TRANSPORTATION, TRANSSHIPMENT, AND ASSIGNMENT PROBLEMS

- There are m supply points with items available to be shipped to n demand points. Specifically, plant i can ship at most S_i items, and demand point j requires at least D_j items.
- The cost of shipping each unit from plant i to demand point j is c_{ij}.
- The objective is to select a routing plan that minimizes total transportation costs.

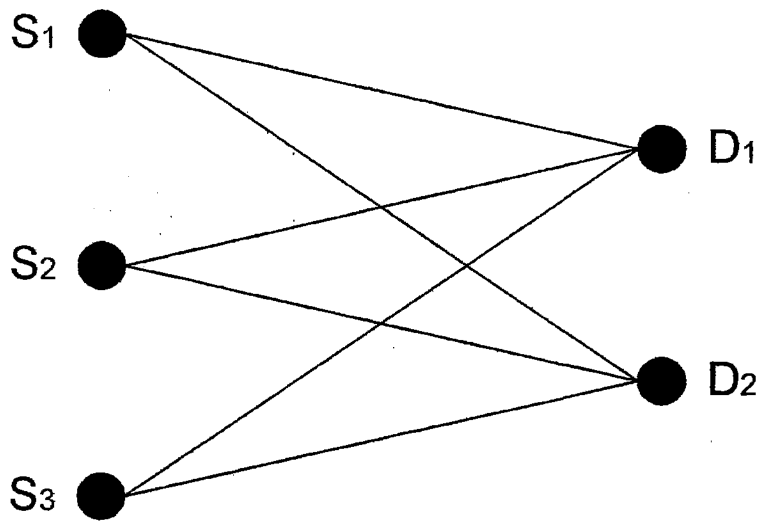
$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} \le S_i \qquad \text{for } i = 1, \ldots, m \qquad \text{supply}$$

$$\sum_{i=1}^{m} x_{ij} \ge D_j \qquad \text{for } j = 1, \ldots, n \qquad \text{demand}$$

$$x_{ij} \ge 0 \qquad \text{for all } i, j$$

TRANSPORTATION PROBLEMS



flow from plants to warehouses

- The following result explains why we formulate the transportation problem as a linear programming problem (LP).

Among all the optimal solutions to the transportation model, there is at least one in which each x_{ij} is *integer-valued* provided the S_i and D_j are all positive integers, which we assume. The regular simplex method will find such a solution.

TRANSPORTATION PROBLEMS

- For the model to possess a feasible solution, total supply must be at least as large as total demand,

$$\sum_{i=1}^{m} S_i \geq \sum_{j=1}^{n} D_j.$$

- In analyzing a standard transportation model and devising an optimizing algorithm, however, it is convenient to assume that total supply equals total demand.

  Create a fictitious destination with a requirement of $\sum_i S_i - \sum_j D_j$. Denote this destination as fict, set $c_{i,fict} = 0$ and $x_{i,fict}$ is the slack capacity at plant i.

- A special case of the transportation problem is the *assignment problem*.

THE ASSIGNMENT PROBLEM

Each of n tasks can be performed by any of n agents. The cost of task *i* being accomplished by agent *j* is c_{ij}. Assign one agent to each task to minimize the total cost.

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} = 1 \qquad \text{for all } i$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for all } j$$

$$x_{ij} \in \{0, 1\}$$

LP AND THE TRANSPORTATION PROBLEM

- The simplex solution to an LP problem always has at most ℓ positive variables, where ℓ is the number of independent constraints.

- Convince yourself that there are m +n - 1 independent constraints in the transportation problem.

- The standard simplex instructions are paraphrased next.

TRANSPORTATION ALGORITHM HIGH LEVEL DESCRIPTION

Step 1. Select a set of $m + n - 1$ routes that provides an initial basic feasible solution.

Step 2. Check whether the solution is improved by introducing a nonbasic variable. If so, go to Step 3; otherwise stop.

Step 3. Determine which routes leave the basis when the variable that you selected in Step 2 enters.

Step 4. Adjust the flows of the other basic routes. Return to Step 2.

TRANSPORTATION ALGORITHM

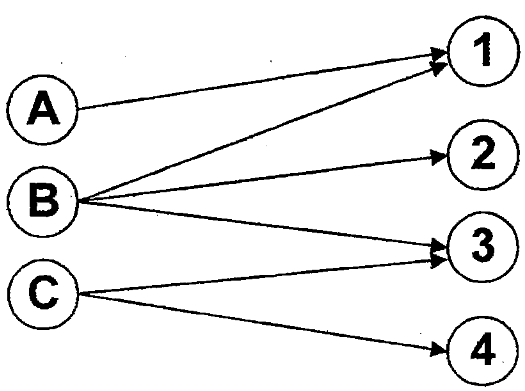
- We need to address the following.

  1. Obtaining an initial solution.
  2. Optimization criterion to be used.
  3. Procedure to improve the current nonoptimal solution.

- First, we consider obtaining a basic feasible solution using the Northwest Corner Rule.

THE NORTHWEST CORNER RULE

	1	2	3	4	Supply
A	8 35	6	10	9	~~35~~ 0
B	9 10	12 20	13 20	7	~~50~~ ~~40~~ ~~20~~ 0
C	14	9	16 10	5 30	~~40~~ ~~30~~ 0
Demand	~~45~~ ~~10~~ 0	~~20~~ 0	~~30~~ ~~10~~ 0	~~30~~ 0	



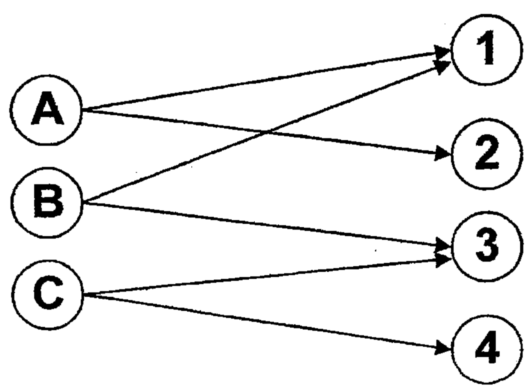
- Initial basis for Northwest Corner Method
- Note connection with spanning trees

Minimum Matrix Method

	1	2	3	4	Supply
A	8	6	10	9	35
B	9	12	13	7	50
C	14	9	16	5	40
Demand	45	20	30	30	

THE MINIMUM MATRIX METHOD

	1	2	3	4	Supply
A	[8] 15	[6] 20	[10]	[9]	~~35~~ ~~15~~ 0
B	[9] 30	[12]	[13] 20	[7]	~~50~~ ~~20~~ 0
C	[14]	[9]	[16] 10	[5] 30	~~40~~ ~~10~~ 0
Demand	~~45~~ ~~30~~ 0	~~20~~ 0	~~30~~ ~~10~~ 0	~~30~~ 0	



- Notice that the last arcs chosen are very expensive.
- Minimum Matrix Method often employs some of the least and most expensive arcs.

VOGEL'S APPROXIMATION

	1	2	3	4	Supply	Row Difference
A	8	6	10	9	35	2
B	9	12	13	7	50	2
C	14	9	16	5 30	~~40~~ 10	4 ⟵
Demand	45	20	30	~~30~~ **0**		
Column Difference	1	3	3	2		

Remove column 4 and continue.

VOGEL'S APPROXIMATION — CONTINUED

	1	2	3	Supply	Row Difference
A	8	6	10	35	2
B	9	12	13	50	3
C	14	9 10	16	~~10~~ 0	5 ⟵
Demand	45	~~20~~ 10	30		
Column Difference	1	3	3		

Remove row C and continue.

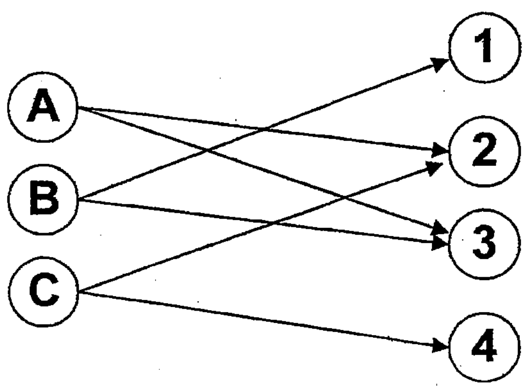
VOGEL'S APPROXIMATION — CONTINUED

	1	2	3	Supply	Row Difference
A	8	6 10	10	~~35~~ 25	2
B	9	12	13	50	3
Demand	45	~~10~~ 0	30		
Column Difference	1	6 ↑	3		

Remove column 2 and continue.

VOGEL'S APPROXIMATION — CONTINUED

	1	3	Supply	Row Difference
A	8	10 25	~~25~~ 0	2
B	9 45	13 5	~~50~~ 0	4 ←
Demand	~~45~~ 0	~~30~~ 0		
Column Difference	1	3		



<u>Total Cost</u>

Northwest Corner Rule	$1180
Minimum Matrix Method	$1080
Vogel's Approximation	$1020

BASIC FEASIBLE SOLUTIONS

- The three procedures for finding an initial basic feasible solution resulted in 3 different bases. However, there are similarities:

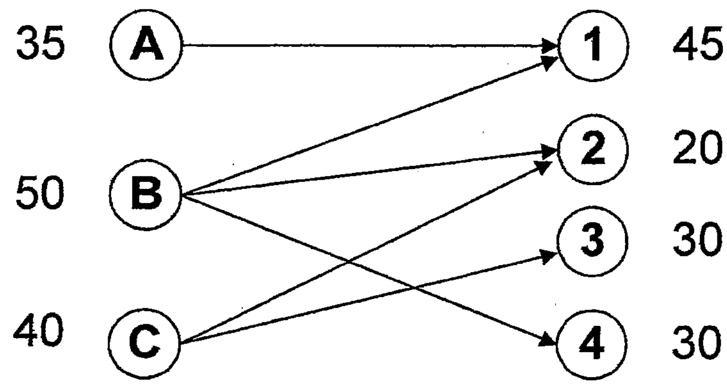
  (1) Each basis consists of exactly $m + n - 1$ arcs, one less than the number of nodes in the network.

  (2) Ignore directions on arcs and we have a spanning tree.

- The fact that the basis corresponds to a spanning tree makes the solution of these problems by the simplex method very efficient.

THE TRIANGULARITY PROPERTY

- Suppose we are given the feasible basis shown below.



- We can read off the corresponding feasible solution. This is known as the triangularity property.
- As long as initial supplies and demands are integers, any basic feasible solution will be an integer solution.

OPTIMALITY CONDITIONS

- For the sample transportation problem that we have worked with, we show that Vogel's method gives the optimal solution.
- Formulate the primal LP problem in equality-constraint form. The dual LP becomes

$$\max \quad \sum_{i=1}^{m} S_i V_i + \sum_{j=1}^{n} D_j W_j$$

$$V_i + W_j \leq C_{ij} \quad \forall (i,j)$$

- The theorem of complementary slackness gives us our optimality conditions.

$$\overline{C}_{ij} = V_i + W_j - C_{ij} \leq 0 \quad \text{if } X_{ij} = 0$$

$$\overline{C}_{ij} = V_i + W_j - C_{ij} = 0 \quad \text{if } X_{ij} > 0.$$

OPTIMALITY CONDITIONS APPLIED

- Apply optimality conditions to see if Vogel's solution is optimal.

$\overline{C}_{ij} = V_i + W_j - C_{ij} \leq 0 \quad \text{if} \quad X_{ij} = 0; \quad \overline{C}_{ij} = V_i + W_j - C_{ij} = 0 \quad \text{if} \quad X_{ij} > 0.$

	1	2	3	4	V_i
A	8 / -2	6 / 0	10 / 0	9 / -7	0
B	9 / 0	12 / -3	13 / 0	7 / -2	3
C	14 / -5	9 / 0	16 / -3	5 / 0	3
W_j	6	6	10	2	

Therefore, $1020 is the best we can do.

	1	2	3	4	V_i
A	8	6 / 0	10 / 0	9	0
B	9 / 0	12	13 / 0	7	
C	14	9 / 0	16	5 / 0	
w_j					

TRANSPORTATION ALGORITHM ILLUSTRATION

	1	2	3	4	Supply
A	2	3	11	7	6
B	1	0	6	1	1
C	5	8	15	9	10
Demand	7	5	3	2	

TRANSPORTATION ALGORITHM — CONTINUED

	1	2	3	4	Supply
A	(2) 6	(3)	(11)	(7)	6
B	(1)	(0) 1	(6)	(1)	1
C	(5) 1	(8) 4	(15) 3	(9) 2	10
Demand	7	5	3	2	

Initial basic feasible solution via minimum matrix method.

Cost = 112

$$\bar{c}_{ij} = v_i + w_j - c_{ij}$$

2 / 0	3	11	7	0
1	0 / 0	6	1	
5 / 0	8 / 0	15 / 0	9 / 0	
				w_j \ v_i

TRANSPORTATION ALGORITHM — CONTINUED

$V_i + W_j - c_{ij}$

2 0	3 2	11 1	7 -1	0
1 -4	0 0	6 1	1 0	-5
5 0	8 0	15 0	9 0	3
2	5	12	6	V_i W_j

Bring route (A, 2) into the basis

6–θ	+θ		
	1		
1+θ	4–θ	3	2

Deciding which variable leaves

TRANSPORTATION ALGORITHM — CONTINUED

2	4		
	1		
5		3	2

New basis cost $= 104$

$V_i + W_j - c_{ij}$

[2] 0	[3] 0	[11] 1	[7] -1	0
[1] -2	[0] 0	[6] 3	[1] 2	-3
[5] 0	[8] -2	[15] 0	[9] 0	3
2	3	12	6	V_i / W_j

Bring route (B, 3) into the basis

TRANSPORTATION ALGORITHM — CONTINUED

$2-\theta$	$4+\theta$		
	$1-\theta$	$+\theta$	
$5+\theta$		$3-\theta$	2

Deciding which variable leaves the basis

1	5		
		1	
6		2	2

New basis cost $=101$

TRANSPORTATION ALGORITHM — CONTINUED

$V_i + W_j - c_{ij}$

[2] 0	[3] 0	[11] 1	[7] -1	0
[1] -5	[0] -3	[6] 0	[1] -1	-6
[5] 0	[8] -2	[15] 0	[9] 0	3
2	3	12	6	W_j \ V_i

Bring route (A, 3) into the basis

$1-\theta$	5	$+\theta$	
		1	
$6+\theta$		$2-\theta$	2

Deciding which variable leaves the basis

TRANSPORTATION ALGORITHM — CONTINUED

	5	1	
		1	
7		1	2

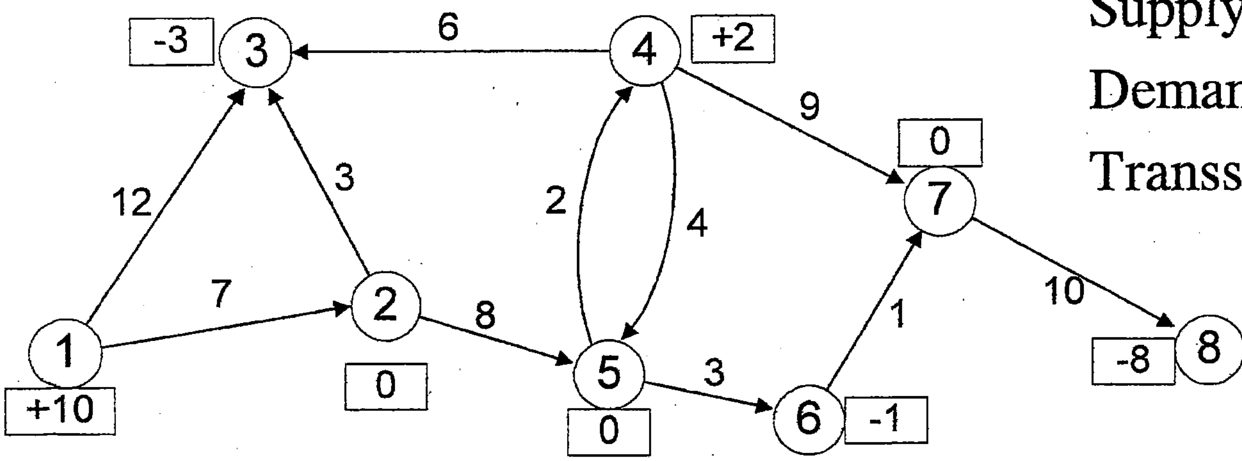
New basis cost = 100

$V_i + W_j - c_{ij}$

[2] -1	[3] 0	[11] 0	[7] -2	0
[1] -5	[0] -2	[6] 0	[1] -1	-5
[5] 0	[8] -1	[15] 0	[9] 0	4
1	3	11	5	V_i / W_j

Optimal, since all $V_i + W_j \leq c_{ij}$

THE TRANSSHIPMENT PROBLEM



Supply nodes: 1, 4

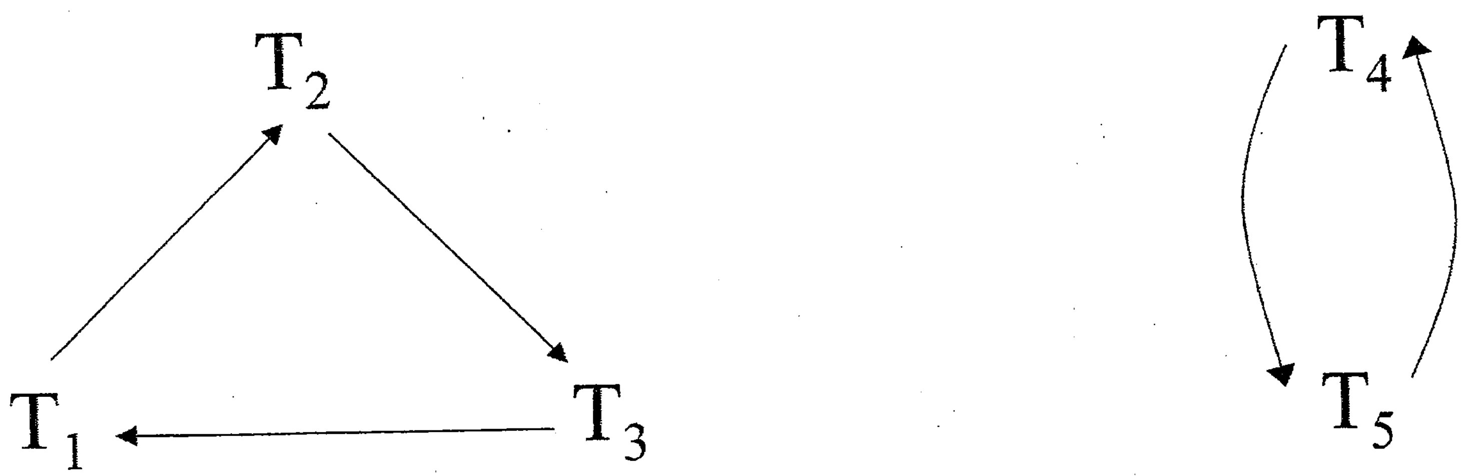
Demand nodes: 3, 6, 8

Transshipment nodes: 2, 5, 7

Equivalence to the transportation problem

	3	6	8	Supply
1	[10] 3	[18] 1	[29] 6	10
4	[6]	[7]	[18] 2	2
Demand	3	1	8	

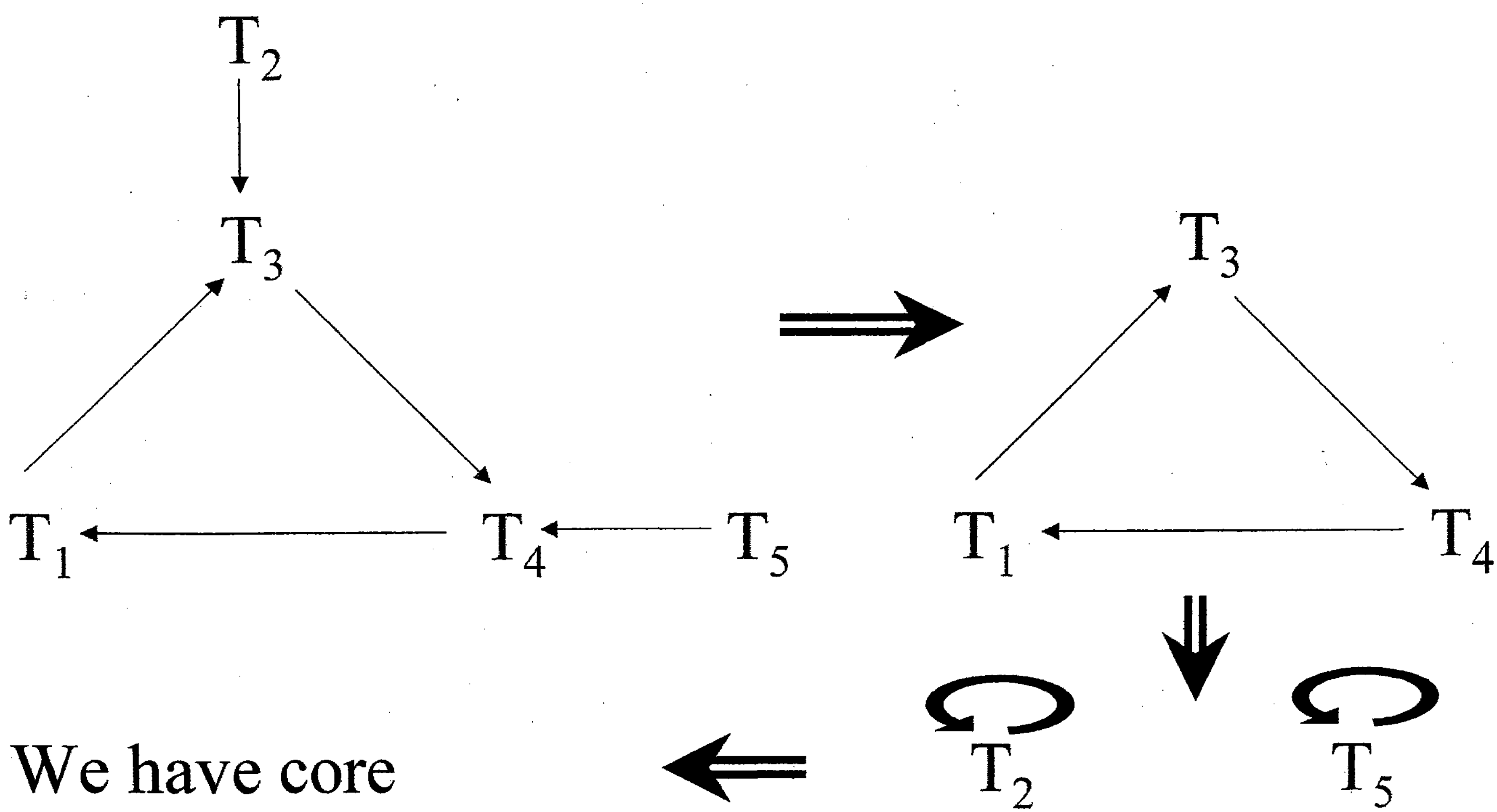
Core Allocations



$T_i \longrightarrow T_j$ means T_i obtains item of T_j

So, T_1 gets a 3, T_4 gets a 2. But if T_1 & T_4 left room and traded, T_1 gets 2, T_4 gets 1!

Core Allocation Algorithm



We have core allocation.

2-25c