Network Flows

- What is the maximum flow that can be transmitted from a source s to a sink t ?
- Let q_{ij} be the capacity of arc (i, j).
- The solution indicates the arcs in the network which are *saturated*.
- Labeling techniques (such as the one developed by Ford and Fulkerson) exist for solving the maximum flow problem, but we will present an algorithm which makes repeated use of a shortest path procedure.

- Formulation is given below

$$\max \quad v$$

$$s.t. \quad \sum_{y \in A(x)} \varphi(x,y) - \sum_{y \in B(x)} \varphi(y,x) = \begin{cases} v & x = s \\ 0 & x \neq s,t \\ -v & x = t \end{cases}$$

$$r_{xy} = 0 \leq \varphi(x,y) \leq q_{xy} \qquad all\ (x,y) \in A.$$

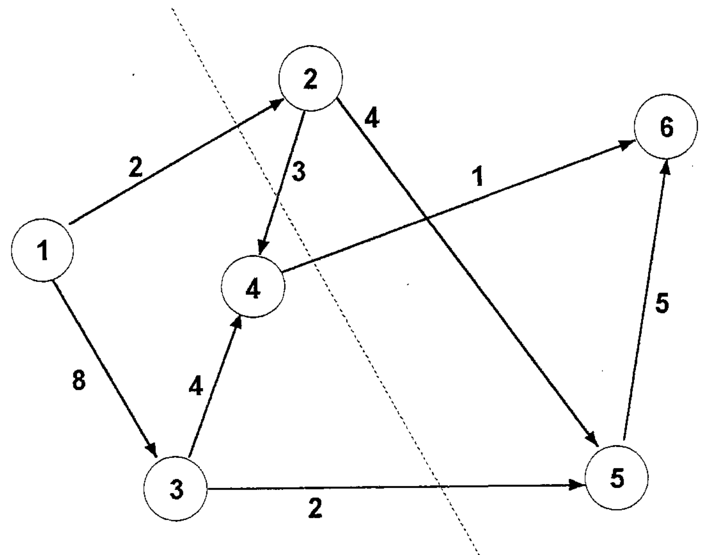
- Next, we state the famous **max-flow, min-cut theorem**.

- **Theorem:** For any network the maximal flow value from s to t is equal to the minimal cut capacity of all cuts separating s and t.

- **Definition**: A cut separating s and t is a set of arcs $(X, \overline{X})$ where $s \in X$ and $t \in \overline{X}$.

- **Definition:** The capacity of the cut $(X, \overline{X})$ is $\Sigma \{q_{xy} : x \in X,\ y \in \overline{X}\}$.

- Illustration of minimal cut concept ($s = 1$, $t = 6$)



Flow paths: {1,2,5,6}, {1,3,4,6}, {1,3,5,6}, {1,2,4,6}

Minimal Cut = { (1, 2), (3, 5), (4, 6)}

$X = \{1, 3, 4\}$

$\overline{X} = \{2, 5, 6\}$

Introduction to Maximal Flow Algorithm

- Let φ_0 denote the starting flow which is identically zero.
- Find a succession of flows $\varphi_1, \varphi_2, \ldots, \varphi_k$ having flow values 1, 2, ..., k, where φ_k is the maximal flow.
- We work from an associated graph or incremental graph I.
- We associate a length λ with every arc in I as follows:

$$\lambda(x,y) = \begin{cases} 0 & \text{if} \quad \varphi(x,y) < q_{xy} \\ \infty & \text{if} \quad \varphi(x,y) = q_{xy} \end{cases}$$

$$\lambda(y,x) = \begin{cases} 0 & \text{if} \quad \varphi(x,y) > r_{xy} \\ \infty & \text{if} \quad \varphi(x,y) = r_{xy} \end{cases}$$

Maximal-Flow Algorithm

1. Initially, set i = 0 and take φ_0 as the flow which is identically zero in every arc.

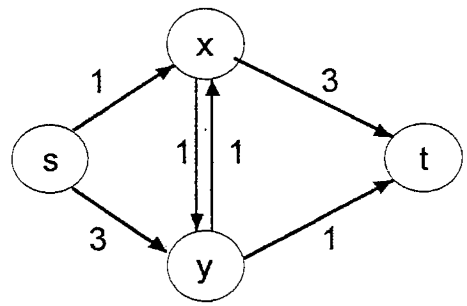
2. Determine the shortest distance from s to t in I, relative to the distance function λ_i.

3. If the distance determined in step 2 is finite, let C denote any simple path from s to t with shortest length and let σ denote the corresponding simple path flow. Define $\varphi_{i+1} = \varphi_i + \sigma_C$ and repeat step 2, with i+1 in place of i.

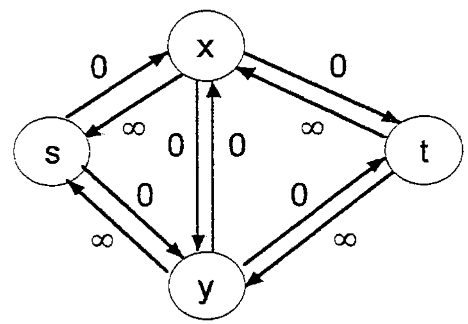
4. If the shortest distance from s to t is ∞, then φ_i is a flow having maximal value, and the algorithm may be terminated.

■ What is the serious drawback to this procedure ?

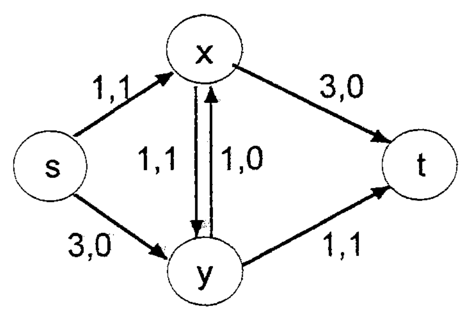
Example of the Maximal-Flow Algorithm



capacities are written adjacent to arcs.

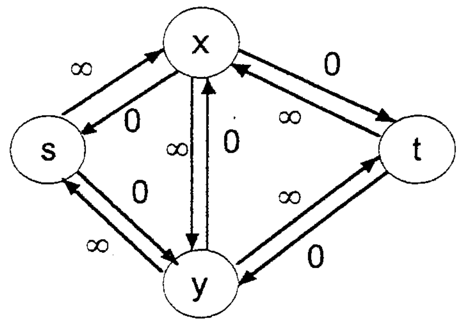


distances are written adjacent to arcs.
C = s, x, y, t.

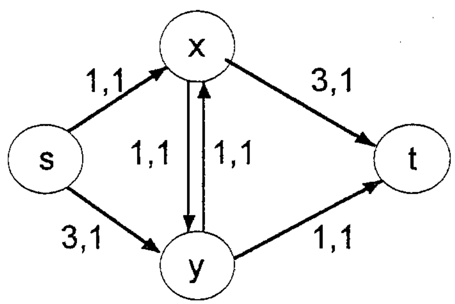


capacity, flow pairs are written adjacent to arcs.

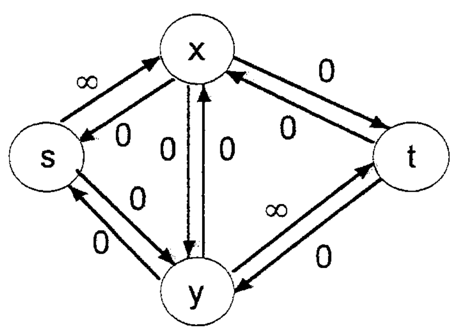
Example of the Maximal-Flow Algorithm (continued)



distances are written adjacent
to arcs.
C = s, y, x, t.

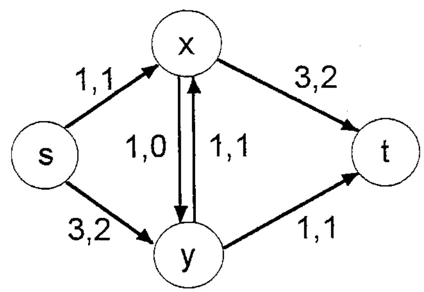


capacity, flow pairs are written
adjacent to arcs.

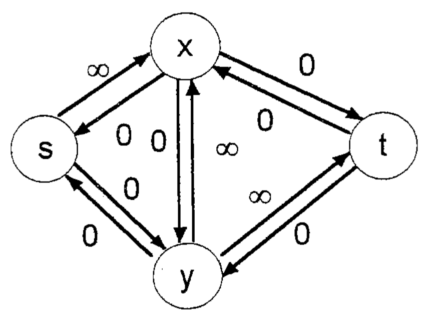


distances are written adjacent
to arcs. C = s, y, x, t. If arcs gets
two distances, choose smaller
of two.

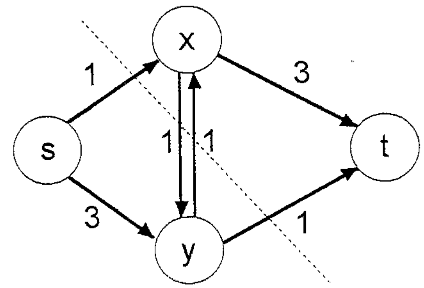
Example of the Maximal-Flow Algorithm (continued)



capacity, flow pairs are written adjacent to arcs.

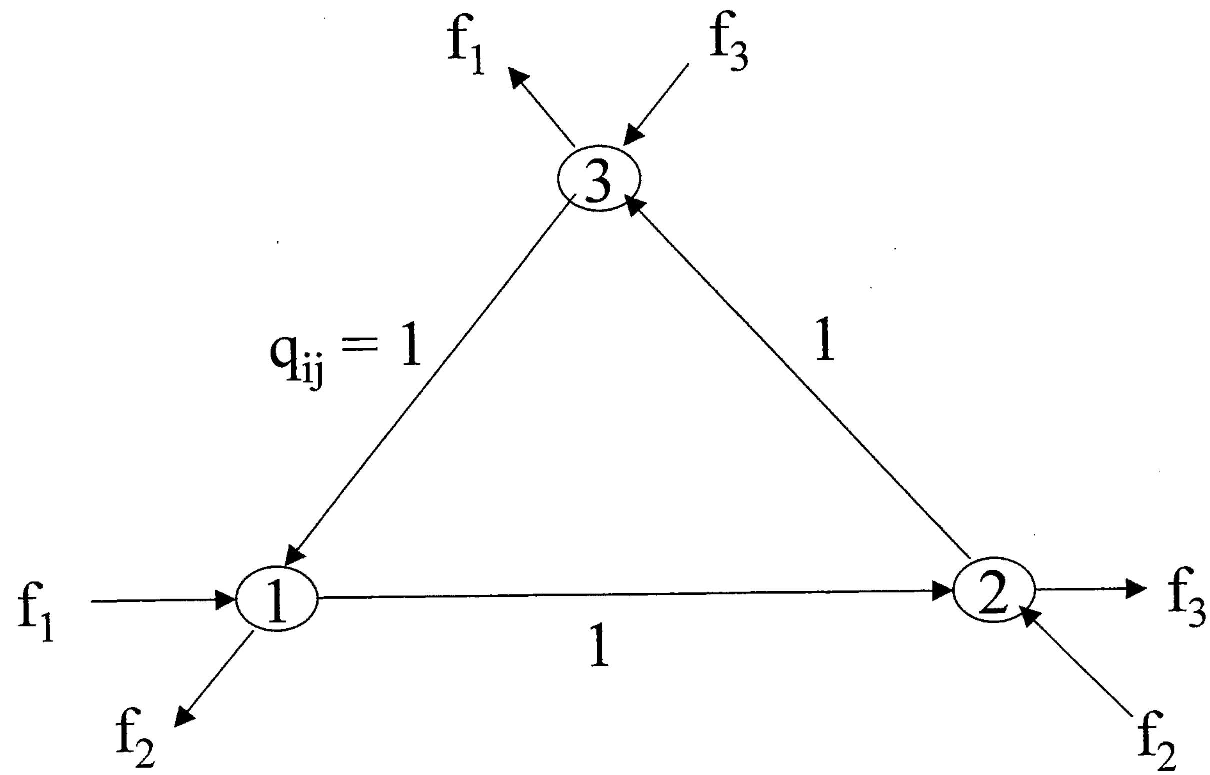


distances are written adjacent to arcs.
Shortest distance from s to t is ∞,
so maximal flow is 3 units.



capacities are written adjacent to arcs. Minimal cut is shown.

Multicommodity Flow Problems



Three commodities: $1 \rightarrow 3$, $2 \rightarrow 1$, and $3 \rightarrow 2$

Maximal flow problem: maximize $f_1 + f_2 + f_3$

Is optimal solution integral?

Min Cost Network Flows

- Find a feasible flow of k units from s to t in a capacitated network, whose cost is minimal

- Terminology

  $\beta\ (i,j) =$ capacity of arc (i,j)

  $c\ (i,j) =$ per unit cost of arc (i,j)

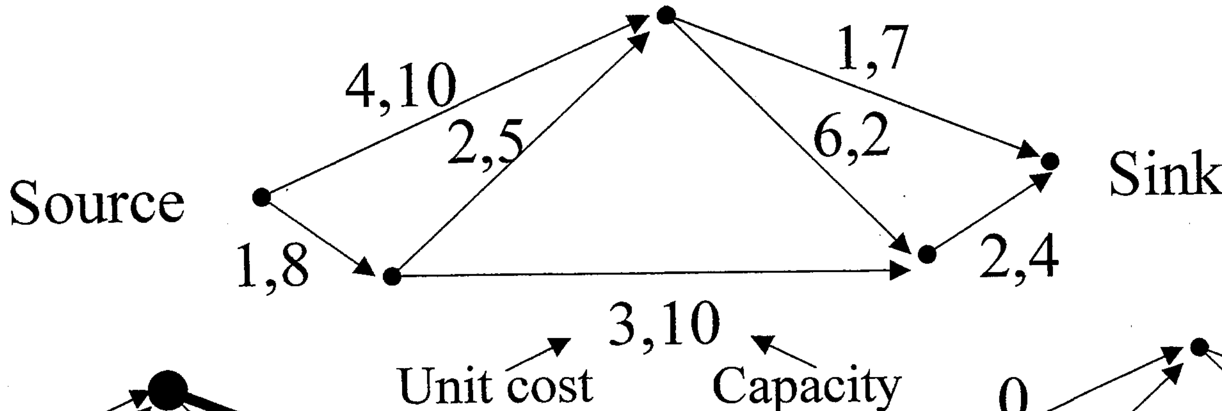
  $\varphi\ (i,j) =$ flow on arc (i,j)

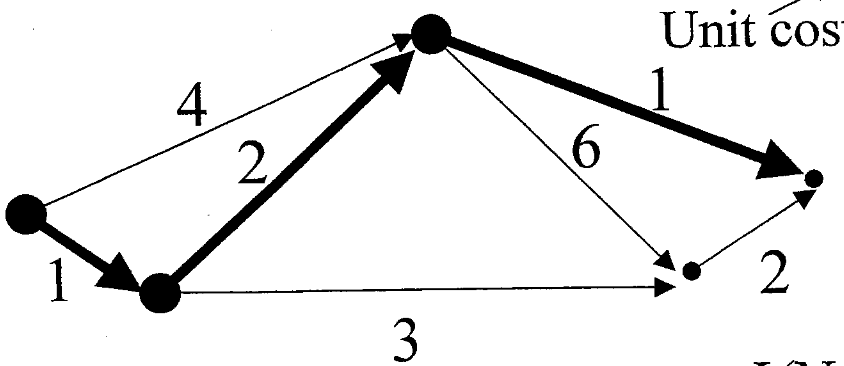
- Solution Procedure
  - Based on incremental graph concept
  - Illustrated on pages 17-13 & 17-14

- Incremental Graph Definitions
  - $\lambda\,(i,j) = c\,(i,j)$ if $\varphi\,(i,j) < \beta\,(i,j)$
    $= \infty$ if $\varphi\,(i,j) = \beta\,(i,j)$
  - $\lambda\,(j,i) = -c\,(i,j)$ if $\varphi\,(i,j) > 0$
    $= \infty$ if $\varphi\,(i,j) = 0$

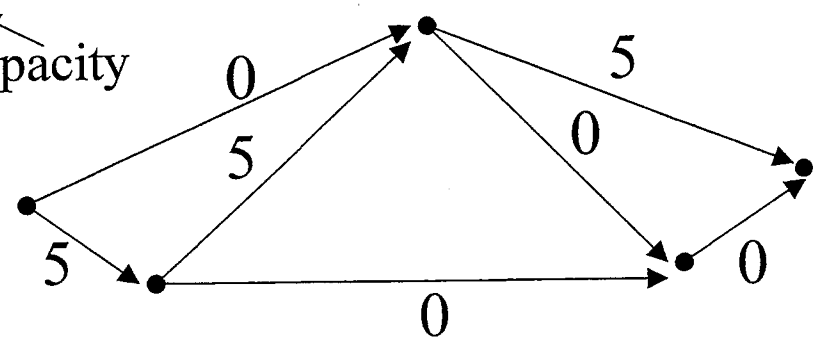
- Theorem: There are no negative cycles in the incremental graph. Therefore, the optimal solution is obtained.

A Small Example

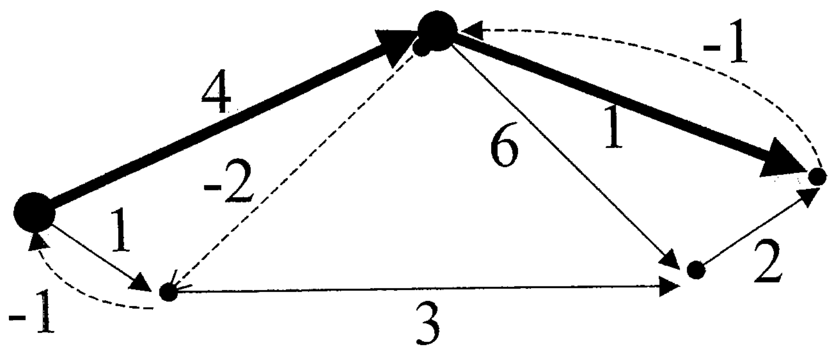




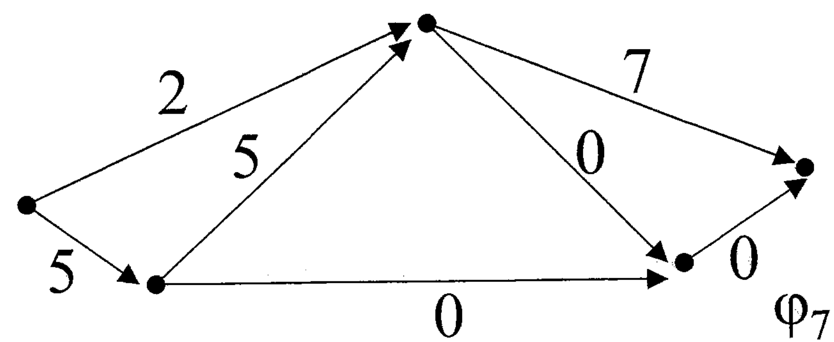
$I(N, \varphi_0)$



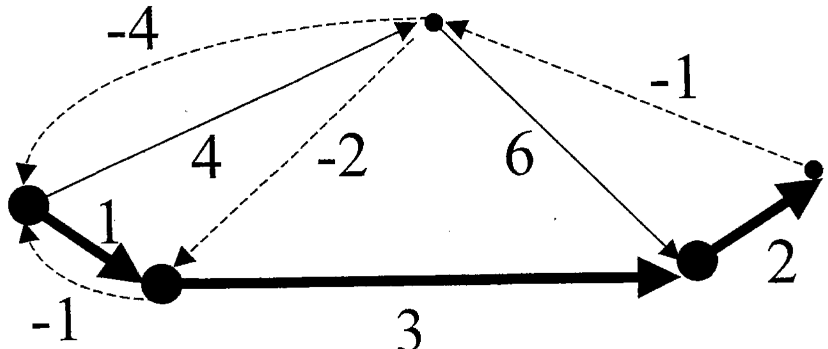
φ_5



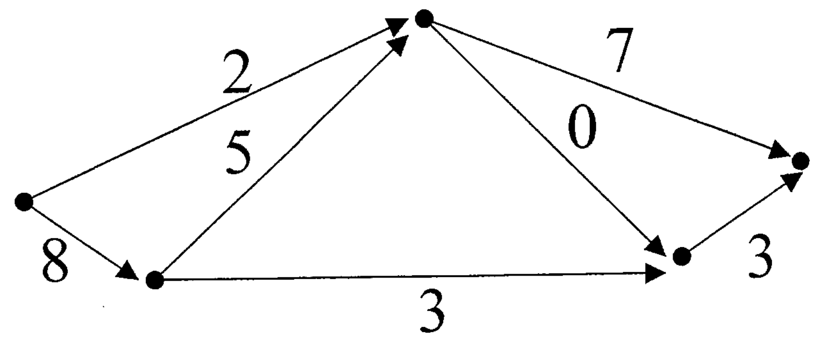
$I(N, \varphi_5)$



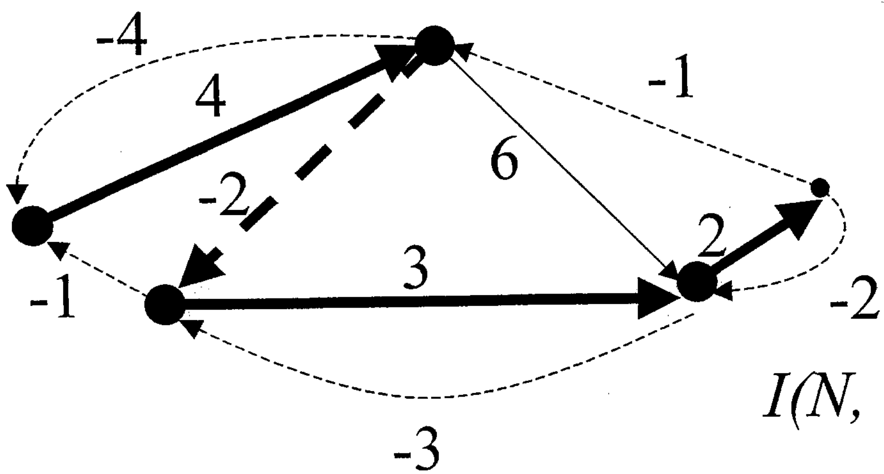
φ_7



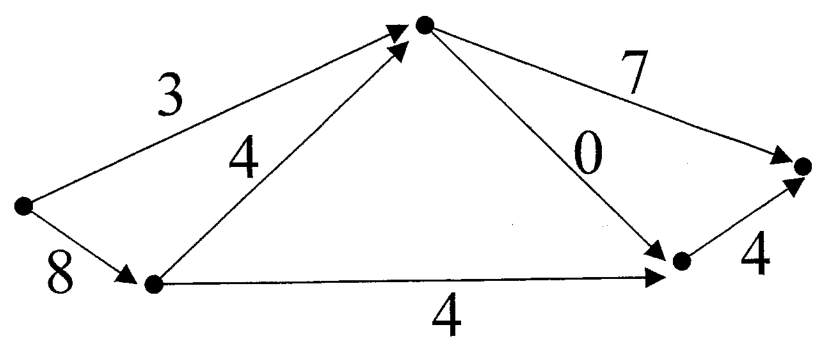
$I(N, \varphi_7)$



φ_{10}



$I(N, \varphi_{10})$



φ_{11}