Christofides' TSP Algorithm

Triangle Inequality Holds

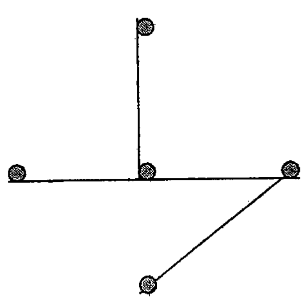
Step 1. Find a minimal spanning tree T of G.

Step 2. Identify all the odd degree nodes in T. Solve a minimum cost perfect matching on the odd degree nodes using the original cost matrix. Add the branches from the matching solution to the branches already in T, obtaining an Eulerian cycle. In this subgraph, every node is of even degree although some nodes may have degree greater than 2.

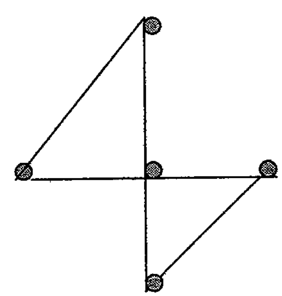
Christofides' Algorithm -- continued

Step 3. Remove polygons over the nodes with degree greater than 2 and transform the Eulerian cycle into a Hamiltonian cycle.

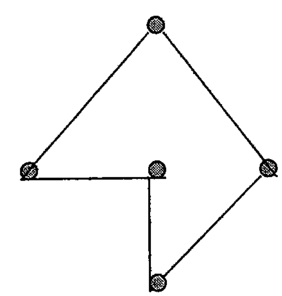
$$\frac{\text{Length of Algorithm Tour}}{\text{Length of Optimal Tour}} < 1.5$$



Minimal Spanning Tree



Eulerian Cycle



Hamiltonian Cycle

Christofides' Algorithm -- continued

THEOREM

let $l(\phi_H)$ = Length of Christofides' Tour,

$l(\phi^*)$ = Length of Optimal Tour,

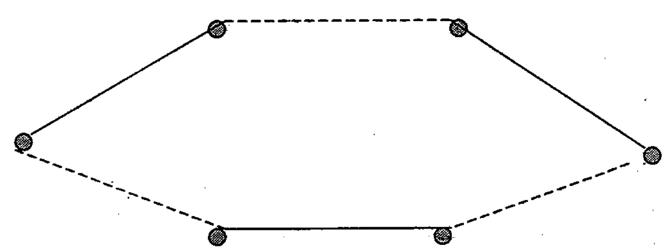
and assume the triangle inequality holds. Then

$l(\phi_H)/ (\phi^*) < 1.5$

Proof: Let T be the minimal spanning tree of G with length $l(T)$. Clearly, $l(T) < l(\phi^*)$, since any Hamiltonian path is a spanning tree. There are an even number of odd degree nodes in T. Let the minimum cost perfect matching over this set S_o of odd nodes be denoted by M_o with length $l(M_o)$.

Christofides' Algorithm -- continued

We will make use of the fact that $l(M_o) \leq l(\phi_o)/2$, where ϕ_o is the optimal TSP tour over S_o and $l(\phi_o)$ is its length. This is easy to demonstrate. Suppose that the tour ϕ_o is shown below.



This tour consists of two alternating matchings M_1 and M_2 such that

$$l(M_1) + l(M_2) = l(\phi_o).$$

Christofides' Algorithm -- continued

Since M_1 and M_2 are defined arbitrarily, we can assume that $l(M_1) \leq l(M_2)$, and so we have

$$l(M_o) \leq l(\phi_o)/2 \leq l(\phi^*)/2.$$

The last inequality follows directly from the triangle inequality. Since the Hamiltonian cycle obtained in Step 3 is shorter in length than the Eulerian cycle obtained in Step 2 of the heuristic, it follows that

$$l(\phi_H) \leq l(T) + l(M_o) < l(\phi^*) + l(\phi^*)/2,$$

which yields the desired result.