GRAPH THEORY DEFINITIONS

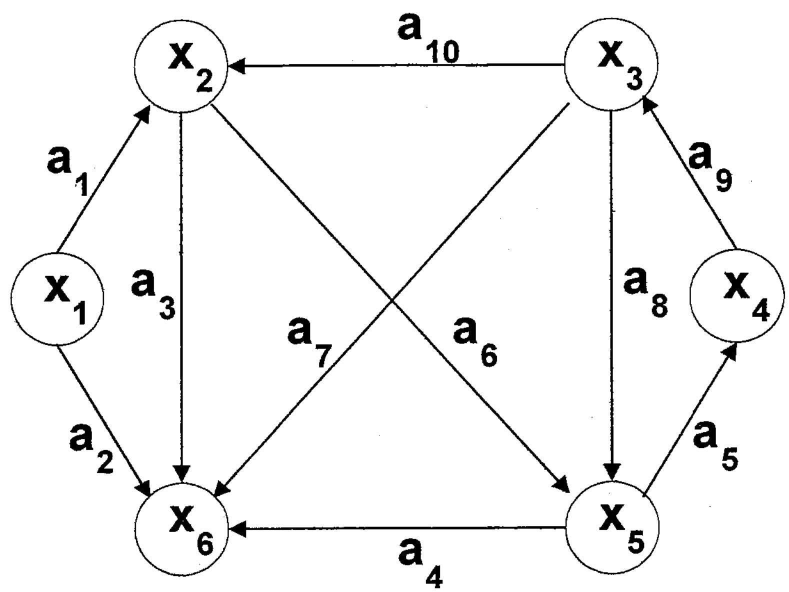
- A *graph* G is a collection of points, nodes, or vertices $x_1, x_2, \ldots, x_n$ (denoted by set X), and a collection of lines, edges, links, or arcs $a_1, a_2, \ldots, a_m$ (denoted by set A) joining all or some of these points.

- The notation (x_1, x_2) or $x_1 - x_2$ refers to a *directed* arc from x_1 to x_2.

- Graphs can be *directed, undirected,* or *mixed* depending on whether all, none, or some of the arcs are oriented.

- A *path* in a directed graph is any sequence of arcs where the final vertex of one is the initial vertex of the next one.

GRAPH THEORY DEFINITIONS



- In the figure, the sequence of arcs

  a_6, a_5, a_9, a_8, a_4,

  a_1, a_6, a_5, a_9, and

  a_1, a_6, a_5, a_9, a_{10} , a_6, a_4 are all paths.
- A *simple path* is a path which does not use the same arc more than once.
- An *elementary* path is a path which does not use the same vertex more than once.

GRAPH THEORY DEFINITIONS

- An elementary path is also simple, but the reverse is not always so.
- A number c_{ij} may be associated with an arc (x_i, x_j). These numbers are called *weights*, *lengths*, or *costs* and the graph is called *arc-weighted.*
- A weight v_i may sometimes be associated with a vertex x_i and the resulting graph is then called *vertex-weighted.*
- Considering a path μ represented by the sequence of arcs $(a_1, a_2, \ldots, a_q)$ the *length (cost) of the path* $\ell(\mu)$ is taken to be the sum of the arc weights on the arcs in μ, i.e.,

$$\ell(\mu) = \sum_{(x_i, x_j) \text{ in } \mu} c_{ij}.$$

GRAPH THEORY DEFINITIONS

- A *loop* is an arc whose initial and final vertices are the same.

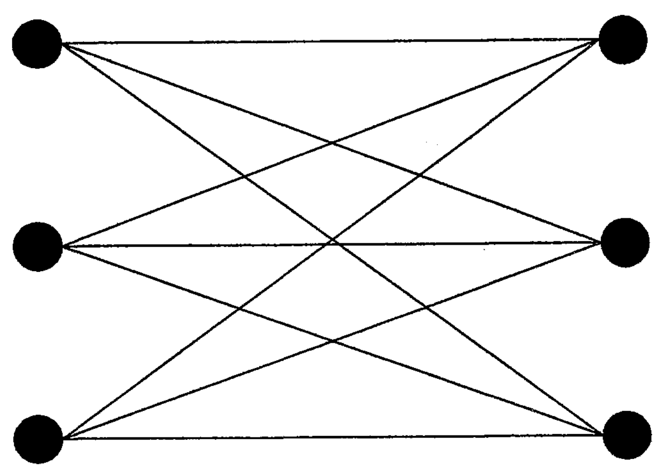
- A *circuit* or *cycle* is a path $a_1, a_2, \ldots, a_q$ in which the initial vertex of a_1 coincides with the final vertex of a_q.

- *Cycles* or *circuits* are *elementary* if they do not use the same vertex more than once (except for the initial and final vertices which are the same).

- An elementary cycle or circuit which passes through all of the n vertices of a graph G is of special significance and is known as a *Hamiltonian cycle*. Not all graphs have a Hamiltonian cycle.

GRAPH THEORY DEFINITIONS

- The number of arcs which have a vertex x_i as their initial vertex is called the *outdegree* of x_i, and similarly the number of arcs which have x_i as their final vertex is called the *indegree* of vertex x_i.
- An undirected graph $G = (X, A)$ is said to be *bipartite*, if the set X of its vertices can be partitioned into subsets X^a and X^b so that all arcs have one terminal vertex in X^a and the other in X^b.
- An example of $K_{3,3}$ is shown below.



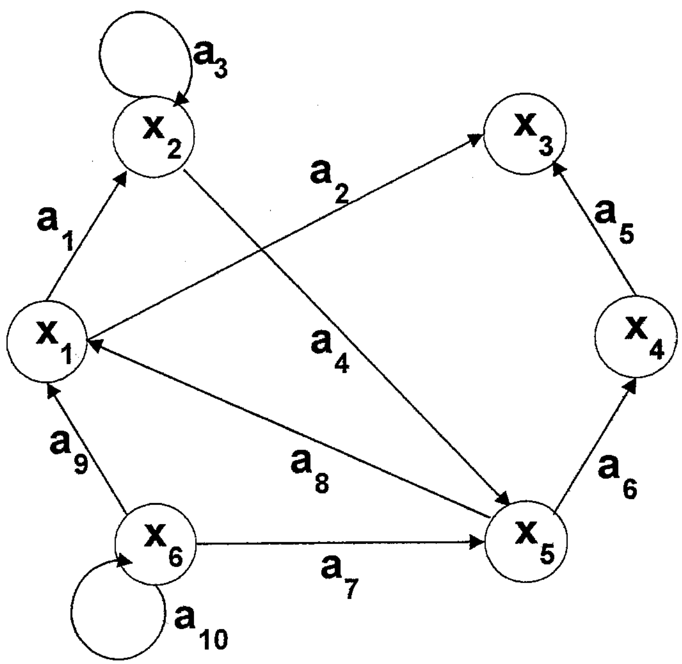
Bipartite Graph

GRAPH THEORY DEFINITIONS

- Given a graph G, its *adjacency matrix* is denoted by $A = [a_{ij}]$ and is given by

$$a_{ij} = \begin{cases} 1 & \text{if arc } \left(x_i, x_j\right) \quad \text{exists in G.} \\ 0 & \text{if arc } \left(x_i, x_j\right) \quad \text{does not exist in G.} \end{cases}$$

<u>Graph</u>



<u>Adjacency Matrix</u>

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	1	1	0	0	0
x_2	0	1	0	0	1	0
x_3	0	0	0	0	0	0
x_4	0	0	1	0	0	0
x_5	1	0	0	1	0	0
x_6	1	0	0	0	1	1

GRAPH THEORY DEFINITIONS

- Given a graph G on n vertices and m arcs, the *incidence matrix* of G is denoted by $B = [b_{ij}]$ and is an n x m matrix defined as follows.

$$b_{ij} = \begin{cases} 1 & \text{if } x_i \text{ is the initial vertex of arc } a_j \\ -1 & \text{if } x_i \text{ is the final vertex of arc } a_j \end{cases}$$

and $b_{ij} = 0$ if neither of the above or if a_j is a loop.

- If G is an undirected graph, then the incidence matrix is defined as before except that all entries of -1 are now changed to +1.

GRAPH THEORY DEFINITIONS

- For the graph on page A- 6, the incidence matrix is:

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
x_1	1	1	0	0	0	0	0	-1	-1	0
x_2	-1	0	0	1	0	0	0	0	0	0
x_3	0	-1	0	0	-1	0	0	0	0	0
x_4	0	0	0	0	1	-1	0	0	0	0
x_5	0	0	0	-1	0	1	-1	1	0	0
x_6	0	0	0	0	0	0	1	0	1	0