) and (d). Explain any relationships you see between the objects plotted on each graph. For which of these groupings is $\mathbf{f}_{\mathsf{mv}} \geq \mathbf{0}$? Compute $\mathbf{f}_{\mathsf{lf}}(\mu)$ for each of these groupings, identifying the nodal portfolios.