Chaos On A Simple Rational Planar Map

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Abstract. The dynamics of a 2-D rational map are studied for various values of it's control parameters. Despite it's simple structure this model is very rich in non-linear phenomena such as, multi-scroll strange attractors, transitions to chaos via period doubling bifurcations, quasi-periodicity as well as intermittency, interior crisis, hyper-chaos etc. In this work, strange attractors, bifurcation diagrams, periodic windows, invariant characteristics are investigated both analytically and numerically.
Keywords: 2-D nonlinear maps, Chaos, Quasiperiodicity, Hyperchaos.

1 Introduction

Over the past few decades, scientists have come to understand that a large variety of systems exhibit complicated evolution both in time and in space. Chaos is such a behaviour; It is met in systems that mathematically appear to be too simple to exhibit high complexity. In the case of maps, there are several examples of simple polynomial maps are thoroughly studied in literature [9]. This is not the case though, with rational systems that to our best knowledge have not thoroughly been studied yet. The first example of a 1-D rational map is discussed in [1]. The map $g_a(x) = \frac{1}{0.1+x^2} - ax$ has come up from the study of evolutionary algorithms and is reported in [1] as a first example of a complex 1-D rational map. In [2], Chang et al. proposed a two dimensional version of $g_a(x)$; $h_{a,b}(x,y) = (\frac{1}{0.1+x^2} - ay, \frac{1}{0.1+y^2} + bx)$.

The model we discuss, in this paper, is

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\dfrac{ax}{1+y^2} \\ x+by \end{pmatrix} \tag{F}$$

where $a, b \in \mathbb{R}$ are the control parameters. The map was first reported in [4] we revisit and discuss (F) presenting some new analytical results as well as numerical measurements. We explore (F)'s dynamics verifying that exhibits exceptionally interesting complex behaviour together various non-linearities.

The paper is organized as follows. In section 2, we present simulations of the phase space for typical values of the parameters a and b. These figures illustrate the state dynamics of (F) and shall be discussed throughout the paper. In section 3, we discuss some elementary properties of the map and report the dynamics of the fixed and period 2 points. Moreover, we state, without proof, few useful propositions that outline the steady state dynamical behaviour of the system for the very interesting, as we will see below, region of $|b| < 1$. In section 4, we discuss the transition of (F) to chaos. We fix parameter b and identify the bifurcations that take place as a increases through critical values. We argue that a rather unusual quasi-periodicity involved scenario takes place as (F) becomes chaotic. In section 5, we report and characterise some other non-linearities such as coexistence of attractors, interior crises and hyper-chaos. In section 6, we estimate the Lyapunov spectrum and the correlation sums of the attractors of section 2. We summarize and discuss our results in section 7. Our effort to make this paper as self-contained as possible competes with the strict page limit. The reader may either believe our claims or consult the technical report for complete proofs and detailed discussion of the work presented here [11].

2 Orbit Simulations

In this section, we illustrate typical attractors of (F) for different values of (a, b). The results are presented in Fig. 1. The variety of the strange attractors motivates a further investigation of this dynamical system.

3 First Remarks and Rigorous Results

The fact that the denominator of (F) is larger than 1, the system is $C^\infty(\mathbb{R}^2)$. Another remark is that the (F) is anti-symmetric, i.e. $F(-x, -y) = -F(x, y)$. Now, the Jacobian matrix, at (x, y), is

$$DF(x,y) = \begin{pmatrix} \dfrac{-a}{1+y^2} & \dfrac{2axy}{(1+y^2)^2} \\ 1 & b \end{pmatrix} \tag{1}$$

The general solution is at the $(n+1)^{th}$ iterate is:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = F\begin{pmatrix} x_n \\ y_n \end{pmatrix} = F^{(n+1)}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \dfrac{(-1)^n a^n}{\prod_{i=0}^{n-1}(1+y_i^2)} x_0 \\ b^{n+1} y_0 + b^n \sum_{i=0}^{n} b^{-i} x_0 \end{pmatrix} \tag{2}$$

From (2) one notes that the y-axis is F-invariant and for $|b| < 1$ (the range of main interest in this work), we have $y_{n+1}/x_n \propto 1$ for large n. Then for $a > 0$, x_n alternates around 0, and so does y_n in anti-phase with x_n. We conclude that for positive values of a the 2^{nd} and the 4^{th} quadrant are eventually F-invariant while for $a < 0$ the 1^{st} and 3^{rd} quadrants are eventually F-invariant. With this observation in mind, it can be verified that $\det(DF) \neq 0$ for $b < 0$. It follows that (F) is invertible and in fact it is possible to extract F^{-1} in closed form [11].It can be shown that for $b < 0$, (F) is a diffeomorphism.

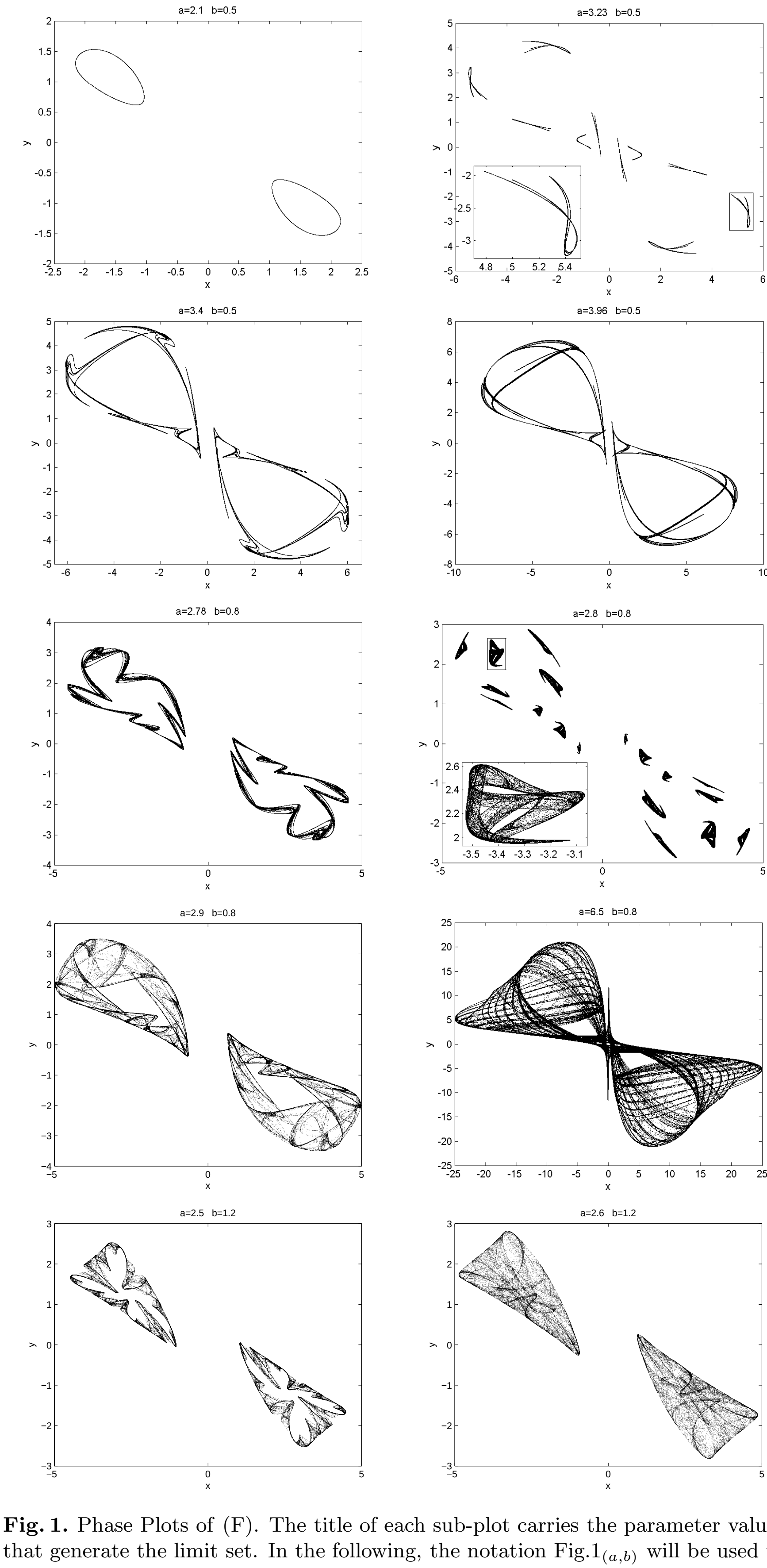

Fig. 1. Phase Plots of (F). The title of each sub-plot carries the parameter values that generate the limit set. In the following, the notation Fig.$1_{(a,b)}$ will be used to enumerate the corresponding sub-plot.

3.1 Fixed and Period 2 Solutions

The fixed points of (F) are:

i. $b \neq 1,\ a \leq -1$: $(0,0)$, $\big(\pm(1-b)\sqrt{-1-a}, \pm\sqrt{-1-a}\big)$.
ii. $b \neq 1,\ a > -1$: $(0,0)$ unique.
iii. $b = 1$: y-axis is a continuum of fixed points.

In this paper we will consider the case $a > -1$. We have the following results.

Proposition 1 *[11] For $|b| < 1$ all solutions of (F) are bounded.*

Proposition 2 *[11] For $|a|, |b| < 1$ the origin $(0,0)$ is globally asymptotically stable for almost every initial condition.*

Moreover, (F) has the unique period 2 orbit $\big(\mp(b+1)\sqrt{-1+a}, \pm\sqrt{-1+a}\big)$ which arises for $a > 1$. and is asymptotically stable for $1 < a < 2$.

Proposition 3 *[11] For any $|b| < 1$, (F) exhibits a period-doubling bifurcation as a increases through 1 and for all but two values of $|b| < 1$, it exhibits a supercritical Naimark-Sacker (i.e. discrete Hopf) bifurcation as a increases through 2.*

Proposition 1 is a first step to argue that the case $|b| < 1$ in fact implies that the existence of a global attractor for almost every initial condition (as unstable periodic orbits may exist). This is also claimed in [4]. Proposition 3 follows the discussion in [6]. There are two critical values of $|b| < 1$ that do not full-fill the assumption of the bifurcation theorem and, technically, they should be avoided. For complete proof consult [11]; and Fig. 2 for numerical validation. The conclusion is that as a increases through 2, the solutions of (F) become quasi-periodic.

4 Chaotic Transitions

Here we present and discuss some numerical results of the types of chaotic transitions of (F) and try to interpret the findings that are theoretically consistent. Numerical evidence suggest that one route to chaos is via the typical period doubling cascade (see Fig. 2). It is also claimed that chaos may occur via a quasiperiodic torus break down [4]. It is true that the quasi-periodic behaviour of the system is ubiquitous on the (a,b) plane and it is reasonable to investigate the possibility that (F) follows the Ruelle-Takens-Newhouse route to chaos [8]. Another chaotic transition that is reported, it is this via intermittency.

4.1 Bifurcation Diagram

A bifurcation diagram for the fixed value of $b = 0.5$ and $a \in (1.8, 4.2)$ is presented in Fig. 2. From the discussion in the previous section we suspect that for $|b| < 1$ any bifurcation diagram is qualitatively similar for a significant part of the range of parameter $a1$. Indeed, for $a \in (-1, 1)$ the origin is asymptotically stable while for $a \in (1,2)$ a period 2 orbit becomes stable. As a increases through zero a Naimark-Sacker bifurcation leads to quasi-periodicity. If one imagines the phase space of (F) as a Poincaré section, on a plane manifold embedded in the 3D space, the period 2 and the Naimark-Sacker bifurcations correspond to 2 subsequent Hopf bifurcations of the abstract continuous system.

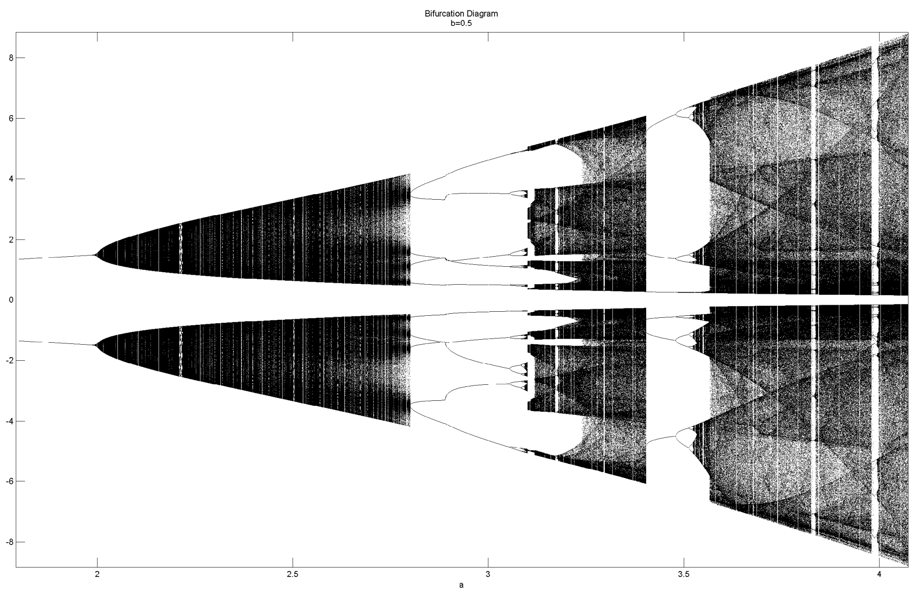

Fig. 2. Bifurcation diagram for $b = 0.5$ and $a \in (1.8, 4.2)$. At $a = 2$ a Naimark-Sacker bifurcation occurs bringing the system into quasi-periodic behaviour.

4.2 Ruelle-Takens-Newhouse route to chaos

If a continuous system undergoes three subsequent Hopf bifurcations, then it is "likely" that the system possesses a strange attractor after the third bifurcation. The power spectrum of such a system will exhibit one, then two and then possibly three independent frequencies. When the third frequency is about to appear some broad band noise will simultaneously appear, if there is a strange attractor. Practically, the three tori can decay into a strange attractor immediately after the critical parameter value for its existence has been reached such that one observes in the power spectrum only two independent frequencies, that is, two Hopf bifurcations and then chaos. This is not the case here though.

4.3 Quasi-periodic destruction $\rightarrow$ period doubling $\rightarrow$ chaos

A chaotic transition is described in Fig. 3. As a increases, the quasiperiodic attractor (3(a)) folds and degenerates continuously (3(c)) to a periodic attractor (3(d)). Then as we continue to increase a a period doubling cascade process leads to chaos (Fig. 3(e),3(f)). This is not what we should expect according to the scenario in section 4.1. Quasi-periodic attractors occurs for many values of (a, b), however they all seem to degenerate according to this scenario. Strange attractors occur and in many cases they take the place and shape of a previous quasi-periodic attractor. We argue that the transition follows a classical cascade.

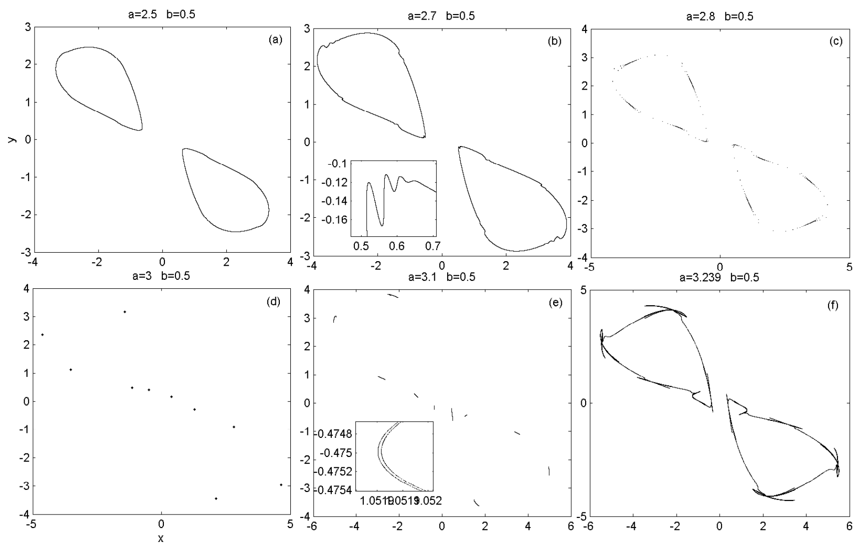

Fig. 3. Chaotic transition via destruction of quasi-periodicity. All simulations suggests that a period doubling cascade intervenes between quasi-periodic and chaotic dynamics.

4.4 Transition to chaos through Intermittency

Another transition for some parameters of (F) is this through intermittency. In Fig. 4 we report such a transition for fixed $b = 0.8$ and a decreasing through 4.2186 (period 8) to 4.218 (chaotic). The plots are the time series $(x(n), n)$ and the intermittency is of Type III following the Pomeau-Manneville classification [9]. The intermittency occurs after an inverse period doubling bifurcation for decreasing a [11].

5 Other Non-linear Phenomena

5.1 Coexistence of Attractors and Hyper-Chaos

For the most part of this paper we focus in the case $|b| < 1$. For such b the unique fixed point at the origin admits a global stable manifold $\mathcal{W}^S(0,0) := \{(x, y) \in \mathbb{R}^2 : x = 0\}$. For $|b| > 1$, $\mathcal{W}^S(0,0)$ vanishes to $(0,0)$ becoming

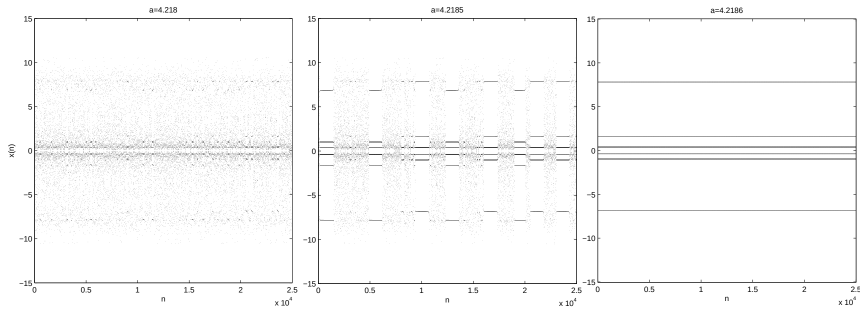

Fig. 4. Chaotic transition via intermittency. The time-series plots are more appropriate to capture this phenomenon than the phase plots.

actually a 'stable manifold' of the point at infinity. Solutions then may become unbounded although there are attractors both non-chaotic and chaotic, depending on the value a. This is the co-existence of attractors phenomenon. The strange attractors in Fig.$1_{(2.5)}$ and Fig.$1_{(2.6,1.2)}$ are in the area of $|b| > 1$. Note that Fig.$1_{(2.6,1.2)}$ is in fact hyper-chaotic (i.e. both Lyapunov exponents are positive- see Table 1).

5.2 Interior Crisis at $a_c = 3.5657$, $b = 0.5$

We report now an interior type of crisis that occurs for $b = 0.5$ at the vicinity of $a_c := 3.5657$. As the parameter a increases through a_c the orbit on the attractor spends long stretches of time in the region to which the attractor was confined before the crisis. At the end of one of long stretches the orbit bursts out of the old region and bounces around chaotically in the new enlarged region made available to it by the crisis. The times between bursts appear to be intermittent and have a long-time exponential distribution [11].

6 Exponents, Dimensions, Entropies

In this section, we present numerical results of the main invariant characteristics of the attractors illustrated in Fig. 1, mainly for consistency reasons. The three measures we approximate are the Lyapunov spectrum, the correlation dimension and the entropy as these are the easiest and most appropriate for strange attractors to compute. Assuming that a natural F-invariant measure exists, the correlation sums were first introduced by Grassberger & Procaccia [5]

$$C(m,\varepsilon,N) = \frac{2}{N(N-1)} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \Theta(\varepsilon - ||\mathbf{x}_i - \mathbf{x}_j||_m) \tag{3}$$

where Θ is the Heavyside step function, m the embedding dimension [7]. So $C(m,\varepsilon,N)$ counts the pairs $(\mathbf{x}_i, \mathbf{x}_j)$ of points on the attractor, embedded in m^{th} dimensional phase space, whose distance is smaller than ε. The ansatz is then that $C(m,\varepsilon,N)$ behaves like a power law as $N \to \infty$, $\varepsilon \to 0$. The

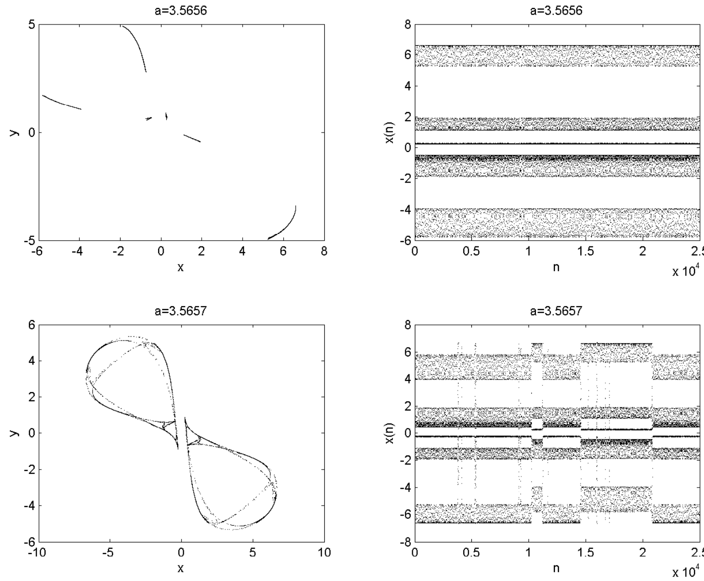

Fig. 5. Interior Crisis. A sudden and significant increase in size and shape of the attractor is illustrated as a passes through a_c.

correlation dimension and entropy are then defined as

$$D_c = \lim_{\varepsilon \to 0} \lim_{m,N \to \infty} \frac{\partial \log C(m,\varepsilon,N)}{\partial \log \varepsilon} \ , \ h_c = \lim_{\varepsilon \to 0} \lim_{m.N \to \infty} \ln \frac{C(m,\varepsilon,N)}{C(m+1,\varepsilon,N)} \quad (4)$$

Numerical results are given in Figure 6 and Table 1. The approximations of the correlation sums were done the use of the TI.SE.AN. package [10,7] while the algorithm of the Lyapunov exponent spectrum is based on the Oseledets multiplicative ergodic theorem and is an implementation of a robust algorithm from [3]. In Fig. 6(b,c) we have outlined with a bold line the range of ε where scaling occurs and is the same for increasing embedding dimensions. A comprehensive analysis of these plots is discussed in [7]

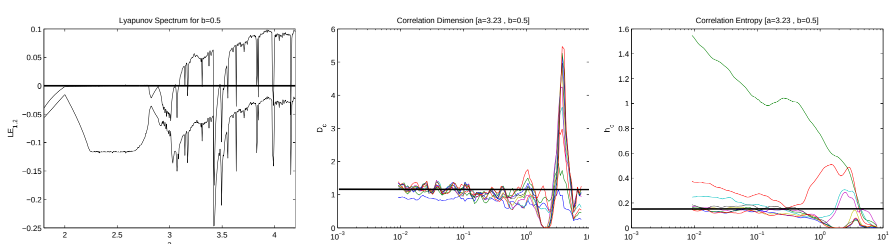

Fig. 6. (a).The Lyapunov exponent spectrum for $b = 0.5$. Compare it with the bifurcation diagram in Fig. 2. (b) The D_c estimator. (c) The h_c estimation.

7 Concluding Remarks

We presented one of the simplest examples of a rational planar map that exhibits chaotic behaviour. For $a > 0$ the system exhibits mainly symmetric

Fig.1$_{(a,b)}$	Lyapynov Spectrum	D_c	h_c
(2.1, 0.5)	(-0.035, 0.000)	1.000	0.000
(3.23, 0.5)	(-0.207, 0.057)	1.162	0.153
(3.4, 0.5)	(-0.043, 0.060)	1.304	0.150
(3.96, 0.5)	(-0.248,0.099)	1.178	0.220
(2.78, 0.8)	(-0.051, 0.032)	1.326	0.110
(2.8, 0.8)	(-0.011, 0.015)	1.500	0.080
(2.9, 0.8)	(-0.045, 0.038)	1.480	0.012
(6.5, 0.8)	(-0.132, 0.116)	1.470	0.350
(2.5, 1.2)	(-0.025, 0.028)	1.458	0.149
(2.6, 1.2)	(0.004, 0.052)	1.686	0.213

Table 1. Estimates of the invariant characteristics of the attractors of Fig. 1

attractors which however do not merge, as long as $|b| < 1$. For $a > 1$ the origin is a saddle fixed point with y-axis to be the global stable manifold which the attractors approach as a gets larger and larger but they never overrun it. A typical orbit will oscillate between the second and third quadrant getting attracted by an anti-symmetric attractor. For $|b| < 1$ all solutions are bounded although it can be verified that as a gets larger and larger the magnitude of the limit set increases. The transition to chaos is made either through intermittency or period-doubling bifurcations. The quasi-periodic behaviour is ubiquitous for a numerous values of the parameters, however there is no convincing evidence that (F) goes to chaos via quasi-periodic behaviour. The full version of this work is [11].

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