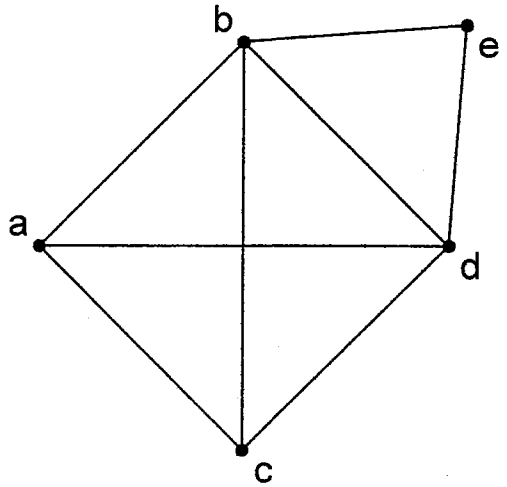
INTRODUCTION: GRAPHS AND NETWORKS

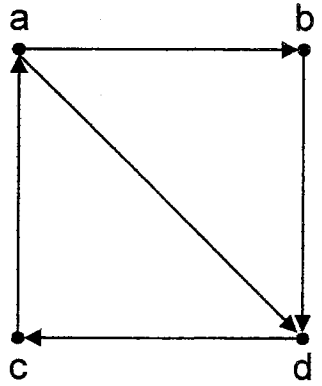
DEFINITION

A graph consists of a nonempty set V, a (possibly empty) set E, and an incidence mapping associated with the graph.

- The elements V and E are called the vertices (nodes) and edges (arc, links) of the graph.
- The incidence mapping indicates the origin and destination of each arc.



Graph



Directed Graph

LINEAR PROGRAMMING AND NETWORK FLOWS

Let vertices v_0 and v_n of a directed graph be designated as the origin and destination of a substance flowing through the arcs. Also assume that an arc from vertex v_i to vertex v_j has a capacity, or upper limit on flow, u_{ij}. Finally, let c_{ij} be the cost per unit flow in the arc. Our flow problem becomes a linear programming problem requiring the minimization of $\sum_{i,j} c_{ij}x_{ij}$ for a total flow c from v_0 to v_n subject to

$$\sum_j \left(x_{0j} - x_{j0}\right) = c$$

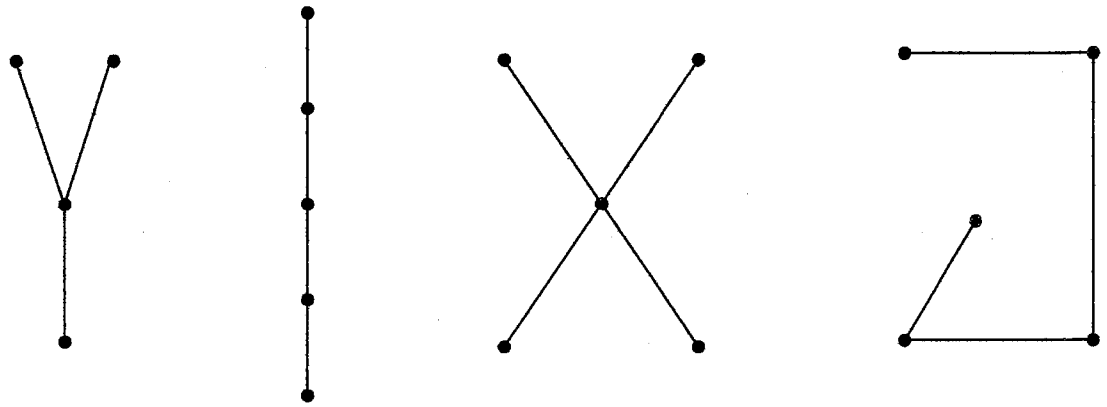
$$\sum_j \left(x_{ij} - x_{ji}\right) = 0 \quad \text{for } i = 1, \ldots, n-1$$

$$\sum_j \left(x_{nj} - x_{jn}\right) = -c$$

$0 \le x_{ij} \le u_{ij}$ for every arc.

DEFINING TREES

- A graph with no cycles is *acyclic*.
- A *tree* is a connected acyclic graph.



Some examples of trees

- A *spanning tree* of a graph G contains or spans all the nodes of G.

DEFINING TREES — CONTINUED

DEFINITIONS

The following definitions of an *undirected tree* can be shown to be all equivalent to each other.

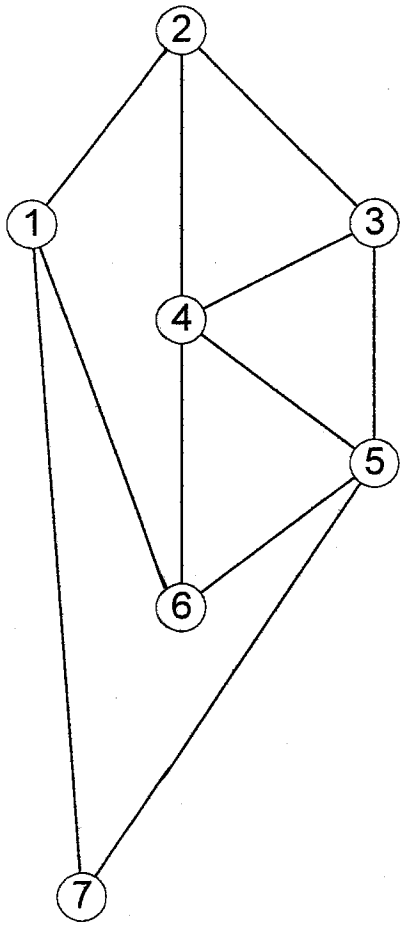
An *undirected tree* is

(i) a connected graph of n vertices and (n-1) links,

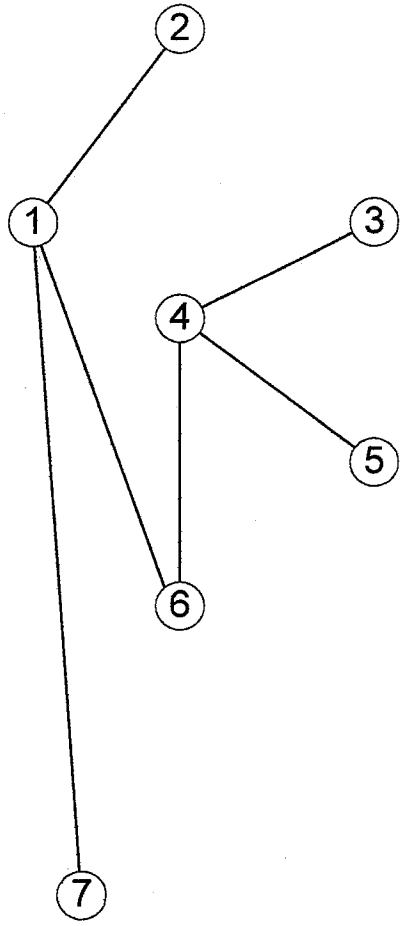
(ii) a connected graph without a cycle,

(iii) a graph in which every pair of vertices is connected with one and only one elementary path.

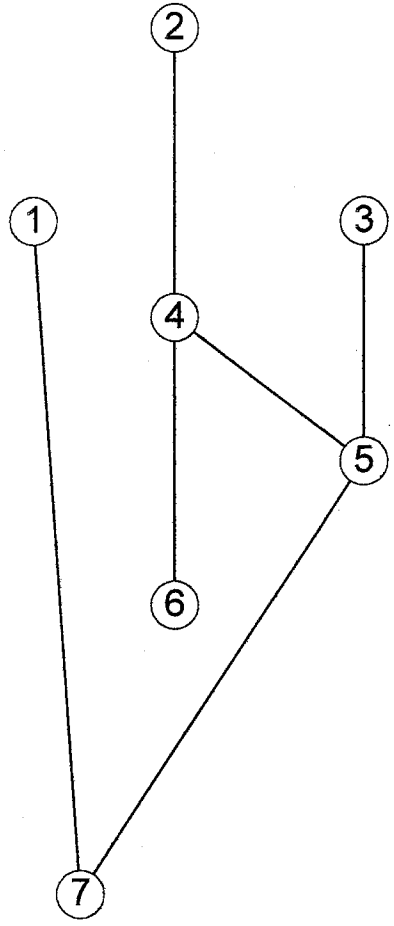
SPANNING TREES



Graph G



A spanning tree of G



Another spanning tree of G

COUNTING TREES

- In 1847, Kirchhoff first developed the theory of trees while studying the simultaneous linear equations associated with the circuitry of an electric network.

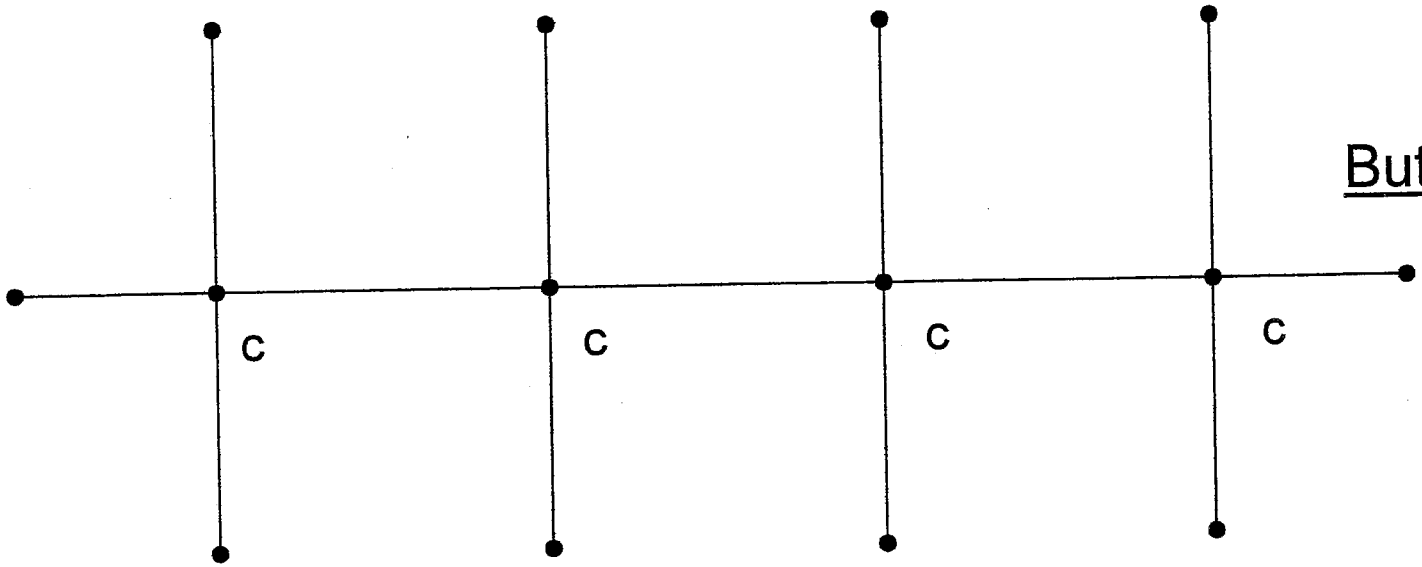
- Cayley discovered trees in 1857 in his work on organic chemistry. He was engaged in enumerating isomers of the saturated hydrocarbons, C_nH_{2n+2}, with a given number n of carbon atoms.

  - Cayley restated the problem abstractly:

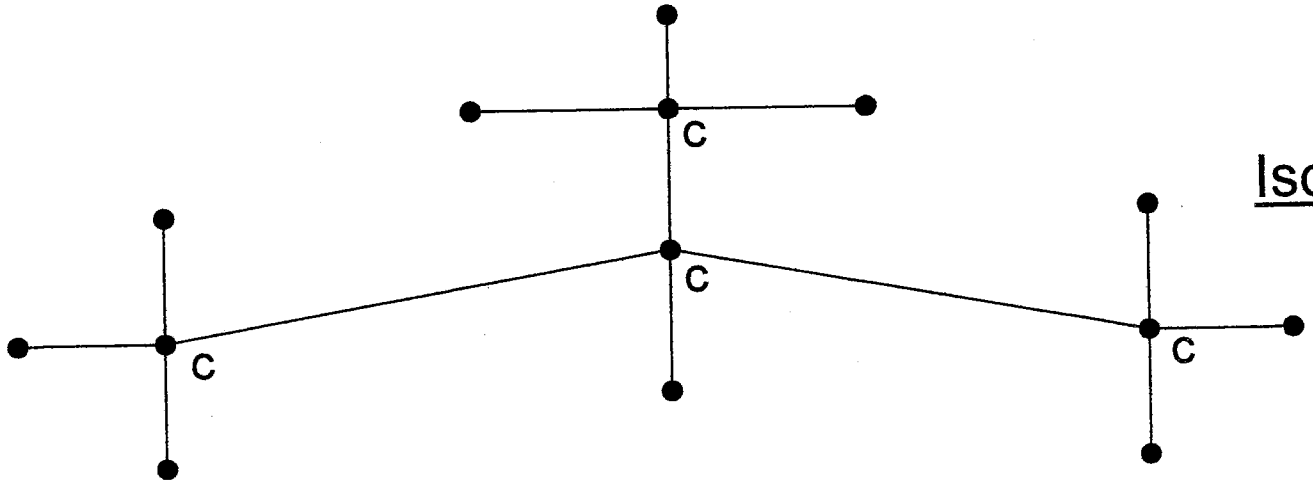
    Find the number of trees in which each point has degree 1 or 4.

    For example, consider C_4H_{10}

COUNTING TREES — CONTINUED



Butane

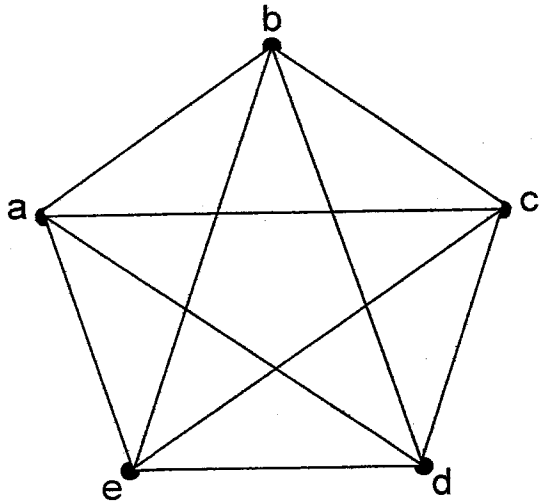


Isobutane

CAYLEY'S FORMULA

Let T(n) be the number of spanning trees with n labeled points $v_1, v_2, \ldots, v_n$.

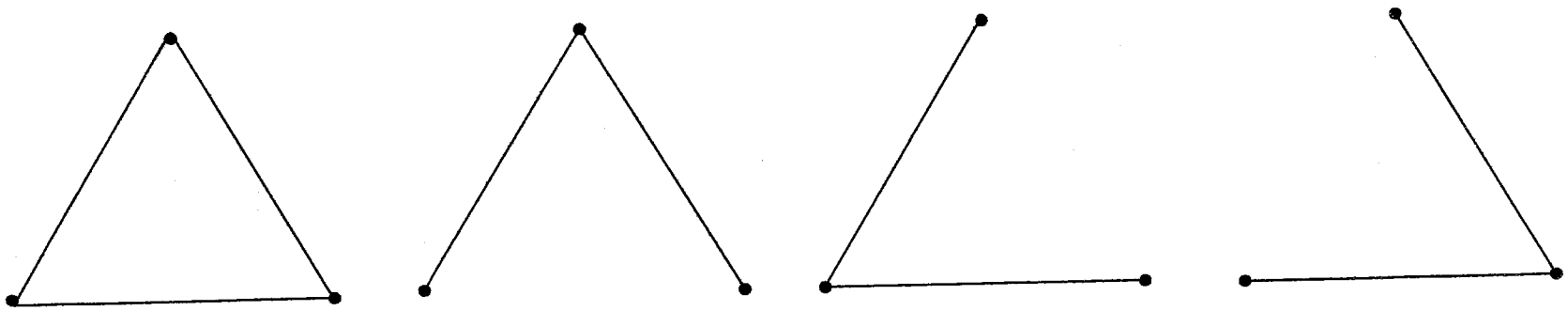
- In 1889, Cayley proved that $T(n) = n^{n-2}$.
- For example, the *complete graph* below has 125 spanning trees.
  - A *complete graph* is one in which every pair of nodes is connected.



Complete Pentagon

CAYLEY'S FORMULA—CONTINUED

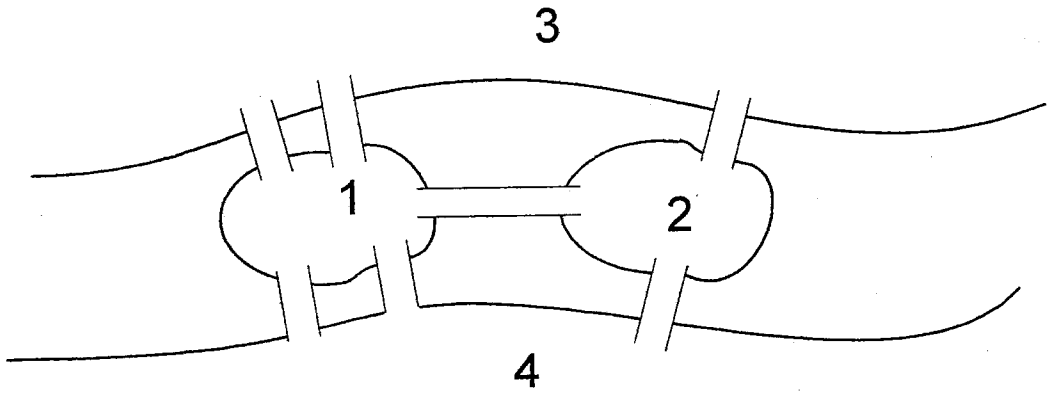
Even more simply, the graph on three nodes has 3 spanning trees.



Complete graph on three nodes and trees

THE KÖNIGSBERG BRIDGES PROBLEM

Consider the problem of the Königsberg Bridges, which was solved by the mathematician Leonhard Euler in 1736. This is regarded as the first problem in graph theory. The city of Königsberg is located on the banks and on the two Islands of the river Pregel. The various sections of the city were joined by seven bridges, as indicated below.



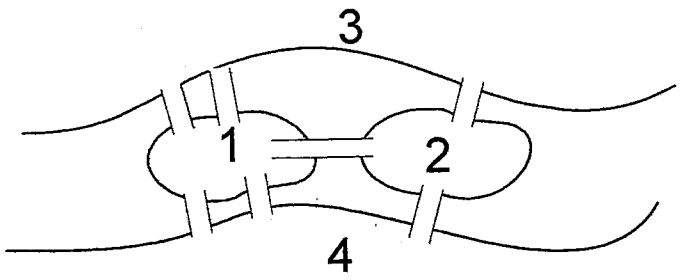
Königsberg Bridges

THE KÖNIGSBERG BRIDGES PROBLEM — CONTINUED

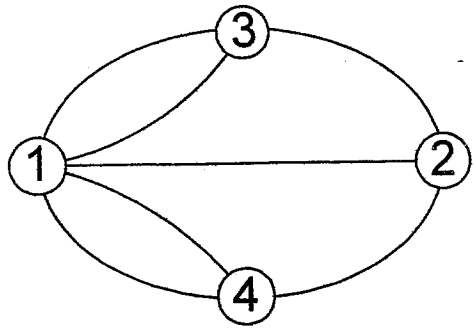
Is it possible to make a walking tour of the city in such a way as to return to the starting point after having crossed each bridge once, and only once?

- Euler's solution of this problem is considered to be the origin of graph theory. Let us see how this problem can be turned into a problem on graphs.

- We represent each of the four sections 1, 2, 3, 4 of the city as a vertex, and each of the bridges by an edge connecting two vertices.



Königsberg Bridges



Königsberg Graph

Is it possible to find a path in the graph which traverses each edge exactly once and returns to its starting point?

- It is easy to see that there is no such path. The path would have to enter each vertex as many times as it leaves it; this is true even for the starting vertex.

- Hence, there must be an even number of edges meeting at each vertex. Since this is not true, in this case, no such path is possible.

EULER CYCLES

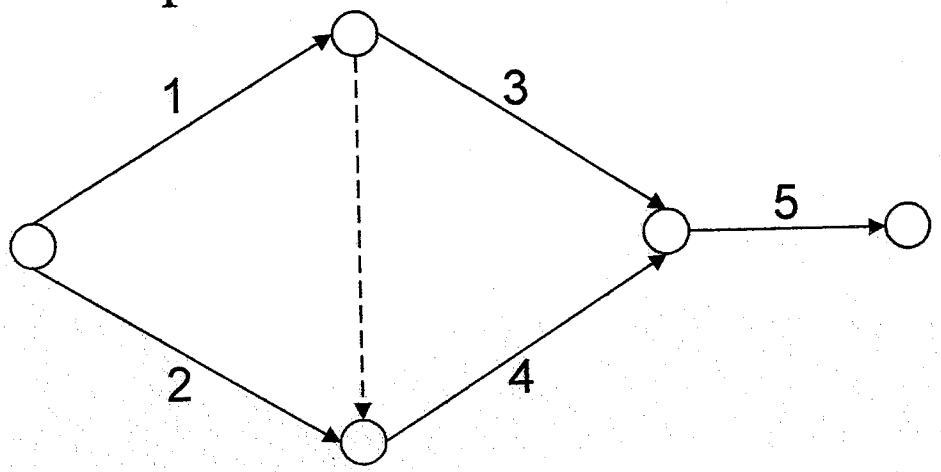
- A path in a graph that traverses each edge exactly once and returns to its starting point is called an Euler (or Eulerian) cycle (or circuit).

- We will see later that the Chinese Postman Problem is a more modern extension of some of the above concepts.

PROJECT PLANNING

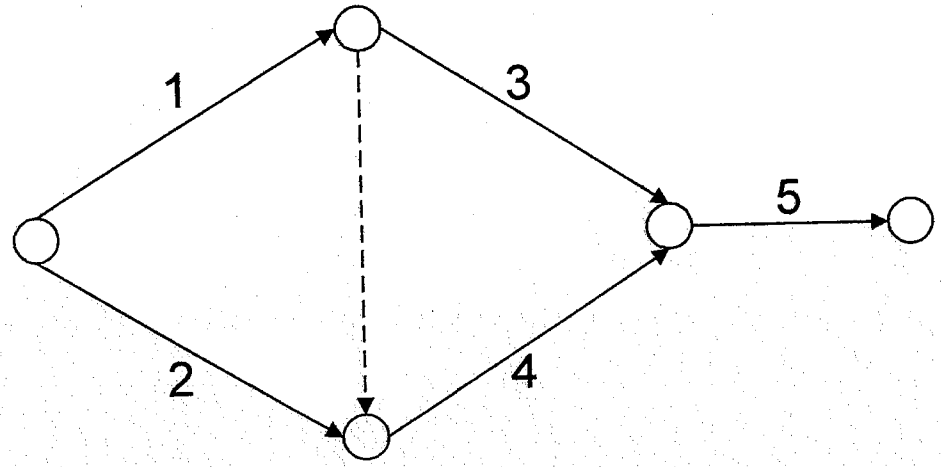
One popular network application deals with planning and scheduling of large complicated projects.

- Suppose that a project of some kind (the construction of a bridge, for example) is broken down into many individual jobs.
- Certain of the jobs will have to be finished before others can be started. We may depict the order relations among the jobs by means of an acyclic network whose arcs represent jobs.
- To take a simple case, suppose there are five jobs with the ordering: 1 precedes 3; 1 and 2 precede 4; 1, 2, 3, and 4 precede 5.



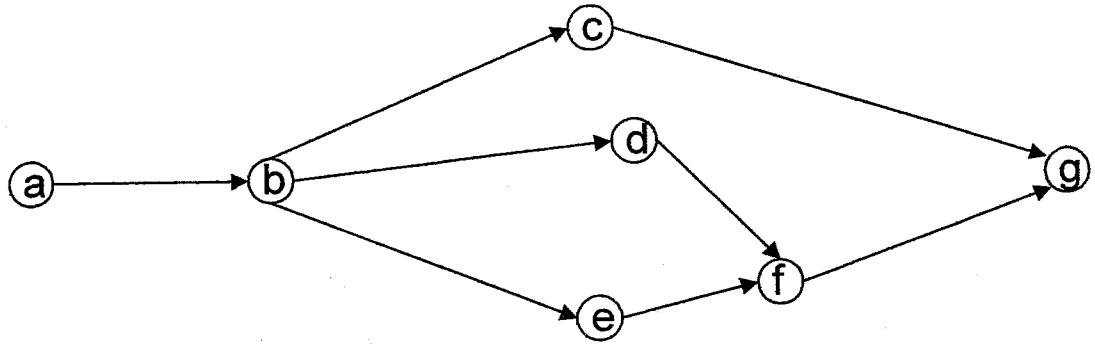
PROJECT PLANNING — CONTINUED

- Notice that we have added a "dummy" job, the dashed vertical arc, to maintain the proper order relations among the jobs.
- Assume that each job has a known duration time (dummies have zero duration time) and that the only scheduling restriction is that all inward-pointing jobs at a node must be finished before any outward pointing job can be started.
- The minimum time to complete the entire project is equal to the length of a longest path of jobs. Hence, the minimum project duration time can be calculated easily, as we shall see.



MAXIMUM FLOW IN A RAILWAY NETWORK

- A large number of commuters travel from **a** to **g** every day. There are three routes as shown below.



- Each section of the line (each arc) has a capacity (number of trains per hour) based on safety considerations. Suppose each capacity is 10.
- How many trains per hour can reach **g**?
- How can the number be increased by changing the capacity of a single arc?

MINIMAL SPANNING TREES

A network problem for which there is a simple solution method is the selection of a minimum spanning subtree from an undirected network.

- Imagine a number n of cities on a map. The cost of installing a communication link between cities i and j is $c_{ij} = c_{ji} \geq 0$.

- Each city must be connected, directly or indirectly, to all others, and this is to be done at minimum total cost.

- Attention can be confined to trees, because if the network contains a cycle, removing one link of the cycle leaves the network connected and reduces cost.

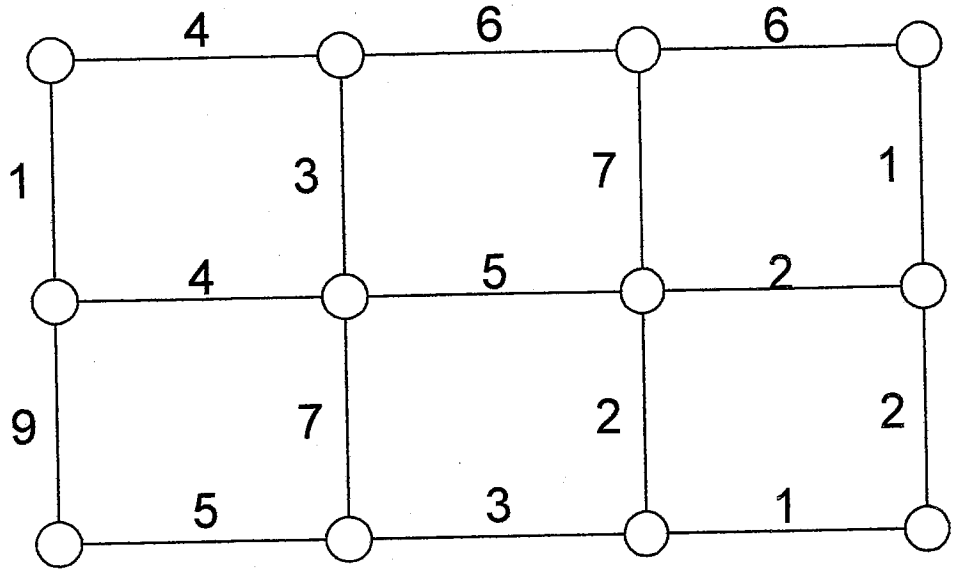
MINIMUM SPANNING TREES — CONTINUED

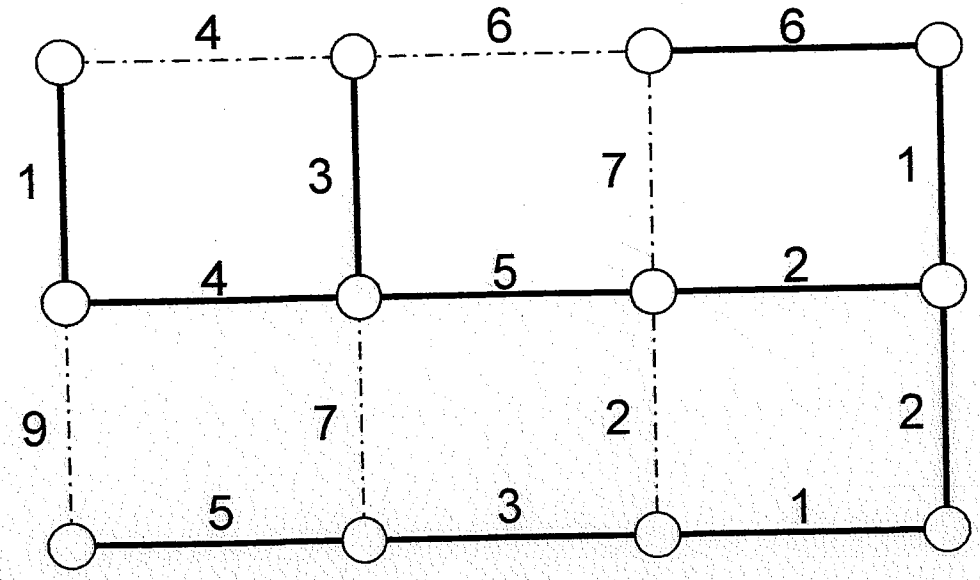
A minimal cost tree for this problem can be found easily.

- Select the cheapest link, then the next cheapest, and so on.
- Make sure at each stage that no subset of the selected links forms a cycle.
- After n - 1 selections, a minimum spanning tree has been constructed.

MINIMUM SPANNING TREES — CONTINUED

Original Network

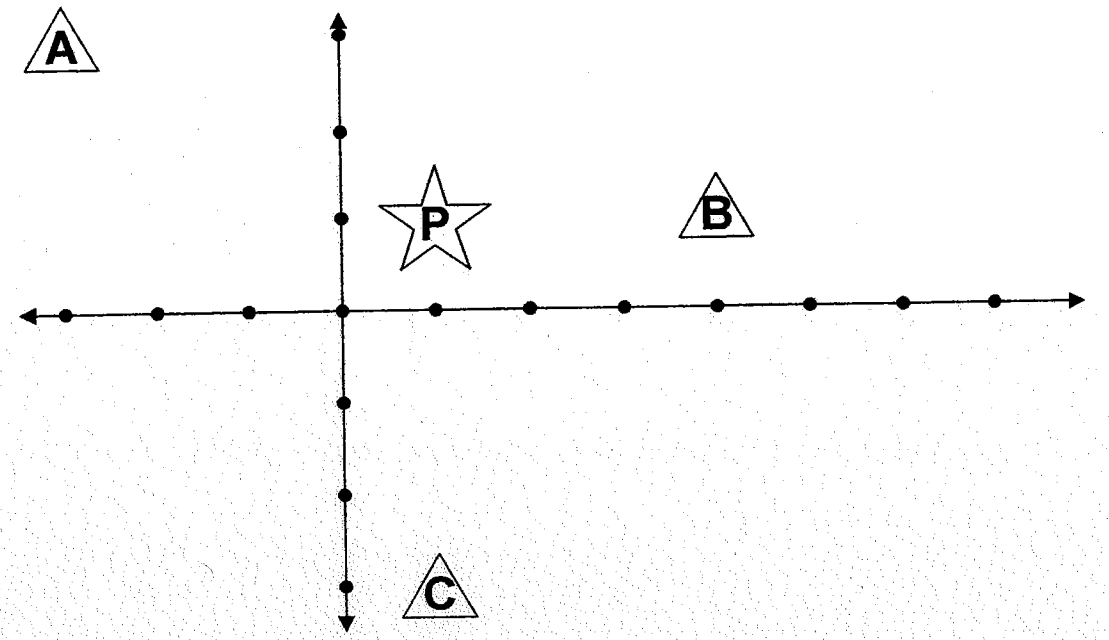




Minimum Spanning Tree

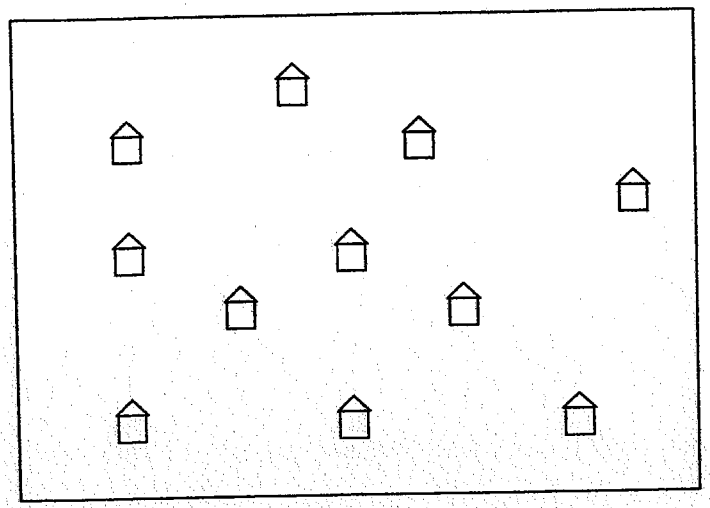
LOCATION ON NETWORKS

- Imagine a military base with three key buildings located at A = (-3, 3), B = (4, 1), and C = (1, -3).
- All streets within the military base run N-S or E-W and "taxicab" distance is used.
- Locate a central facility (at point P) from which A, B, and C will be visually monitored and from which inspectors will randomly inspect A, B, and C.
- Find the point P which will minimize $d_T(P, A)+d_T(P, B)+d_T(P, C)$.

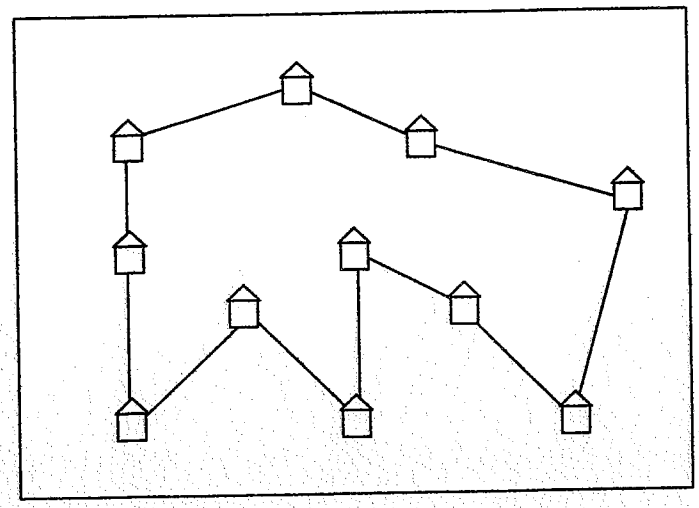


THE TRAVELING SALESMAN PROBLEM

- Imagine a small, suburban college campus with 40 separate buildings scattered over 200 acres of land.
- To promote safety, an experienced security guard must inspect each building every evening.
- The goal is to sequence the 40 buildings so that the total time (travel time plus inspection time) is minimized.
- This is an example of the well-known TSP.



Original problem



Possible solution

CORE ALLOCATIONS

- Consider the following trading problem. There are n traders who enter a market each with an indivisible item (e.g., a car or a house) to offer in trade.

- Each trader establishes ordinal preferences for the n items brought to the market by the traders, including his own item.

- The goal of the market is to redistribute the ownership of the items according to the preferences of the traders. Such a redistribution of the n items is called an *allocation*.

CORE ALLOCATIONS — CONTINUED

- An allocation will be called a *core allocation,* if there is no set S of fewer than *n* traders such that it would have been possible for the traders in S to trade amongst themselves in such a way that each was better off than in the given allocation.

- There always exists at least one core allocation, and we can use graph theory to find it.