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by Pedram Hovareshti, John S. Baras

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Decentralized control of networks of dynamic agents with invariant nodes: A probabilistic view

Pedram Hovareshti and John S. Baras

Abstract—**This paper addresses agreement problems in networks of dynamic agents in presence of invariant nodes. The network consists of a set of integrator agents which can communicate according to an underlying topology. Invariant nodes are nodes whose dynamics are invariant of their connection to the network. These nodes can be thought of as local leaders, adversaries or intruders trying to break the group's consensus or simply nodes of more importance than regular nodes. The convergence of the networks of dynamic agents with presence of invariant nodes has been studied and it has been shown that determination of the steady sate values of regular nodes in presence of invariant nodes results in solving a discrete Dirichlet problem with boundary values given by the invariant nodes.**

I. Introduction

THIS paper addresses agreement problems in networks of dynamic agents in presence of invariant nodes.

Decentralized control of dynamic agents has been studied by many researchers in recent years. A very interesting branch of research has been dedicated to the study of "nearest neighbor rules" for control of a group of dynamic agents[1-6].

Inspired by an idea introduced by Reynolds [1] for coordinated and decentralized alignment of a flock of computer agents, Jadbabaie et. al. [3] have studied a model of autonomous agents moving in a plane with the same speed but with different headings. Each agent's heading is updated using a local rule based on the average of its own heading and the headings of its neighbors. They have provided a theoretical explanation of why all agents will eventually move in the same direction despite the absence of a centralized control scheme.

Olfati Saber and Murray[4][5] have studied agreement problems in networks of dynamic agents and showed how using simple" nearest neighbor rules" in networks of integrator agents will result in all nodes' convergence to a consensus.

The contribution of this work is to introduce invariant nodes in the context of aforementioned papers and to show how these invariant nodes affect the dynamics of the regular nodes. These nodes can be thought of as "local and regional leaders", nodes with more "importance", or even "adversaries" trying to break the group's consensus.

By considering proper invariant nodes in the network of dynamic agent, one can guide the network to perform a specific task in a distributed manner.

Convergence properties of networks of dynamic agents with presence of invariant nodes has been studied in the present work and it has been shown that the determination of the steady sate values of regular nodes in presence of invariant nodes results in solving a discrete Dirichlet problem with boundary values given by the invariant nodes. For the foregoing discussion we limit our focus to fixed topology. Further study is going on in case of changing topology.

II. Problem Setting

Consider a group of dynamic agents $\{x_i \mid i = 1,...,N\}$.

Each agent is considered as a node of a graph $G(V, E)$. There is an edge between any two neighboring nodes. The graph is assumed to be connected. The dynamics of each node is determined by the value of the node and its neighbors. There are two types of nodes:

Regular nodes with indices coming from a set *D,* whose dynamics is determined by:

$$\dot{x}_i = \sum_{j \in N_i} (x_j(t) - x_i(t)) \Leftrightarrow i \in D. \tag{1}$$

There is also another set of nodes which are called invariant nodes. The value of each of these nodes is fixed throughout all the time. The set of indices of invariant nodes is denoted by ∂D and :

$$i \in \partial D \Leftrightarrow \forall t \geq 0 \qquad \dot{x}_i = 0 \Leftrightarrow x_i(t) = \varphi_i \in R. \tag{2}$$

Without loss of generality we will assume that the nodes $i = 1,2,..., N_R$ are regular nodes and the nodes

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Pedram Hovareshti is with the Institute for Systems Research and Department of Electrical & Computer Engineering, University of Maryland, College Park, MD 20742, USA. (email: hovaresp@eng.umd.edu)

John S Baras is with the Institute for Systems Research and Department of Electrical and Computer Engineering, University of Maryland, College Park, MD 20742, USA. (email: baras@isr.umd.edu)

$i = N_R + 1, \ldots, N$ are the invariant nodes i.e.

$D = \{1, \ldots, N_R\}$ is the set of indices for regular nodes and $\partial D = \{N_R + 1, \ldots, N\}$ is the set of indices for invariant nodes.

III. MATRIX REPRESENTATION

The system equations (1) can be rewritten as

$i \in D$:

$$\dot{x}_i = \sum_{\substack{j \in N_i \\ j \in D}} (x_j - x_i) + \sum_{\substack{j \in N_i \\ j \in \partial D}} (\varphi_j - x_i) \tag{3}$$

Also, (3) can be rewritten as**:**

$i \in D$

$$\dot{x}_i = -n_i x_i + \sum_{\substack{j \in N_i \\ j \in D}} (x_j - x_i) + \sum_{\substack{j \in N_i \\ j \in \partial D}} \varphi_j \tag{4}$$

In which $n_i \geq 0$ denotes number of the invariant nodes which are neighbor to $i'th$ (regular) node.

Let M be the $N_R \times N_R$ diagonal matrix. $M = diag(n_i)$

Then we can write

$$\dot{x}_R = -(M + L_R) x_R + Bu \tag{5}$$

where $x_R = \left[x_1, \ldots, x_{N_R}\right]^T$, L_R is the N_R by N_R Laplacian matrix corresponding to the graph consisting of regular nodes, neglecting the existence of invariant nodes, $u = \left[\varphi_{N_R+1}, \ldots, \varphi_N\right]^T \in R^{N_I}$ can be regarded as an input vector and $B \in R^{N_R \times N_I}$ is a matrix whose entries B_{ij} are either 1 or 0 depending on whether the i'th regular node is connected to the *j*'th invariant node(i.e. node N_R+j)or not.

IV. MAIN RESULTS

In this section the main results for stability of regular nodes are presented and the steady-state behavior of regular nodes is investigated.

Theorem 1

Let G be a connected graph with fixed topology and suppose that the state of each regular node of graph evolves as:

$$\dot{x}_i = \sum_{j \in N_i} (x_j(t) - x_i(t)) \qquad i = 1, \ldots, N$$

Suppose that there are $N_I > 0$ invariant nodes where $N_I + N_R = N$ is the total number of nodes and for each invariant node:

$$x_i(t) = x_i(0) = \varphi_i \in \partial D \qquad i = N_R + 1, \ldots, N$$

Then the value of each regular node can be found as a solution of the equation (5):

$$\dot{x}_R = -(M + L_R) x_R + Bu$$

In addition the system

$$\begin{cases} \dot{x}_R = -(M + L_R) x_R + Bu \\ y_R = x_R \end{cases}$$

is L_p-stable $\forall p \in [1, \infty]$. Furthermore the state of each node will reach a finite steady-state value $x_i(\infty) < \infty$.

Proof:

As shown in the previous section, the dynamic equations of the regular nodes can be expressed by the state space model

$$\dot{x}_R = -(M + L_R) x_R + Bu$$

Since G is connected L_R is a positive semi-definite matrix [4], $-L_R$ is negative semi-definite. M is a diagonal matrix with $n_i \geq 0$ where at least one of the n_i's is strictly positive. So:

$$\phi(x_R) = x_R^T(-M - L_R)x_R = -x_R^T L_R x_R - x^T M x =$$
$$-\sum_{\substack{i \approx j \\ i,j \in \{0,\ldots,N_R\}}} (x_i - x_j)^2 - \sum_{i=1}^{N_R} n_i x_i^2$$

$\phi(x_R) \equiv 0 \Rightarrow x_1 = \ldots = x_{N_R}$ and $\exists x_i ; x_i = 0 \Rightarrow$ $x_1 = \ldots = x_{N_R} = 0$

So $-(M + L_R)$ is negative definite.

Now consider the state space model:

$$\begin{cases} \dot{x}_R = -(M + L_R) x_R + Bu \\ y_R = x_R \end{cases}$$

The output of this system is given by:

$$y_R(t) = x_R(t) = e^{-(L_R + M)t} x_0 + \int_0^t e^{-(L_R + M)(t-\tau)} Bu(\tau) d\tau$$

Note that $u(\tau) = u$ is time-independent. Since $-(M + L_R)$ is Hurwitz, we have $\left\| e^{-(M+L_R)t} \right\| \leq k e^{-at}$ $\forall t \geq 0$ for some positive constants k and a. so

$$\|y_R(t)\| \leq k_1 e^{-at} + k_2 \int_0^t e^{-a(t-\tau)} d\tau . \|u\|$$

Where:

$k_1 = k\|x_0\|$ and $k_2 = k\|B\|$

It can be shown [7;pp 267-268] since $u \in L_{Pe}^{N_I}$ for all $p \in [1, \infty]$

$$\|(y_R)_\tau\|_{L_p} \leq k_1 \rho + \frac{k_2}{a} \|u_\tau\|_{L_p}$$

Where

$$\rho = \begin{cases} 1 & p = \infty \\ (\frac{1}{ap})^{1/p} & p \in [1, \infty) \end{cases}$$

So the system is finite gain L_p-stable for each $p \in [1, \infty]$. Furthermore the bias term $k_1 \rho$ is proportional to $\|x_0\|$.

Furthermore since

$$x_R(t) = e^{-(L_R+M)t} x_0 + \left[\int_0^t e^{-(L_R+M)(t-\tau)} d\tau \right] Bu$$

And *–(L_R+M)* is strictly negative-definite the $e^{-(L_R+M)t} x_0$ term will tend to zero as $t \to \infty$. Also since *–(L_R+M)* is negative–definite, as $t \to \infty$ $\int_0^t e^{-(L_R+M)(t-\tau)} d\tau$ will tend to a constant matrix so the state of each node will reach a finite value $x_i(\infty) < \infty$.

Remark:

Olfati-Saber and Murray [4] have shown that in absence of invariant nodes with all nodes regular interacting using nearest neighbor rules, the dynamics of the system is described by $\dot{x}_i = \sum_{j \in N_i} (x_j(t) - x_i(t)) \Leftrightarrow \dot{x} = -Lx$ where *L* is the Laplacian matrix. For a connected network since *L* is positive semi-definite with only one eigen-value at 0, they have shown that the value $\sum_{i=1}^{N} x_i(t)$ is time-invariant and all the nodes will reach the same "consensus".

In theorem 1 it was shown that in the presence of invariant nodes the value of regular nodes will converge to steady-state limits. Now, the objective is to determine these limits.

At steady state the following equations should hold:

For regular nodes:

$$\sum_{j \in N_i} (x_i - x_j) = 0 \tag{6}$$

For invariant nodes:

$$x_i = \varphi_i \tag{7}$$

If the Laplacian Matrix of the whole set of nodes (including invariant nodes) is denoted by *L* (which is different from L_R), then:

For $i = 1, \ldots, N_R$

$$(Lx)_i = 0 \tag{8}$$

Equation (8) means that the first N_R elements of the vector Lx will be equal to zero. (The remaining N_I entries are not important and we discard them.)

Similarly if the whole number of nodes (regular and invariant) which are neighbor to the i^{th} regular node is denoted by $n_i^* \geq 0$, equation (6) can be rewritten as

$$n_i^* x_i = \sum_{j \in N_i} x_j \Rightarrow (n_i^* + 1) x_i = x_i + \sum_{j \in Ni} x_j \Rightarrow$$

$$x_i = \frac{1}{n_i^* + 1} \left(x_i + \sum_{j \in Ni} x_j \right) \qquad i = 1, \ldots, N_R \tag{9}$$

Now, let $P := (I + D)^{-1} (A + D)$, where A is the adjacency matrix of the graph G and D is the diagonal matrix with *i*'th diagonal element equal to the degree of node i (number of its neighbors). Equation (9) can be rewritten as:

$$\begin{aligned} x_i &= \{[(I + D)^{-1} (A + D)] x\}_i \Rightarrow \\ x_i &= (Px)_i \qquad i = 1, 2, \ldots N_R \end{aligned} \tag{10}$$

Note that *P* is a stochastic matrix and the equation is valid for the first N_R components of *x*.

Let $\{Z_n\}_{n \geq 0}$ be a homogeneous Markov chain with state space $V = \{1, 2, \ldots, N_R, \ldots, N\}$. Let $D = \{1, 2, \ldots N_R\}$ be the sub-graph of regular nodes, and denote by ∂D the complement of D in V, i.e. $\partial D = \{N_{R+1}, \ldots, N\}$. Let $\varphi : \partial D \to \mathbb{R}$ be a function defined by $\varphi(i) = \varphi_i$. Let T be the hitting time of ∂D. For each state $i \in V$ define:

$$h_i = E\left[\varphi(Z_T) 1_{\{T < \infty\}} \mid Z_0 = i \right] \tag{11}$$

It will be shown that the function $h : V \to R^N$ is finite. Since the underlying graph G is connected, P is irreducible. Also $\forall i \in V \quad p_{ii} > 0$ which means the chain is aperiodic. The number of states is finite and therefore the chain is positive recurrent and $P(T < \infty \mid Z_0 = i) = 1$. The following theorem describes the steady state behavior of the nodes.

Theorem 2

Let $h \in R^N$ be defined by equation (11). Then $x_i < \infty$ can be determined uniquely; Furthermore, $x_i(\infty) = h(i) \qquad \forall i \in \{1, \ldots, N\}$.

Proof:

It was shown that

$$x_i = \begin{cases} (Px)_i & i = 1,2,\ldots N_R \\ \varphi_i & i = N_R + 1,\ldots,N \end{cases}$$

x can be represented as:

$$x = \begin{cases} Px & on \ \ D \\ \varphi & on \ \ \partial D \end{cases} \tag{12}$$

By definition, $h = \varphi$ on ∂D and $x = \varphi$ on ∂D.

By first step analysis:

$h(i) = \sum_{j \in V} p_{ij} h(j)$ on *D*.

So:

$$h = \begin{cases} Ph & on \ \ D \\ \varphi & on \ \ \partial D \end{cases} \tag{13}$$

So $h = x$ on $D \cup \partial D$.

It can be shown that since:

$\forall i \in V \qquad P(T < \infty \mid Z_0 = i) = 1$,

(12) has at most one bounded solution.

If there is another solution u to (12) then:

$M_n = u(Z_n) - u(Z_0) - \sum_{k=0}^{n-1} (P - I)u(Z_k)$ is a Levy Martingale with respect to $\{Z_n\}_{n \geq 0}$.

Let $M_{T \wedge k}$ denote process M_k stopped at *T*. Then by optional sampling theorem for all integers $K \geq 0$:

$E[M_{T \wedge K} \mid Z_0 = i] = E[M_0 \mid Z_0 = i]$=0, and therefore:

$$u(i) = E\left[u(Z_{T \wedge K}) \mid Z_0 = i\right] - E\left[\sum_{k=0}^{T \wedge K - 1} (P - I)u(Z_k) \mid Z_0 = i\right]$$

$$= E\left[u(Z_{T \wedge K}) \mid Z_0 = i\right]$$

because:

$(I - P)u = 0$ on *D*.

Also:

$P(T < \infty) \mid Z_0 = i) = 1$

and

$\lim_{K \uparrow \infty} E[u(Z_{T \wedge K}) \mid Z_0 = i] = E[u(Z_T) \mid Z_0 = i]$

by dominated convergence. Therefore :

$u(i) = E[\varphi(Z_T) \mid Z_0 = i] = x_i(\infty)$.

So for all $i \in D$:

$$x_i(\infty) = E[\varphi(Z_T) \mid Z_0 = i] \tag{14}$$

So since φ is bounded the function

$i \to E[\varphi(Z_T) \mid Z_0 = i]$

is bounded and therefore since

$P(T < \infty) \mid Z_0 = i) = 1$ for all $i \in V$,

the unique bounded solution of (12) is:

$x_i(\infty) = E[\varphi(Z_T) \mid Z_0 = i]$.

Remarks:

1) The proof is –with minor differences-basically the same as [8 theorem 5-2.1, pp 180-182]. With some more restrictive conditions like positive-ness of φ the above theorem can be extended as a special case of Maximum principle in discrete time.

2) L is a discrete Laplacian. As a limit the continuous case can be considered. There is a continuous counterpart for the above theorem:

Let $U \subseteq \mathbb{R}^N$ be a smooth, bounded domain and $\varphi : \partial U \to \mathbb{R}$ a given continuous function. It is known from classical PDE theory that there exists a (harmonic) function $u \in C^2(U) \cap C(U)$ satisfying the boundary value problem:

$$\begin{cases} \Delta u = 0 & on \ \ U \\ u = \varphi & on \ \ \partial U \end{cases}$$

Then we have for each point

$x \in U : u(x) = E[\varphi(\mathrm{X}(T_x))]$

For $\mathbf{X}(.) := \mathbf{W}(.) + x$ a Brownian motion starting at x.

For a proof see [9].

V. Illustration

A simple example is given to illustrate the results:

Consider the network of dynamic agents given by Figure 1, where $\mathrm{D} = \{1,2,3\}$ is the set of regular nodes and $\partial \mathrm{D} = \{4,5\}$ is the set of invariant nodes. We have shown the set of invariant nodes by a circle around the nodes. The Laplacian matrix of the overall graph is denoted by *L* and the Laplacian of the regular sub-graph is denoted by L_R as before.

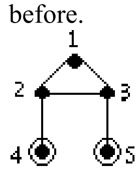

Figure 1

The matrices *L*, L_R, *M*, *B*, *P* and the vector *u* as defined in previous sections are calculated:

$$L=\begin{bmatrix}2 & -1 & -1 & 0 & 0\\ -1 & 3 & -1 & 0 & -1\\ -1 & -1 & 3 & -1 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & -1 & 0 & 0 & 1\end{bmatrix};$$

$$P=\begin{bmatrix}1/3 & 1/3 & 1/3 & 0 & 0\\ 1/4 & 1/4 & 1/4 & 0 & 1/4\\ 1/4 & 1/4 & 1/4 & 1/4 & 0\\ 0 & 0 & 1/2 & 1/2 & 0\\ 0 & 1/2 & 0 & 0 & 1/2\end{bmatrix}$$

$$L_R=\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix};M=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix};B=\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}$$

The differential equations of the system evolution can be written as:

$$\dot{x}_R=(-M-L_R)x_R+Bu$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=-\left(\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}+\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}\right)\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+$$

$$+\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\varphi_4\\ \varphi_5\end{bmatrix}\Rightarrow$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-2 & 1 & 1\\ 1 & -3 & 1\\ 1 & 1 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ \varphi_4\\ \varphi_5\end{bmatrix}$$

At the steady state for $i\in D$:

$$\left[\begin{array}{ccc:cc}-2 & -1 & -1 & 0 & 0\\ -1 & 2 & 0 & 0 & -1\\ -1 & 0 & 2 & -1 & 0\\ \hdashline 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{array}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}=\left[\begin{array}{c}0\\ 0\\ 0\\ \hdashline \varphi_4\\ \varphi_5\end{array}\right] \qquad (15)$$

Also for $i\in D=\{1,2,3\}$, $(L*x)_i=0$ or equivalently $(P*x)_i=0$. For $i\in\partial D=\{4,5\}$:

$$x_i=\varphi_i\Rightarrow\begin{bmatrix}x_4\\ x_5\end{bmatrix}=\begin{bmatrix}\varphi_4\\ \varphi_5\end{bmatrix}=\varphi.$$

So overall:

$$x=\begin{cases}Px & on\ \ D\\ \varphi & on\ \ \partial D\end{cases}\Rightarrow \qquad x_i=E\left[\varphi(Z_T)\,|\,Z_0=i\right] \qquad (16)$$

Solving the system of equations (15) yields:

$$\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}=\begin{bmatrix}\dfrac{\varphi_4+\varphi_5}{2}\\ \dfrac{\varphi_{4+}3\varphi_5}{2}\\ \dfrac{3\varphi_4+\varphi_5}{2}\\ \varphi_4\\ \varphi_5\end{bmatrix} \qquad (17)$$

The values of x_i's can be also obtained from equation (16) as will be shown in following. For $i\in D$ and $j\in\partial D$ let p_{ij} denote $P[Z_T=j\,|\,Z_0=i]$. Then:

$$\begin{cases}p_{15}=\frac{1}{2}p_{25}+\frac{1}{2}p_{35}\\ p_{25}=\frac{1}{2}+p_{15}\\ p_{35}=\frac{1}{2}p_{25}\end{cases}\Rightarrow\begin{bmatrix}p_{15}\\ p_{25}\\ p_{35}\end{bmatrix}=\begin{bmatrix}1/2\\ 3/4\\ 1/4\end{bmatrix}\Rightarrow$$

$$\begin{bmatrix}p_{14}\\ p_{24}\\ p_{34}\end{bmatrix}=\begin{bmatrix}1/2\\ 1/4\\ 3/4\end{bmatrix}\Rightarrow$$

$$E\left[\varphi(Z_T)\,|\,Z_0=i\right]=$$
$$E\left[\varphi(Z_T)\,|\,Z_0=i,Z_T=4\right]P\left[Z_T=4\,|\,Z_0=i\right]$$
$$+E\left[\varphi(Z_T)\,|\,Z_0=i,Z_T=5\right]P\left[Z_T=5\,|\,Z_0=i\right]$$
$$+E\left[\varphi(Z_T)\,|\,Z_0=i,Z_T=5\right]P\left[Z_T=5\,|\,Z_0=i\right].$$

$$\Rightarrow\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}=\begin{bmatrix}\dfrac{\varphi_4+\varphi_5}{2}\\ \dfrac{\varphi_{4+}3\varphi_5}{2}\\ \dfrac{3\varphi_4+\varphi_5}{2}\\ \varphi_4\\ \varphi_5\end{bmatrix}$$ which is the same as (17).

VI. Conclusion

Invariant nodes have been proposed in the context of a network of dynamic nodes. The effects of these invariant nodes on the behavior of the regular nodes have been studied thoroughly. It has been shown that the resulting system under a set of constraints is stable. Also we have shown how the behavior of the regular nodes in steady state can be regarded as a solution to a certain Dirichlet-type problem with boundary values coming from the behavior of invariant nodes. An example has been given to clarify the approach. As a simple example for the usefulness of invariant nodes we can consider a single invariant node. A single invariant node can be regarded as a group leader and it can be shown that with a single invariant node the group will reach a "consensus" which is the value of the "leader". Also invariant nodes can be regarded as adversaries trying to break the group's consensus. Various other behaviors can also be regarded as the result of introducing proper invariant nodes. The future work is to consider the problem setting for the case of changing topology.

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