

# The Minimum Label Spanning Tree Problem: Some Genetic Algorithm Approaches

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# Outline of Lecture

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- A Short Tribute to Ben Franklin on his 300<sup>th</sup> Birthday
- Introduction to the MLST Problem
- A GA for the MLST Problem
- Four Modified Versions of the Benchmark Heuristic
- A Modified Genetic Algorithm
- Results, Conclusions, and Related Work

# Ben Franklin and the Invention of America

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- Born in Boston on January 17, 1706
- Best scientist, inventor, diplomat, writer, and businessman (printer and publisher) in America in the 1700s
- Great political and practical thinker
- Proved that lightning was electricity
- Inventions include bifocal glasses, the clean-burning stove, and the lightning rod
- Founded a library, college, fire department, and many other civic associations

# More about Ben Franklin

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- Only person to sign all of the following
  - The Declaration of Independence
  - The Constitution of the United States
  - The Treaty of Alliance with France
  - The Treaty of Peace with Great Britain
- Retired from business at age 42, lived 84 years
- He also made significant contributions to recreational mathematics
  - Magic squares
  - Magic circles

# Franklin Magic Squares

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52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

# Franklin Magic Squares

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16	1	64	49	48	33	32	17

Each row sum = each column sum = 260

# Properties of Franklin Magic Squares

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The shaded entries sum to 260

# Properties of Franklin Magic Squares

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16	1	64	49	48	33	32	17

Any half-row or half-column totals 130



# Properties of Franklin Magic Squares

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Each of the two bent diagonals above totals 260

# Properties of Franklin Magic Squares

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Each of the two bent diagonals above totals 260

# Franklin Magic Squares: Final Remarks

- Franklin's most impressive square is 16 by 16
- It has many additional properties
- See the June-July 2001 issue of The American Mathematical Monthly for details
- Mathematicians today are trying to determine how Franklin constructed these squares
- Note the connection to integer programming

# Introduction

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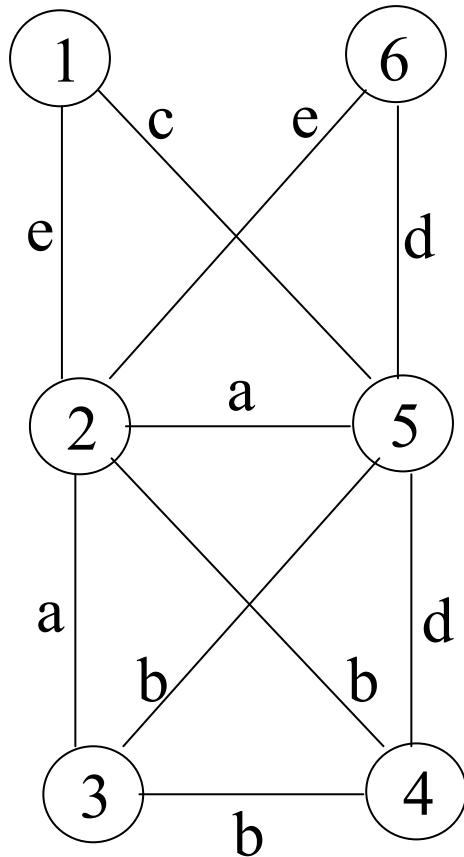
- The Minimum Label Spanning Tree (MLST) Problem
  - Communications network design
  - Edges may be of different types or media (e.g., fiber optics, cable, microwave, telephone lines, etc.)
  - Each edge type is denoted by a unique letter or color
  - Construct a spanning tree that minimizes the number of colors

# Introduction

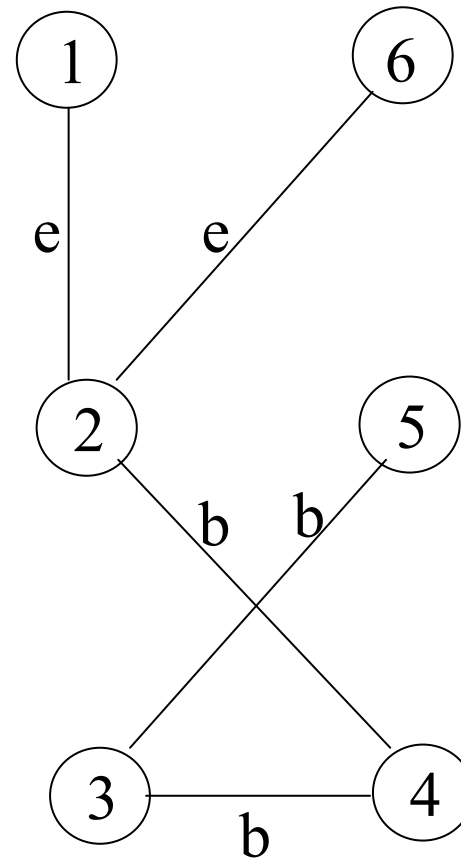
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## ■ A Small Example

**Input**



**Solution**



# Literature Review

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- Where did we start?
  - Proposed by Chang & Leu (1997)
  - The MLST Problem is NP-hard
  - Several heuristics had been proposed
  - One of these, MVCA (maximum vertex covering algorithm), was very fast and effective
  - Worst-case bounds for MVCA had been obtained

# Literature Review

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- An optimal algorithm (using backtrack search) had been proposed
- On small problems, MVCA consistently obtained nearly optimal solutions
- A description of MVCA follows

# Description of MVCA

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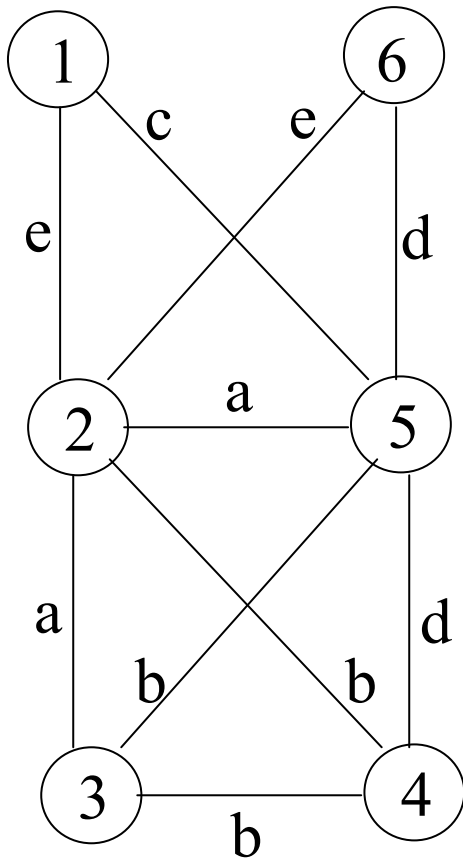
0. Input:  $G (V, E, L)$ .
1. Let  $C \leftarrow \{ \}$  be the set of used labels.
2. repeat
3.     Let  $H$  be the subgraph of  $G$  restricted to  $V$  and edges with labels from  $C$ .
4.     for all  $i \in L - C$  do
5.         Determine the number of connected components when inserting all edges with label  $i$  in  $H$ .
6.     end for
7.     Choose label  $i$  with the smallest resulting number of components and do:  $C \leftarrow C \cup \{i\}$ .
8. Until  $H$  is connected.



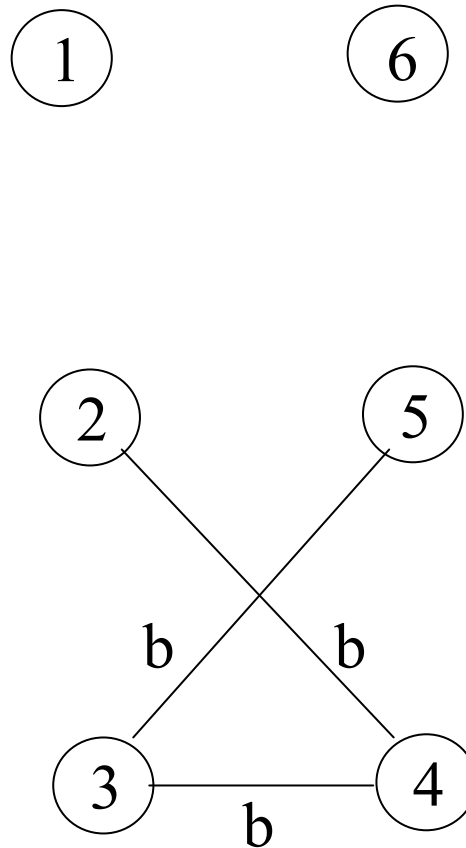
# How MVCA Works

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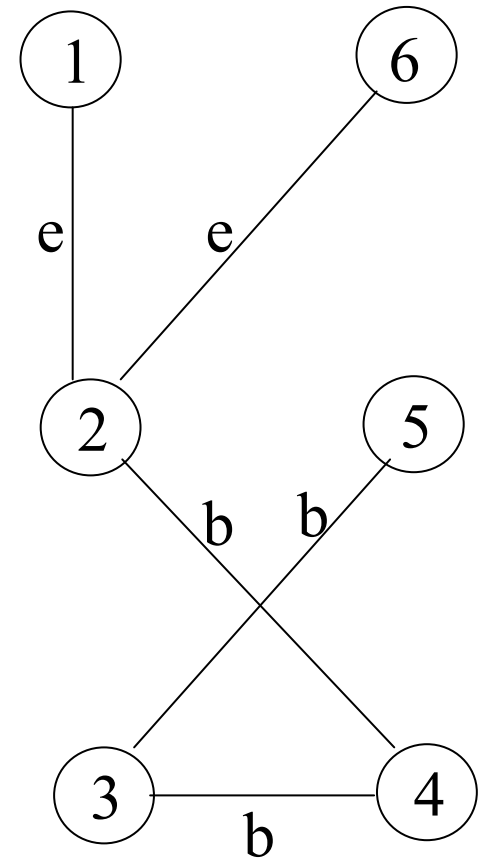
**Input**



**Intermediate  
Solution**



**Solution**



# Worst-Case Results

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1. Krumke, Wirth (1998):

$$\frac{\text{MVCA}}{\text{OPT}} \leq 1 + 2 \ln n$$

2. Wan, Chen, Xu (2002):

$$\frac{\text{MVCA}}{\text{OPT}} \leq 1 + \ln(n-1)$$

3. Xiong, Golden, Wasil (2005):

$$\frac{\text{MVCA}}{\text{OPT}} \leq H_b = \sum_{i=1}^b \frac{1}{i} < 1 + \ln b$$

where  $b = \max$  label frequency, and

$H_b = b^{\text{th}}$  harmonic number

# Some Observations

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- The Xiong, Golden, Wasil worst-case bound is tight
- Unlike the MST, where we focus on the edges, here it makes sense to focus on the labels or colors
- Next, we present a genetic algorithm (GA) for the MLST problem

# Genetic Algorithm: Overview

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- Randomly choose  $p$  solutions to serve as the initial population
- Suppose  $s[0], s[1], \dots, s[p-1]$  are the individuals (solutions) in generation 0
- Build generation  $k$  from generation  $k-1$  as below

For each  $j$  between 0 and  $p-1$ , do:

$t[j] = \text{crossover} \{ s[j], s[(j+k) \bmod p] \}$

$t[j] = \text{mutation} \{ t[j] \}$

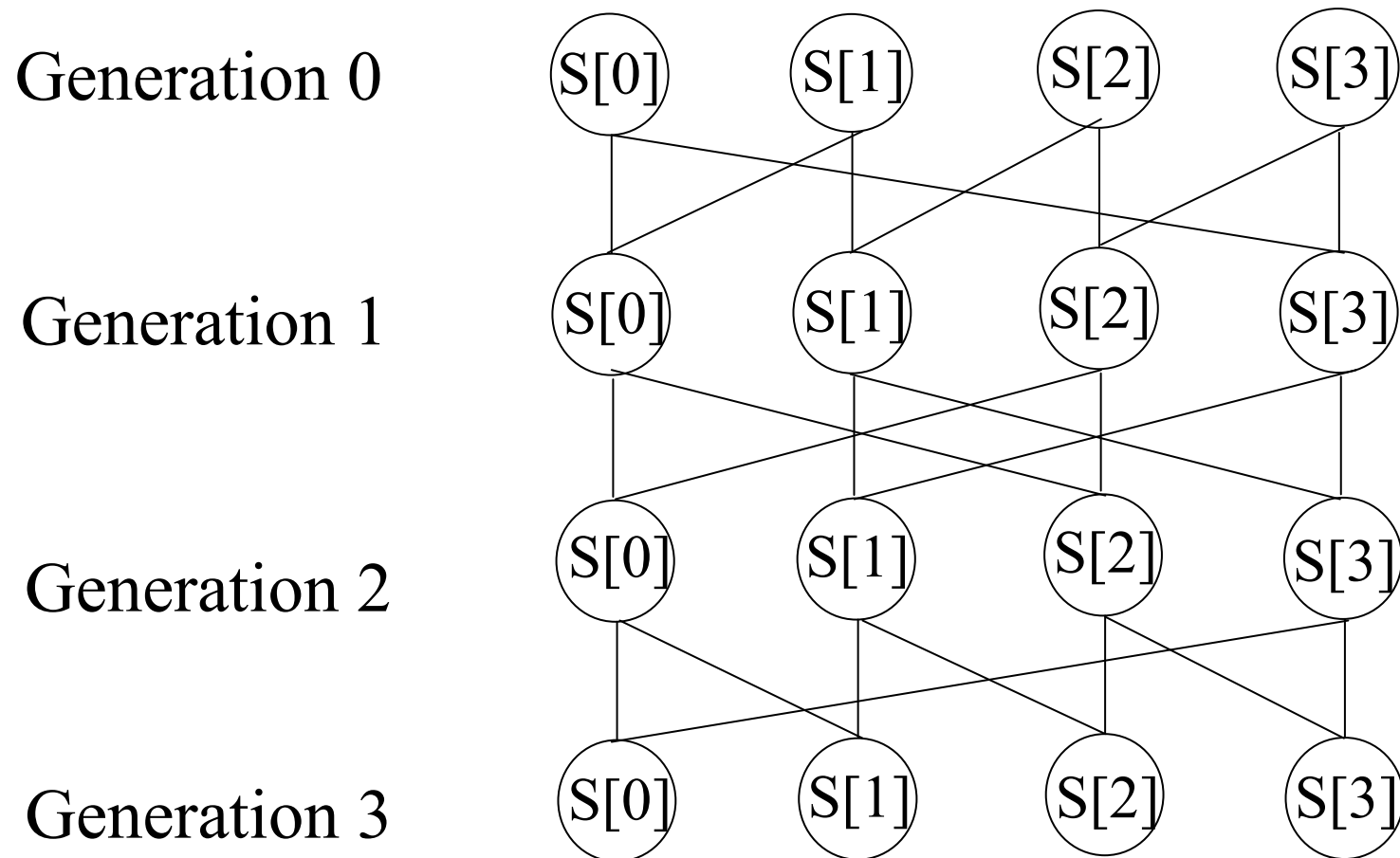
$s[j] = \text{the better solution of } s[j] \text{ and } t[j]$

End For

- Run until generation  $p-1$  and output the best solution from the final generation

# Crossover Schematic (p = 4)

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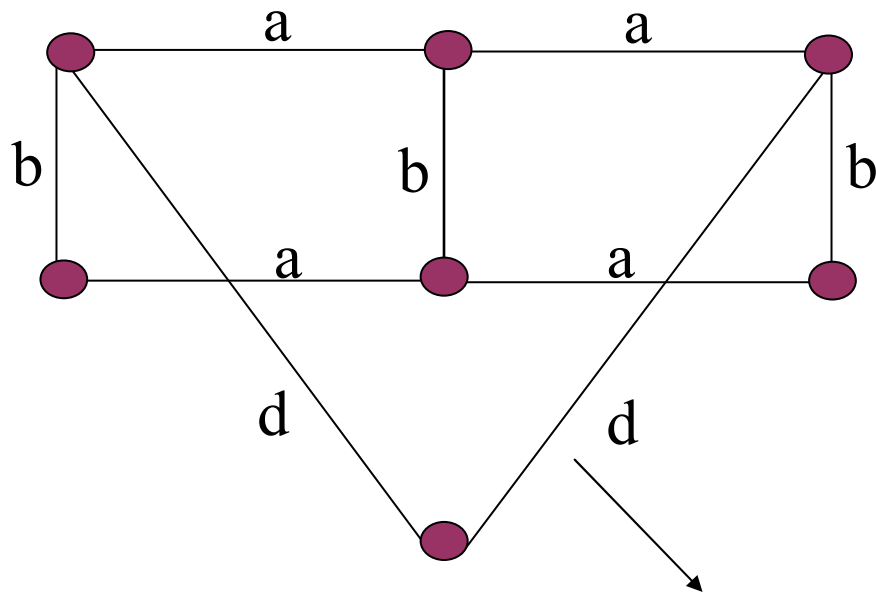
# Crossover

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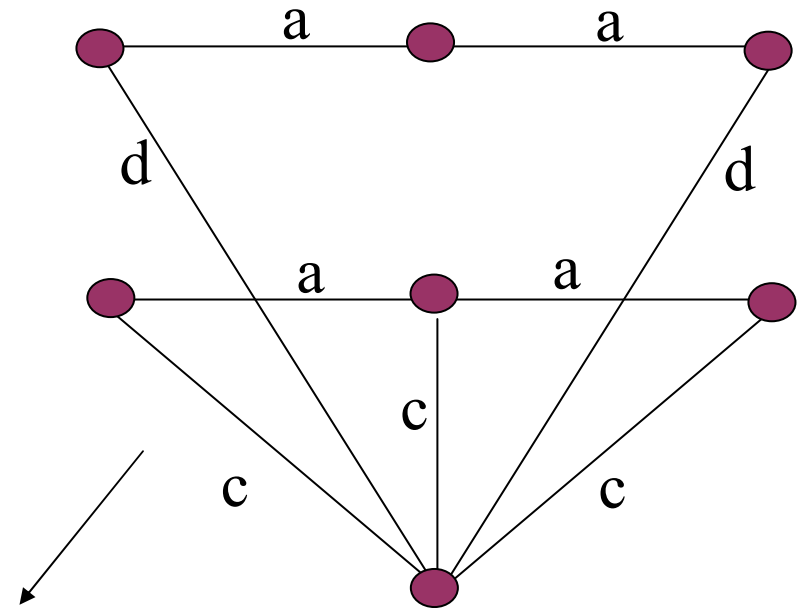
- Given two solutions  $s [ 1 ]$  and  $s [ 2 ]$ , find the child  $T = \text{crossover} \{ s [ 1 ], s [ 2 ] \}$
- Define each solution by its labels or colors
- Description of Crossover
  - a. Let  $S = s [ 1 ] \cup s [ 2 ]$  and  $T$  be the empty set
  - b. Sort  $S$  in decreasing order of the frequency of labels in  $G$
  - c. Add labels of  $S$ , from the first to the last, to  $T$  until  $T$  represents a feasible solution
  - d. Output  $T$

# An Example of Crossover

$s[1] = \{a, b, d\}$



$s[2] = \{a, c, d\}$



$T = \{ \}$

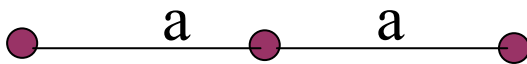
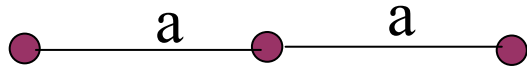
$S = \{a, b, c, d\}$

Ordering: a, b, c, d

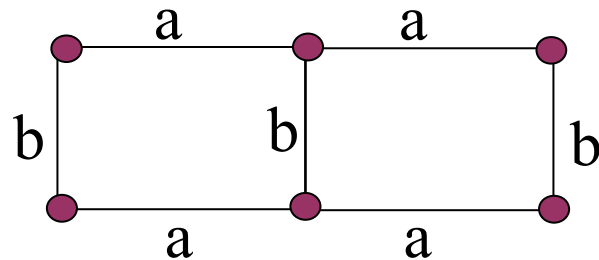
# An Example of Crossover

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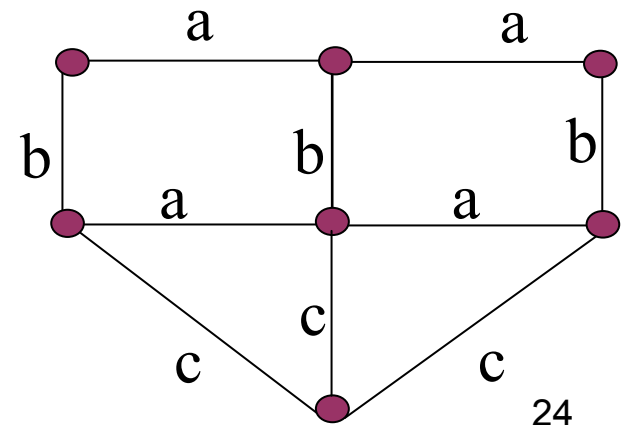
$T = \{ a \}$



$T = \{ a, b \}$



$T = \{ a, b, c \}$





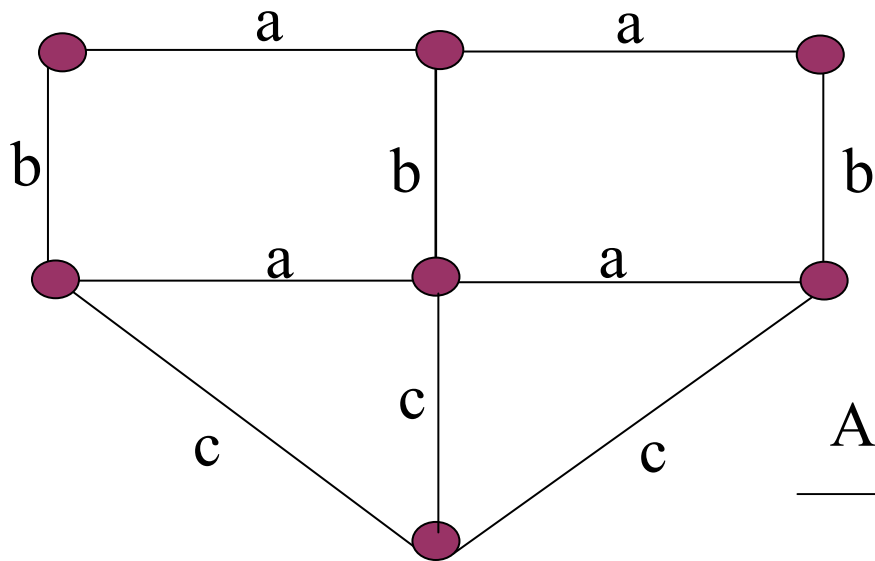
# Mutation

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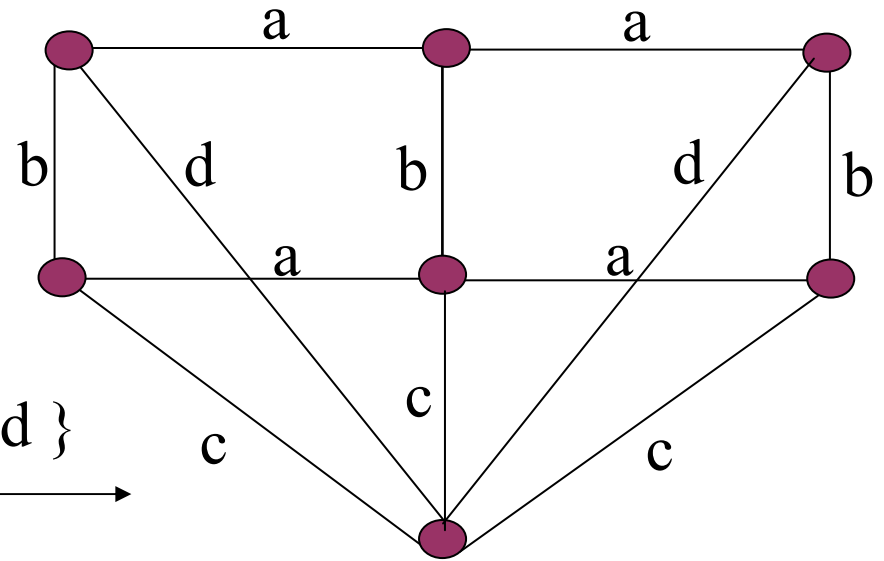
- Given a solution  $S$ , find a mutation  $T$
- Description of Mutation
  - a. Randomly select  $c$  not in  $S$  and let  $T = S \cup c$
  - b. Sort  $T$  in decreasing order of the frequency of the labels in  $G$
  - c. From the last label on the above list to the first, try to remove one label from  $T$  and keep  $T$  as a feasible solution
  - d. Repeat the above step until no labels can be removed
  - e. Output  $T$

# An Example of Mutation

$S = \{ a, b, c \}$



$S = \{ a, b, c, d \}$



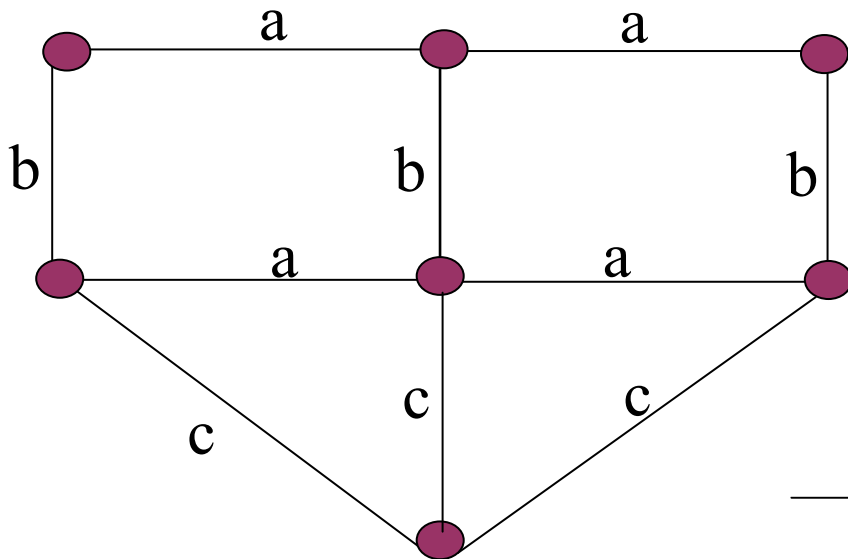
Add { d }

Ordering: a, b, c, d

# An Example of Mutation

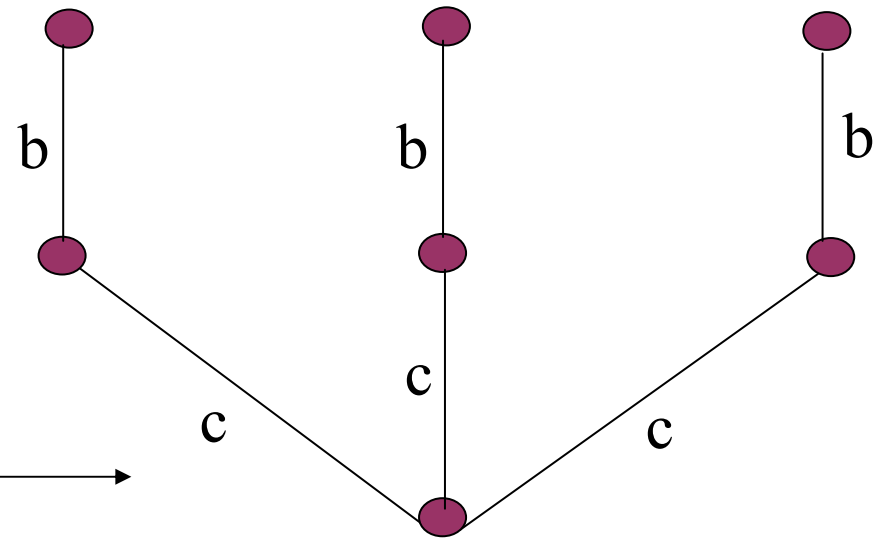
Remove { d }

$S = \{ a, b, c \}$



Remove { a }

$S = \{ b, c \}$



$T = \{ b, c \}$

# Three Modified Versions of MVCA

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- Voss et al. (2005) implement MVCA using their pilot method
- The results were quite time-consuming
- We added a parameter ( % ) to improve the results
- Three modified versions of MVCA
  - MVCA1 uses % = 100
  - MVCA2 uses % = 10
  - MVCA3 uses % = 30

# MVCA1

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- We try each label in  $L$  ( $\% = 100$ ) as the first or pilot label
- Run MVCA to determine the remaining labels
- We output the best solution of the  $l$  solutions obtained
- For large  $l$ , we expect MVCA1 to be very slow

# MVCA2 (and MVCA3)

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- We sort all labels by their frequencies in  $G$ , from highest to lowest
- We select each of the top 10% ( $\% = 10$ ) of the labels to serve as the pilot label
- Run MVCA to determine the remaining labels
- We output the best solution of the  $l/10$  solutions obtained
- MVCA2 will be faster than MVCA1, but not as effective
- MVCA3 selects the top 30% ( $\% = 30$ ) and examines  $3l/10$  solutions
- MVCA3 is a compromise approach

# A Randomized Version of MVCA (RMVCA)

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- We follow MVCA in spirit
- At each step, we consider the three most promising labels as candidates
- We select one of the three labels
  - The best label is selected with prob. = 0.4
  - The second best label is selected with prob. = 0.3
  - The third best label is selected with prob. = 0.3
- We run RMVCA 50 times for each instance and output the best solution

# A Modified Genetic Algorithm (MGA)

- We modify the crossover operation described earlier
- We take the union of the parents (i.e.,  $S = S_1 \cup S_2$ ) as before
- Next, apply MVCA to the subgraph of  $G$  with label set  $S$  ( $S \subseteq L$ ), node set  $V$ , and the edge set  $E'$  ( $E' \subseteq E$ ) associated with  $S$
- The new crossover operation is more time-consuming than the old one
- The mutation operation remains as before



# Computational Results

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- 48 combinations:  $n = 50$  to  $200$  /  $l = 12$  to  $250$  / density =  $0.2, 0.5, 0.8$
- 20 sample graphs for each combination
- The average number of labels is compared

# Performance Comparison

	MVCA	GA	MGA	MVCA1	MVCA2	MVCA3	RMVCA	Row Total
MVCA	-	3	0	0	0	0	0	3
GA	30	-	0	1	9	4	2	46
MGA	33	30	-	10	20	16	16	125
MVCA1	35	30	10	-	24	20	18	137
MVCA2	31	20	5	0	-	0	6	62
MVCA3	34	27	8	0	23	-	11	103
RMVCA	35	30	7	3	20	10	-	105

Summary of computational results with respect to accuracy for seven heuristics on 48 cases. The entry  $(i, j)$  represents the number of cases heuristic  $i$  generates a solution that is better than the solution generated by heuristic  $j$ .

# Running Times

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	MVCA	GA	MGA	MVCA1	MVCA2	MVCA3	RMVCA
$n = 100, l = 125, d = 0.2$	0.05	1.80	7.50	8.25	0.80	2.30	3.85
$n = 150, l = 150, d = 0.5$	0.10	1.85	4.90	11.85	1.15	3.45	4.75
$n = 150, l = 150, d = 0.2$	0.15	3.45	13.55	21.95	2.15	6.35	8.45
$n = 150, l = 187, d = 0.5$	0.15	2.20	6.70	21.70	2.00	6.15	7.50
$n = 150, l = 187, d = 0.2$	0.20	3.95	17.55	39.35	3.60	11.20	11.90
$n = 200, l = 100, d = 0.2$	0.15	3.75	11.40	11.25	1.15	3.35	6.75
$n = 200, l = 200, d = 0.8$	0.25	2.45	5.80	26.70	2.70	8.00	8.65
$n = 200, l = 200, d = 0.5$	0.25	3.45	10.15	38.65	3.90	10.15	12.00
$n = 200, l = 200, d = 0.2$	0.35	6.20	26.65	68.25	6.85	20.35	20.55
$n = 200, l = 250, d = 0.8$	0.30	3.05	7.55	52.25	5.25	15.35	12.95
$n = 200, l = 250, d = 0.5$	0.30	3.95	12.60	69.90	6.80	20.35	16.70
$n = 200, l = 250, d = 0.2$	0.50	6.90	33.15	124.35	12.10	35.80	28.80
Average running time	0.23	3.58	13.13	41.20	4.04	11.90	11.90

Running times for 12 demanding cases (in seconds).

# One Final Experiment for Small Graphs

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- 240 instances for  $n = 20$  to 50 are solved by the seven heuristics
- Backtrack search solves each instance to optimality
- The seven heuristics are compared based on how often each obtains an optimal solution

Procedure	OPT	MVCA	GA	MGA	MVCA1	MVCA2	MVCA3	RMVCA
% optimal	100.00	75.42	96.67	99.58	95.42	87.08	93.75	97.50

# Conclusions

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- We presented three modified (deterministic) versions of MVCA, a randomized version of MVCA, and a modified GA
- All five of the modified procedures generated better results than MVCA and GA, but were more time-consuming
- With respect to running time and performance, MGA seems to be the best

# Related Work

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- The Label-Constrained Minimum Spanning Tree (LCMST) Problem
  - We show the LCMST problem is NP-hard
  - We introduce two local search methods
  - We present an effective genetic algorithm
  - We formulate the LCMST as a MIP and solve for small cases
  - We introduce a dual problem