Some Recent Developments in the Analytic Hierarchy Process

by

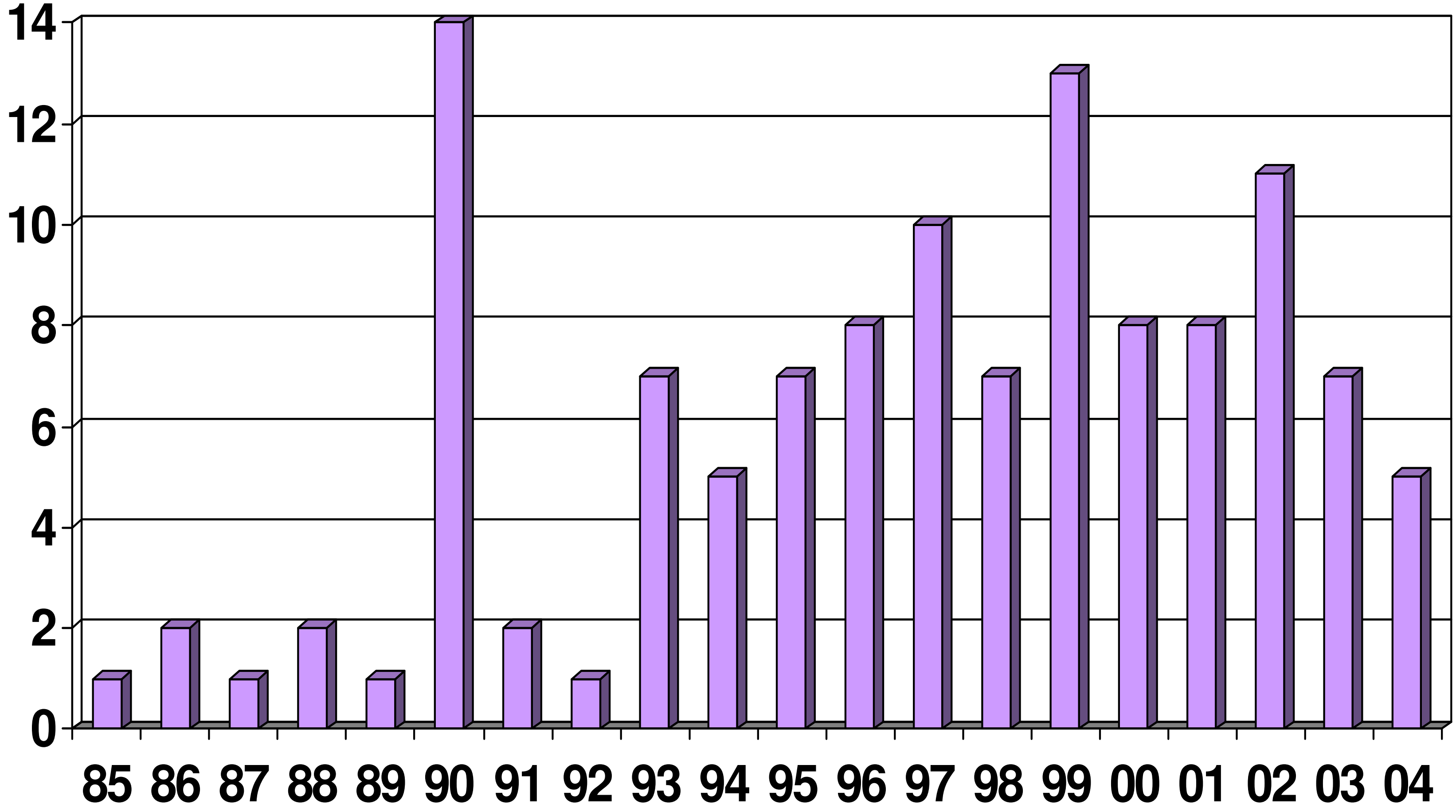
Bruce L. Golden
RH Smith School of Business
University of Maryland

CORS/INFORMS International Conference in Banff
May 16, 2004

Focus of Presentation

- Celebrating nearly 30 years of AHP-based decision making
- AHP overview
- Linear programming models for AHP
- Computational experiments
- Conclusions

Number of AHP Papers in <u>EJOR</u> (last 20 years)



AHP Articles in Press at EJOR

- Solving multiattribute design problems with the analytic hierarchy process and conjoint analysis: An empirical comparison
- Understanding local ignorance and non-specificity within the DS/AHP method of multi-criteria decision making
- Phased multicriteria preference finding
- Interval priorities in AHP by interval regression analysis
- A fuzzy approach to deriving priorities from interval pairwise comparison judgments
- Representing the strengths and directions of pairwise comparisons

A Recent Special Issue on AHP

- Journal: Computers & Operations Research (2003)
- Guest Editors: B. Golden and E. Wasil
- Articles
  - Celebrating 25 years of AHP-based decision making
  - Decision counseling for men considering prostate cancer screening
  - Visualizing group decisions in the analytic hierarchy process
  - Using the analytic hierarchy process as a clinical engineering tool to facilitate an iterative, multidisciplinary, microeconomic health technology assessment
  - An approach for analyzing foreign direct investment projects with application to China's Tumen River Area development
  - On teaching the analytic hierarchy process

A Recent Book on AHP

- Title: Strategic Decision Making: Applying the Analytic Hierarchy Process (Springer, 2004)

- Authors: N. Bhushan and K. Rai

- Contents

Book Contents -- continued

AHP and Related Software

- Expert Choice (Forman)

  EC Resource Aligner combines optimization with AHP to select the optimal combination of alternatives or projects subject to a budgetary constraint

- Criterium DecisionPlus (Hearne Scientific Software)

- HIPRE 3+ (Systems Analysis Laboratory, Helsinki)

- Web-HIPRE

  The first web-based multiattribute decision analysis tool

- Super Decisions (Saaty)

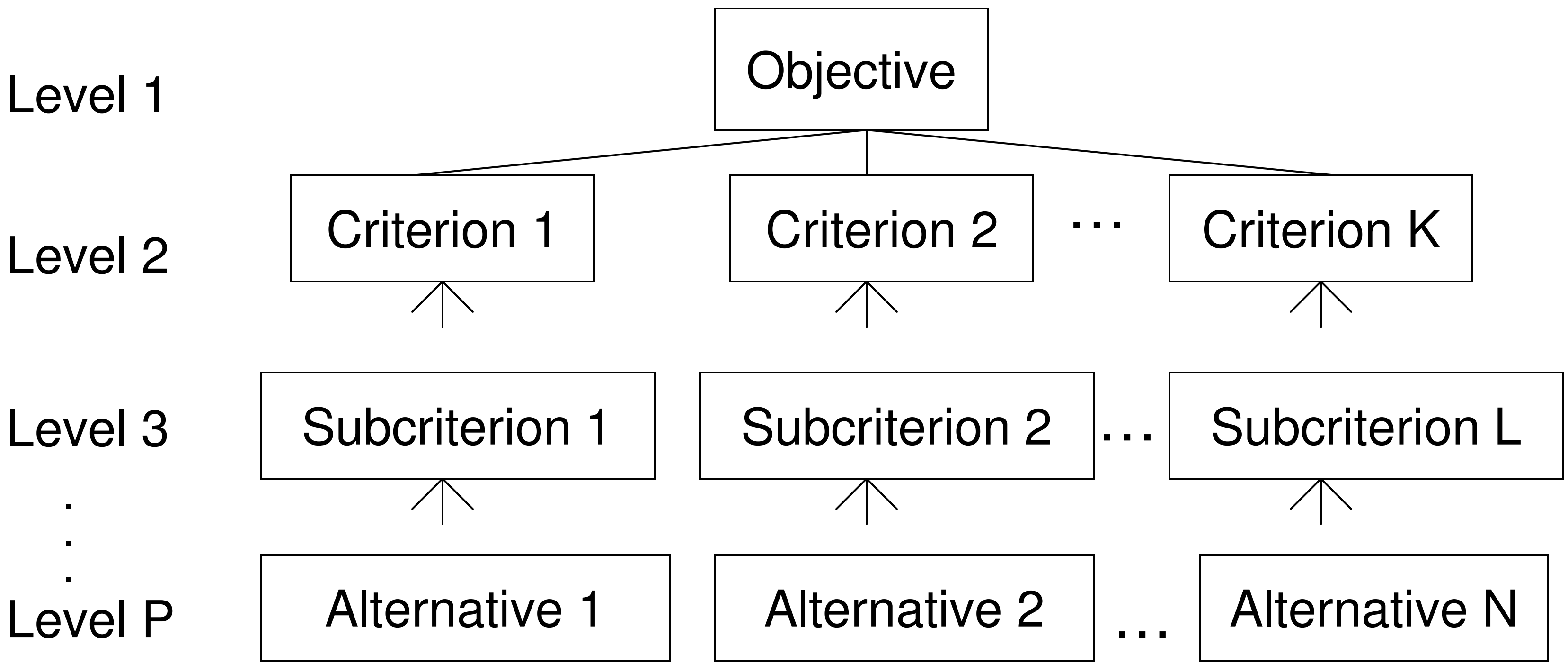
  This software implements the analytic network process (decision making with dependence and feedback)

AHP Overview

- Analysis tool that provides insight into complex problems by incorporating qualitative and quantitative decision criteria
- Hundreds of published applications in numerous different areas
- Combined with traditional OR techniques to form powerful “hybrid” decision support tools
- Four step process

The Analytic Hierarchy Process

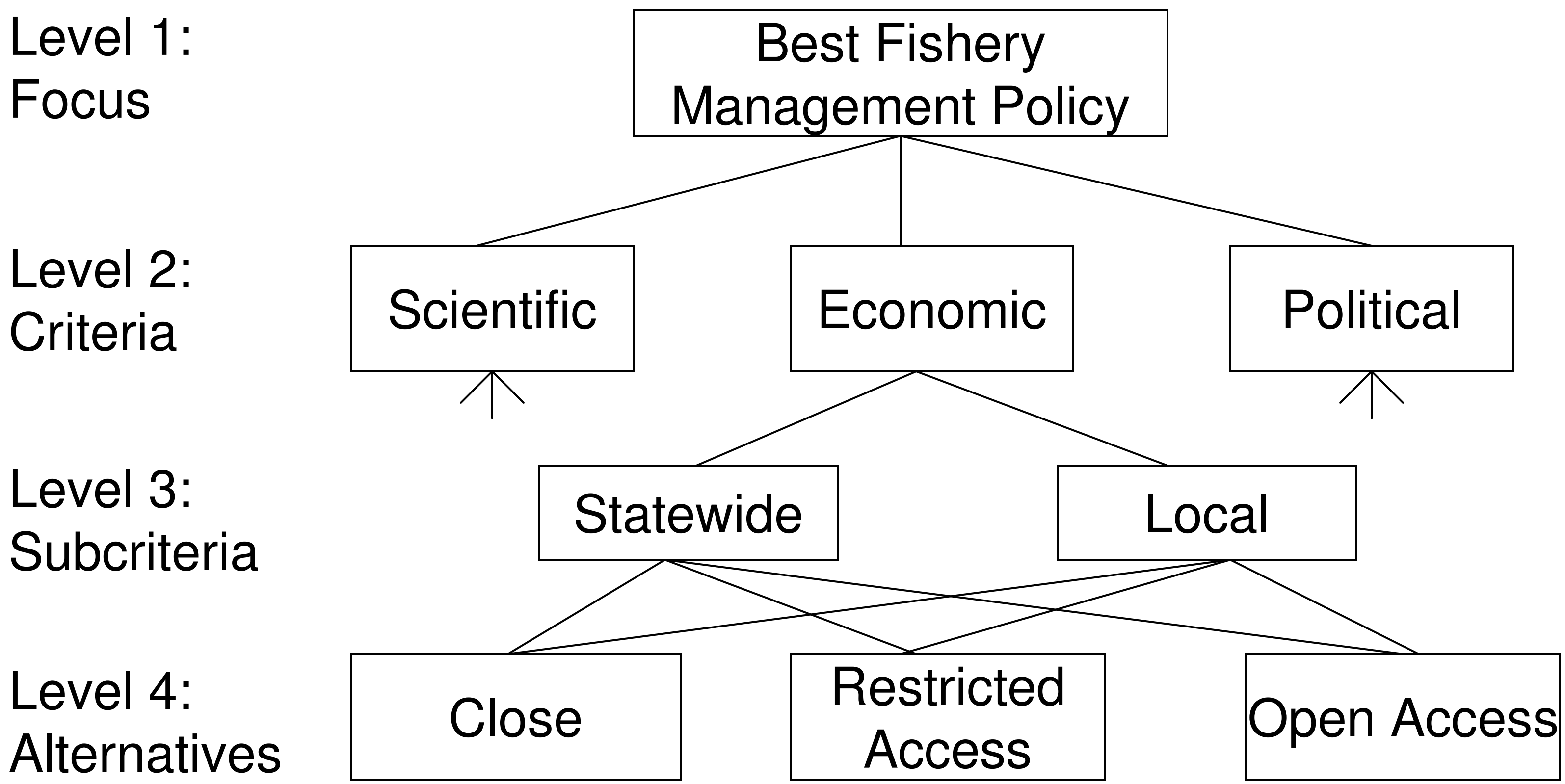
- Step 1. Decompose the problem into a hierarchy of interrelated decision criteria and alternatives



Hierarchy with P Levels

The Analytic Hierarchy Process

- Illustrative example



Partial Hierarchy: Management of a Fishery

The Analytic Hierarchy Process

- Step 2. Use collected data to generate pairwise comparisons at each level of the hierarchy

Illustrative Example

	Scientific	Economic	Political
Scientific	1	a_{SE}	a_{SP}
Economic	$1/a_{SE}$	1	a_{EP}
Political	$1/a_{SP}$	$1/a_{EP}$	1

Pairwise Comparison Matrix: Second Level

The Analytic Hierarchy Process

- Compare elements two at a time
- Generate the a_{SE} entry
  - With respect to the overall goal, which is more important – the scientific or economic factor – and how much more important is it?
  - Number from 1/9 to 9
  - Positive reciprocal matrix

The Analytic Hierarchy Process

- Illustrative Example

	Scientific	Economic	Political
Scientific	1	2	5
Economic	1/2	1	2
Political	1/5	1/2	1

- AHP provides a way of measuring the consistency of decision makers in making comparisons

- Decision makers are not required or expected to be perfectly consistent

The Analytic Hierarchy Process

- Step 3. Apply the eigenvalue method (EM) to estimate the weights of the elements at each level of the hierarchy

- The weights for each matrix are estimated by solving

$$A \cdot \hat{w} = \lambda_{MAX} \cdot \hat{w}$$

where

A is the pairwise comparison matrix

λ_{MAX} is the largest eigenvalue of A

$\hat{w}$ is its right eigenvector

The Analytic Hierarchy Process

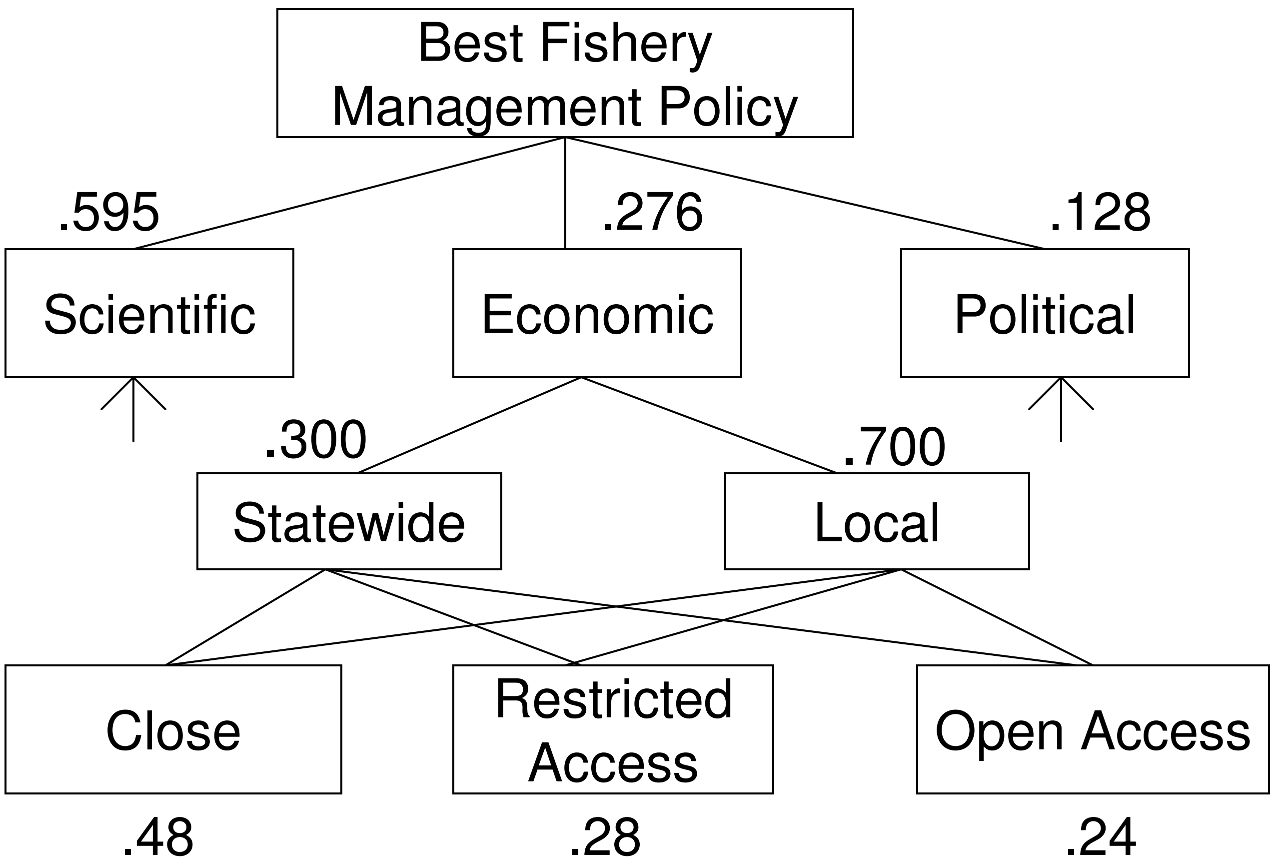
- Illustrative Example

	Scientific	Economic	Political	Weights
Scientific	1	2	5	.595
Economic	1/2	1	2	.276
Political	1/5	1/2	1	.128

Pairwise comparison matrix: Second level

The Analytic Hierarchy Process

- Step 4. Aggregate the relative weights over all levels to arrive at overall weights for the alternatives



Estimating Weights in the AHP

- Traditional method: Solve for $\hat{w}$ in $A\hat{w} = \lambda_{MAX} \hat{w}$
- Alternative approach (Logarithmic Least Squares or LLS): Take the geometric mean of each row and then normalize
- Linear Programming approach (Chandran, Golden, Wasil, Alford)
  - Let $w_i / w_j = a_{ij} \varepsilon_{ij}$ ($i, j = 1, 2, \ldots, n$) define an error ε_{ij} in the estimate a_{ij}
  - If the decision maker is perfectly consistent, then $\varepsilon_{ij} = 1$ and $\ln \varepsilon_{ij} = 0$
  - We develop a two-stage LP approach

Linear Programming Setup

- Given: A = [a_{ij}] is n x n

- Decision variables
  - w_i = weight of element i
  - ε_{ij} = error factor in estimating a_{ij}

- Transformed decision variables
  - $x_i = \ln (w_i)$
  - $y_{ij} = \ln (\varepsilon_{ij})$
  - $z_{ij} = | y_{ij} |$

Some Observations

- Take the natural log of $w_i \,/\, w_j = a_{ij}\,\varepsilon_{ij}$ to obtain

$$x_i - x_j - y_{ij} = \ln a_{ij}$$

- If a_{ij} is overestimated, then a_{ji} is underestimated
  - $\varepsilon_{ij} = 1/\,\varepsilon_{ji}$
  - $y_{ij} = -\,y_{ji}$

- $z_{ij} \geq y_{ij}$ and $z_{ij} \geq y_{ji}$ identifies the element that is overestimated and the magnitude of overestimation

- We can arbitrarily set $w_1 = 1$ or $x_1 = \ln(w_1) = 0$ and normalize the weights later

First Stage Linear Program

Minimize $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} z_{ij}$ — minimize inconsistency

subject to

$x_i - x_j - y_{ij} = \ln a_{ij}$,	$i, j = 1, 2, \ldots, n; i \neq j$,	error term def.
$z_{ij} \geq y_{ij}$,	$i, j = 1, 2, \ldots, n; i < j$,	degree of overestimation
$z_{ij} \geq y_{ji}$,	$i, j = 1, 2, \ldots, n; i < j$,	degree of overestimation
$x_1 = 0$,		set one w_i
$x_i - x_j \geq 0$,	$i, j = 1, 2, \ldots, n; a_{ij} > 1$,	element dominance
$x_i - x_j \geq 0$,	$i, j = 1, 2, \ldots, n; a_{ik} \geq a_{jk}$ for all k; $a_{iq} > a_{jq}$ for some q,	row dominance
$z_{ij} \geq 0$,	$i, j = 1, 2, \ldots, n$,	
x_i, y_{ij} unrestricted	$i, j = 1, 2, \ldots, n$	

Element and Row Dominance Constraints

- ED is preserved if $a_{ij} > 1$ implies $w_i \geq w_j$

EM and LLS do not preserve ED

- RD is preserved if $a_{ik} \geq a_{jk}$ for all k and $a_{ik} > a_{jk}$ for some k implies $w_i \geq w_j$

Both EM and LLS guarantee RD

- We capture these constraints explicitly in the first stage LP

The Objective Function (OF)

- The OF minimizes the sum of logarithms of positive errors in natural log space

- In the nontransformed space, the OF minimizes the product of the overestimated errors ($\varepsilon_{ij} \geq 1$)

- Therefore, the OF minimizes the geometric mean of all errors ≥ 1

- In a perfectly consistent comparison matrix, $z^* = 0$ (since $\varepsilon_{ij} = 1$ and $y_{ij} = 0$ for all i and j)

The Consistency Index

- The OF is a measure of the inconsistency in the pairwise comparison matrix
- The OF minimizes the sum of $n\,(n-1)\,/\,2$ decision variables (z_{ij} for $i < j$)
- The OF provides a convenient consistency index

$$\text{CI (LP)} = 2\ z^* / n\,(n-1)$$

- CI (LP) is the average value of z_{ij} for elements above the diagonal in the comparison matrix

Multiple Optimal Solutions

- The first stage LP minimizes the product of errors ε_{ij}
- But, multiple optimal solutions may exist
- In the second stage LP, we select from this set of alternative optima, the solution that minimizes the maximum of errors ε_{ij}
- The second stage LP is presented next

Second Stage Linear Program

Minimize $z_{\max}$

subject to

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} z_{ij} = z^*,$$

$$z_{\max} \geq z_{ij}, \quad i, j = 1, 2, \ldots, n; \quad i < j,$$

and all first stage LP constraints

- z^* is the optimal first stage solution value
- $z_{\max}$ is the maximum value of the errors z_{ij}

Illustrating Some Constraints

Fig. 1. 3 x 3 pairwise comparison matrix

1	2	3
1/2	1	1
1/3	1	1

- Error term def. constraint (a_{12})

$$x_1 - x_2 - y_{12} = \ln a_{12} = 0.693$$

- Element dominance constraints (a_{12} and a_{13})

$$x_1 - x_2 \geq 0 \text{ and } x_1 - x_3 \geq 0$$

- Row dominance constraints

$$x_1 - x_2 \geq 0, \quad x_1 - x_3 \geq 0, \quad \text{and} \quad x_2 - x_3 \geq 0$$

Advantages of LP Approach

- Simplicity
  - Easy to understand
  - Computationally fast
  - Readily available software
  - Easy to measure inconsistency

- Sensitivity Analysis
  - Which a_{ij} entry should be changed to reduce inconsistency?
  - How much should the entry be changed?

More Advantages of the LP Approach

- Ensures element dominance and row dominance

Limited protection against rank reversal

- Generality
  - Interval judgments
  - Mixed pairwise comparison matrices
  - Group decisions
  - Soft interval judgments

Modeling Interval Judgments

- In traditional AHP, a_{ij} is a single number that estimates w_i / w_j
- Alternatively, suppose an interval [l_{ij} , u_{ij}] is specified
- Let us treat the interval bounds as hard constraints
- Two techniques to handle interval judgments have been presented by Arbel and Vargas
  - Preference simulation
  - Preference programming

Preference Simulation

- Sample from each interval to obtain a single a_{ij} value for each matrix entry
- Repeat this *t* times to obtain *t* pairwise comparison matrices
- Apply the EM approach to each matrix to produce *t* priority vectors
- The average of the feasible priority vectors gives the final set of weights

Preference Simulation Drawbacks

- This approach can be extremely inefficient when most of the priority vectors are infeasible
- This can happen as a consequence of several tight interval judgments
- How large should *t* be?
- Next, we discuss preference programming

Preference Programming

- It begins with the linear inequalities and equations below

$$l_{ij} \leq w_i / w_j \leq u_{ij}, \qquad i, j = 1, 2, \ldots, n; \quad i < j,$$

$$\sum_{i=1}^{n} w_i = 1 \ ,$$

$$w_i \geq 0 \ , \ i = 1, 2, \ldots, n$$

- LP is used to identify the vertices of the feasible region
- The arithmetic mean of these vertices becomes the final priority vector
- No attempt is made to find the best vector in the feasible region

More on the Interval AHP Problem

Fig. 2. 3 x 3 pairwise comparison matrix with lower and upper bounds [l_{ij} , u_{ij}] for each entry

1	[5,7]	[2,4]
[1/7,1/5]	1	[1/3,1/2]
[1/4,1/2]	[2,3]	1

- Entry a_{12} is a number between 5 and 7
- The matrix is reciprocal
- Entry a_{21} is a number between 1/7 and 1/5
- The first stage LP can be revised to handle the interval AHP problem

A New LP Approach for Interval Judgments

- Set a_{ij} to the geometric mean of the interval bounds

$$a_{ij} = (l_{ij} \ x \ u_{ij})^{½}$$

- This preserves the reciprocal property of the matrix
- If we take natural logs of $l_{ij} \leq w_i / w_j \leq u_{ij}$, we obtain

$$x_i - x_j \geq \ln l_{ij}, \qquad i, j = 1, 2, \ldots, n; \quad i < j,$$

$$x_i - x_j \leq \ln u_{ij}, \qquad i, j = 1, 2, \ldots, n; \quad i < j$$

Further Notes

- When $l_{ij} > 1$, $\boxed{x_i - x_j \geq \ln\ l_{ij}} \Rightarrow x_i - x_j \geq 0 \Rightarrow w_i \geq w_j$ and behaves like an element dominance constraint

- When $u_{ij} < 1$, $\boxed{x_i - x_j \leq \ln\ u_{ij}} \Rightarrow x_i - x_j \leq 0 \Rightarrow w_j \geq w_i$ and behaves like an element dominance constraint

- Next, we formulate the first stage model for handling interval judgments

First Stage Linear Program for Interval AHP

Minimize $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} z_{ij}$		minimize inconsistency
subject to		
$x_i - x_j - y_{ij} = \ln a_{ij}$,	$i, j = 1, 2, \ldots, n;\ i \neq j$,	error term def. (GM)
$z_{ij} \geq y_{ij}$,	$i, j = 1, 2, \ldots, n;\ i < j$,	degree of overestimation
$z_{ij} \geq y_{ji}$,	$i, j = 1, 2, \ldots, n;\ i < j$,	degree of overestimation
$x_1 = 0$,		set one w_i
$x_i - x_j \geq \ln l_{ij}$,	$i, j = 1, 2, \ldots, n;\ i < j$,	lower bound constraint
$x_i - x_j \leq \ln u_{ij}$,	$i, j = 1, 2, \ldots, n;\ i < j$,	upper bound constraint
$z_{ij} \geq 0$,	$i, j = 1, 2, \ldots, n$,	
x_i, y_{ij} unrestricted	$i, j = 1, 2, \ldots, n$	

Note: The second stage LP is as before

Mixed Pairwise Comparison Matrices

Fig. 3. 3 x 3 mixed comparison matrix

1	[8,9]	2
[1/9,1/8]	1	[1/7,1/5]
1/2	[5,7]	1

- Suppose, as above, some entries are single numbers a_{ij} and some entries are intervals [l_{ij}, u_{ij}]
- Our LP approach can easily handle this mixed matrix problem
- The first stage LP is nearly the same as for the interval AHP
- We add element dominance constraints, as needed

$$x_1 - x_3 \geq 0$$

Modeling Group Decisions

- Suppose there are *n* decision makers
- Most common approach
  - Have each decision maker *k* fill in a comparison matrix independently to obtain [a^k_{ij}]
  - Combine the individual judgments using the geometric mean to produce entries A = [a_{ij}] where

$$a_{ij} = [\, a^1_{ij} \times a^2_{ij} \times \ldots \times a^n_{ij} \,]^{1/n}$$

  - EM is applied to A to obtain the priority vector

Modeling Group Decisions using LP

- An alternative direction is to apply the LP approach to mixed pairwise comparison matrices

- We compute interval bounds as below (for $i < j$)

$$l_{ij} = \min \{ a^1_{ij}, a^2_{ij}, \ldots, a^n_{ij} \}$$

$$u_{ij} = \max \{ a^1_{ij}, a^2_{ij}, \ldots, a^n_{ij} \}$$

- If $l_{ij} = u_{ij}$, we use a single number, rather than an interval

- If n is large, we can eliminate the high and low values and compute interval bounds or a single number from the remaining $n - 2$ values

Soft Interval Judgments

- Suppose we have interval constraints, but they are too tight to admit a feasible solution
- We may be interested in finding the "closest-to-feasible" solution that minimizes the first stage and second stage LP objective functions
- Imagine that we multiply each upper bound by a stretch factor $\lambda_{ij} \geq 1$ and that we multiply each lower bound by the inverse $1/\lambda_{ij}$
- The geometric mean given by $a_{ij} = (l_{ij}\, u_{ij})^{1/2} = (l_{ij}/\lambda_{ij} \times u_{ij}\, \lambda_{ij})^{1/2}$ remains the same as before

Setup for the Phase 0 LP

- Let $g_{ij} = \ln(\lambda_{ij})$, which is nonnegative since $\lambda_{ij} \geq 1$
- We can now solve a Phase 0 LP, followed by the first stage and second stage LPs
- The Phase 0 objective is to minimize the product of stretch factors or the sum of the natural logs of the stretch factors
- If the sum is zero, the original problem was feasible
- If not, the first and second stage LPs each include a constraint that minimally stretches the intervals in order to ensure feasibility

Stretched Upper Bound Constraints

- Start with $w_i / w_j \le u_{ij} \lambda_{ij}$
- Take natural logs to obtain

$$x_i - x_j \le \ln (u_{ij}) + \ln (\lambda_{ij})$$

$$x_i - x_j \le \ln (u_{ij}) + g_{ij}$$

$$x_i - x_j - g_{ij} \le \ln (u_{ij})$$

- Stretched lower bound constraints are generated in the same way

The Phase 0 LP

Minimize $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} g_{ij}$ — minimize the stretch

$x_i - x_j + g_{ij} \geq \ln(l_{ij})$, $i, j = 1, 2, \ldots, n;\ i < j$,

$x_i - x_j - g_{ij} \leq \ln(u_{ij})$, $i, j = 1, 2, \ldots, n;\ i < j$,

} stretched lower and upper bound constraints

error term def. (GM),

degree of overestimation,

set one w_i,

$z_{ij}, g_{ij} \geq 0$ $i, j = 1, 2, \ldots, n$,

x_i, y_{ij} unrestricted $i, j = 1, 2, \ldots, n$

Two Key Points

- We have shown that our LP approach can handle a wide variety of AHP problems
  - Traditional AHP
  - Interval judgments
  - Mixed pairwise comparison matrices
  - Group decisions
  - Soft interval judgments
- As far as we know, no other single approach can handle all of the above variants

Computational Experiment: Inconsistency

Fig. 4. Matrix 1

1	5	1	4	2	6	7
1/5	1	1/8	1	1/3	4	2
1	8	1	5	3	3	3
1/4	1	1/5	1	1/2	1/2	2
1/2	3	1/3	2	1	7	2
1/6	1/4	1/3	2	1/7	1	1/2
1/7	1/2	1/3	1/2	1/2	2	1

- We see that element 4 is less important than element 6
- We expect to see $w_4 \leq w_6$
- Upon closer examination, we see $a_{46} = a_{67} = a_{74} = \frac{1}{2}$
- We expect to see $w_4 = w_6 = w_7$

The Impact of Element Dominance

Table 1

Priority vectors for Matrix 1

Weight	EM	LLS	Second-stage LP model
	RD	RD	ED and RD
w_1	0.291	0.312	0.303
w_2	0.078	0.073	0.061
w_3	0.300	0.293	0.303
w_4	0.064	0.064	0.061
w_5	0.159	0.157	0.152
w_6	0.051	0.044	0.061
w_7	0.058	0.057	0.061

ED: Element Dominance, RD: Row Dominance

Another Example of Element Dominance

Fig. 5. Matrix 2

1	2	2.5	8	5
1/2	1	1/1.5	7	5
1/2.5	1.5	1	5	3
1/8	1/7	1/5	1	1/2
1/5	1/5	1/3	2	1

- The decision maker has specified that $w_2 \leq w_3$
- EM and LLS violate this ED constraint
- As with Matrix 1, the weights from EM, LLS, and LP are very similar

Computational Results for Matrix 2

Table 2

Priority vectors for Matrix 2

Weight	EM	LLS	Second-stage LP model
	RD	RD	ED and RD
w_1	0.419	0.422	0.441
w_2	0.242	0.239	0.221
w_3	0.229	0.227	0.221
w_4	0.041	0.041	0.044
w_5	0.070	0.071	0.074

ED: Element Dominance, RD: Row Dominance

Computational Experiment: Interval AHP

Fig. 6. Matrix 3

1	[2,5]	[2,4]	[1,3]
[1/5,1/2]	1	[1,3]	[1,2]
[1/4,1/2]	[1/3,1]	1	[1/2,1]
[1/3,1]	[1/2,1]	[1,2]	1

Table 3

Priority vectors for Matrix 3

	Preference simulation[a]			Preference programming[a]	Second-stage LP model
Weight	Minimum	Average	Maximum		
w_1	0.369	0.470	0.552	0.469	0.425
w_2	0.150	0.214	0.290	0.201	0.212
w_3	0.093	0.132	0.189	0.146	0.150
w_4	0.133	0.184	0.260	0.185	0.212

[a] Results from Arbel and Vargas

Computational Experiment with a Mixed Pairwise Comparison Matrix

Fig. 7. Matrix 4

1	[2,4]	4	[4.5,7.5]	1
[1/4,1/2]	1	1	2	[1/5,1/3]
1/4	1	1	[1,2]	1/2
[1/7.5,1/4.5]	1/2	[1/2,1]	1	1/3
1	[3,5]	2	3	1

- We converted every interval entry into a single a_{ij} entry by taking the geometric mean of the lower bound and upper bound
- We applied EM to the resulting comparison matrix
- We compared the EM and LP results

Computational Results for Matrix 4

Table 4

Priority vectors for Matrix 4

Weight	EM	Second-stage LP model
w_1	0.377	0.413
w_2	0.117	0.103
w_3	0.116	0.103
w_4	0.076	0.071
w_5	0.314	0.310

- We point out that the weights generated by EM violate one of the four interval constraints
- The interval [1/5, 1/3] is violated

Group AHP Experiment

- Four graduate students were given five geometric figures (from Gass)
- They were asked to compare (by visual inspection) the area of figure *i* to the area of figure *j* ($i < j$)
- Lower and upper bounds were determined, as well as geometric means
- Since $l_{34} = u_{34} = 4.00$, we use a single number for a_{34}
- Otherwise, we have interval constraints

Geometry Experiment Results

Table 5

Priority vectors for geometry experiment

Weight	EM	LLS	Second-stage LP model	Actual geometric areas
w_1	0.272	0.272	0.277	0.273
w_2	0.096	0.096	0.095	0.091
w_3	0.178	0.178	0.172	0.182
w_4	0.042	0.042	0.041	0.045
w_5	0.412	0.412	0.414	0.409

- The three priority vectors and the actual geometric areas (normalized to sum to one) are presented above
- They are remarkably similar

Computational Experiment with Soft Intervals

Fig. 8. Matrix 5

(above the diagonal)

1	[2,5]	[2,4]	[1,2]
	1	[2.5,3]	[1,1.5]
		1	[0.5,1]
			1

- We observe that several intervals are quite narrow
- We apply Phase 0 and the two-stage LP approach

Soft Interval (Matrix 5) Results

- The optimal stretch factors are

$$\lambda_{12} = 1.2248, \qquad \lambda_{23} = 1.0206,$$

$$\lambda_{13} = \lambda_{14} = \lambda_{24} = \lambda_{34} = 1$$

- The a_{12} and a_{23} intervals stretch from

[2,5] to [1.6329, 6.124]

[2.5,3] to [2.4495, 3.0618]

- The optimal weights are

$w_1 = 0.4233$, $w_2 = 0.2592$, $w_3 = 0.1058$, $w_4 = 0.2116$

Conclusions

- We have presented a compact LP approach for estimating priority vectors in the AHP
- In general, the weights generated by EM, LLS, and our LP approach are similar
- The LP approach has several advantages over EM and LLS
  - LPs are easy to understand
  - Sensitivity analysis
  - Our measure of inconsistency is intuitively appealing
  - Ensures ED and RD conditions
  - Our approach is more general

The End (Really)

- The LP approach can handle a wide variety of AHP problems
  - Traditional AHP
  - Interval entries
  - Mixed entries
  - Soft intervals
  - Group AHP
- We hope to explore extensions and new applications of this approach in future research
- Thank you for your patience