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Average Consensus over Small World Networks: A Probabilistic Framework

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Abstract— **It has been observed that adding a few long range edges to certain graph topologies can significantly increase the rate of convergence for consensus algorithms. A notable example is the class of ring-structured Watts-Strogatz small world graphs. Building on probabilistic methods for analyzing 'small-world phenomena', developed in our earlier work, we provide here a probabilistic framework for analyzing this effect. We investigate what graph characteristics lead to such a significant improvement and develop bounds to analyze consensus problems on randomly varying graphs.**

I. Introduction

Consensus problems have become an area of increasing research focus in recent years (e.g. see [2], [23], [17], [11] and the references therein). Many applications including distributed estimation [29], [4], motion coordination [21] and load balancing in multiple processes [6] have been analyzed in this framework. While the convergence properties of the consensus algorithms are fairly well understood now, the problem of understanding the impact of topology on the convergence rate needs further study. In this paper, we propose and analyze probabilistic models and methods as a first step towards this end. In particular, we study convergence over small world graphs which have recently been shown experimentally to exhibit high convergence rates [22], [15]. Earlier results demonstrating such convergence speed-up in consensus over small world topologies were presented in [18], which treated the seemingly unrelated problem of convergence of dynamic trust (and mistrust) propagation in infrastructure-less networks, like mobile ad hoc networks. Building on the methods and results of [1] we provide a possible probabilistic interpretation of the large increase in the rates of convergence seen in such graphs.

The paper is organized as follows. Beginning with a brief background of the problem formulation in the next section, we move on to a description of the probabilistic framework in Section III. In Section IV, we build on the probabilistic methods and results of [1] and we apply the combined results to small world graphs. In Section V we demonstrate the broader utility of these methods and tools by applying them on some other classes of dynamic iterative problems on random graphs. Due to space limitation we omit the proofs here. The detailed proofs can be found at [16]

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II. Background and Motivation

Consider n nodes that aim to reach consensus over a scalar value. Denote the value held by the i-th node at time k as $x_i(k)$. Also denote by $x(k)$ the n-dimensional vector obtained by stacking the values of all the nodes in a column vector. The interconnection topology of the nodes can be described by a graph, with an edge present between two nodes if and only if they can exchange information. We assume undirected edges, unless noted otherwise. Denote the neighbor set of node i at time k by $\mathcal{N}_i(k)$.

At every time step k, each node i updates its value according to the equation

$$x_i(k+1) = \sum_{j \in \mathcal{N}_i(k) \cup i} w_{ij}(k) x_j(k) \tag{1}$$

where $w_{ij}(k)$ refers to the weight assigned by the node i to the value of node j and $\sum_{j \in \mathcal{N}_i(k) \cup i} w_{ij} = 1, \quad \forall i$. Denote the interconnection graph at time k by $\mathcal{G}(k)$. We can associate a matrix $F(k)$ with this graph as follows

$$F_{ij}(k) = \begin{cases} w_{ij}(k) & j \in \mathcal{N}_i(k) \cup i \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The system thus evolves according to the discrete time equation

$$x(k+1) = F(k)x(k), \qquad x(0) = x_0. \tag{3}$$

We will denote $F(.)$ as the consensus matrix. Various consensus algorithms can be obtained by choosing the matrix $F(k)$ appropriately, see, e.g., [11] for a good overview. In this paper we will use the important class of weight matrices

$$F(k) = I - hL(k),$$

where h is a small positive parameter and $L(k)$ corresponds to the Laplacian of the graph $\mathcal{G}(k)$ [23] for the presentation of our results in this paper[1]. We will also assume $h < \frac{1}{2d_{max}}$ where d_{max} is the maximum degree corresponding to any node in the graph. This assumption causes the F matrices to be symmetric -resulting in average consensus upon convergence- with nonnegative eigenvalues and simplifies the analysis.

Convergence of consensus algorithms have been extensively studied and different sufficient conditions have been

[1]It may be noted, however, that extension of the results to other choices of the matrix F can readily be done.

proposed (see [11] and the references therein). The problem is in fact identical to the problem of convergence of the products of nonhomogeneous matrices [13]

Desai and Rao [8] explicitly show that the rate of convergence is a function of graph topology as well as the weights on the edges. In the case of a static graph topology (i.e., $F(k) = F$ for all time k), it can be shown (see, e.g., [26]) that the convergence of the consensus protocol is geometric, with relative rate equal to the second largest eigenvalue modulus (SLEM). We can thus compare convergence on two graphs by comparing the spectral gap of the graphs, defined as $S.G. = 1 - SLEM(\mathcal{G})$. For the more general case when the topology varies with time, Blondel *et al* [2] showed that the joint spectral radius of a set of matrices derived from the F matrices determines the convergence speed.

Since agents usually have energy constraints, the number of agents with which they can communicate and the long-range interconnections that they can maintain is limited. Thus it is important to investigate convergence over 'efficient' graph topologies. In particular, there has been interest in studying the convergence rates on 'small world' graphs. Olfati-Saber [22] did a simulation study on small world graphs with continuous time consensus protocols, which showed a marked improvement in the rate of convergence. Further experimental evidence has been presented in our earlier work [18], [15]. However analytical verification of this result is largely lacking. In [27], the authors used some results by Durrett [9] on the mixing time of Markov Chains on small world graphs and provided some high probability bounds on the rate of convergence. In the earlier paper [1] Baras and Hovareshti developed a new method for investigating the effects of small world topologies by building on the probabilistic models of Higham [14], that established an equivalent representation of small world topologies as rare transitions among non-neighboring states in the Markov chain associated with a graph. In this model and associated method, small but nonzero positive weights were assigned to the entries of F corresponding to the nodes that are not neighbors [1]. It was observed that for very small perturbation of the values of the weights, there was a marked increase in the rate of convergence. By performing a quantitative analysis of the eigenvalues of the resulting matrices and by employing an appropriate parametrization of these small positive weights, a complete characterization was given as to when small world phenomena (manifested by convergence speed-up) will occur. In [14] small world phenomena were analyzed using these slightly randomly perturbed Markov chains and a 'mean field' approach for the Markov random field associated with the Markov chain. In [1], exploiting the circulant nature of the matrices involved in these dynamic iterative algorithms, in the ϕ-model of Watts and Strogatz [28] (i.e. adding 'short cuts' on top of a ring lattice) rigorous results on the speed-up were obtained.

The above interpretation promotes a probabilistic viewpoint towards understanding and quantifying small world effects on consensus convergence rates: one can allow time-varying topologies, in which every node nominally communicates according to a pre-defined topology, corresponding to the original graph from which a small world network is obtained. The probabilistic interpretation of the small world topology via perturbations of the associated Markov chain, is that with a small probability the node communicates with non-adjacent nodes at every time step. Thus the model may also be looked at as switching between multiple topologies to increase the convergence rate. In such an interpretation, communication with remote nodes is done with a very small probability to conserve the node power. Over the course of writing this paper, we also encountered some recent work on a class of regular expander graphs -known as Ramanujan graphs- which show excellent spectral properties but are usually difficult to construct. An exposition on these graphs in a related framework can be found in [24] and [20].

Before explaining these models in detail, we need a framework for studying consensus problems with probabilistic switching between topologies. We develop this framework in the next section.

III. PROBABILISTIC FRAMEWORK

Consider the same framework as described above. However the graph $\mathcal{G}(k)$ can now be chosen from a finite set

$$\mathbb{G} = \{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_m\}.$$

In this paper, we concentrate on the case when this choice is done in an independent and identically distributed (i.i.d.) manner, with the graph $\mathcal{G}_i$ being chosen at any time step k with a probability p_i. More complicated models, when this choice can be carried out according to a Markov chain, can readily be analyzed (See, e.g., Proposition 3.1). Denote the consensus matrix corresponding to the graph $\mathcal{G}_i$ by F_i and denote the set of F_i's by $\mathbb{F}$. Since the state $x(k)$ now evolves stochastically, the convergence of the node values to the average consensus value occurs in a probabilistic fashion. Conditions for convergence of such schemes have been derived, e.g., in [27], [12], [10] for probabilistic or almost-sure convergence and in [4] for second moment convergence. The rate of convergence was also studied in [4].

In this paper, we are interested in second moment convergence, i.e., the convergence of the covariance of the state vector $x(k)$ to its final value $\mathbf{1}\mathbf{1}^T\mu^2$, where μ is the average of the initial values $x_i(0)$. This is equivalent to studying the convergence of the vector $x(k)$ to its final value $\mu\mathbf{1}$. This can be established using the Lyapunov function $V(x(k)) = \frac{1}{n}[\sum_{i\neq j} E(||x_i(k) - x_j(k)||^2)] = E[x^T(k)\hat{L}x(k)]$, where $\hat{L} = (I - \frac{\mathbf{1}\mathbf{1}^T}{n})$ can be viewed as the Laplacian of the complete graph. $\hat{L}$ is a projection, i.e., $\hat{L}^2 = \hat{L}$.

We have the following result about the conditions for convergence and the rate of convergence. The results about the i.i.d. case have been presented before in [4].

Proposition 3.1: Consider the consensus algorithms of Section II, with the consensus matrix being chosen from the set $\mathbb{F}$ either in an i.i.d. or according to a Markov chain with transition probability matrix Q.

1) For the i.i.d. case, the system converges in the second moment sense if

$$\rho\left(E[F \otimes F] - \frac{1}{n^2}\mathbf{1}\mathbf{1}^T\right) < 1,$$

where $\rho(.)$ denotes the spectral radius and $\otimes$ denotes the Kronecker product. Further, the rate of convergence is governed by $\rho\left(E[F \otimes F] - \frac{1}{n^2}\mathbf{1}\mathbf{1}^T\right)$ or the SLEM of the matrix $E[F \otimes F]$.

2) For the Markovian case, the system converges in the second moment sense if

$$\rho\left((Q^T \otimes I)\,(\mathrm{diag}(E[F_i \otimes F_i])) - \frac{1}{n^2}\mathbf{1}\mathbf{1}^T\right) < 1,$$

where $\mathrm{diag}(A_i)$ denotes a block diagonal matrix with blocks A_i. Further, the rate of convergence is governed by $\rho\left((Q^T \otimes I)\,(\mathrm{diag}(E[F_i \otimes F_i])) - \frac{1}{n^2}\mathbf{1}\mathbf{1}^T\right)$ or the SLEM of the matrix $(Q^T \otimes I)\,(\mathrm{diag}(E[F_i \otimes F_i]))$.

The result given above characterizes the rate of convergence. The calculation of this rate is difficult for arbitrary graphs. Therefore, we will concentrate on graphs with certain degree of symmetry for further analytic results and insights. Moreover, calculating the Kronecker product requires $n^2 \times n^2$ matrix operations for n agents. Because of the presence of the expectation operator, even for symmetric graphs, the eigenvalue calculations can quickly become complicated. This complexity has also been recognized, e.g., [4] where instead other metrics are used as a proxy for such eigenvalues. In contrast, we continue to focus on the SLEM of these matrices by calculating lower and upper bounds, which yield insightful results as shown in the next two sections. The following proposition provides such bounds.

Proposition 3.2: Denote $A = E[F \otimes F]$, where $F = I - hL$. Also denote the average value of the Laplacian $E[L]$ by $\bar{L}$. Finally, let λ_i be the i^{th} largest eigenvalue of a Laplacian matrix L,[2]. i.e.

$$\lambda_1(\bar{L}) \geq \lambda_2(\bar{L}) \geq ... \geq \lambda_n(\bar{L}).$$

Then,

$$1 - h\lambda_{n-1}(\bar{L}) \leq \lambda_2(A) \leq 1 - h\lambda_{n-1}(\bar{L}) + h^2\lambda_1(E[L \otimes L]).$$

The result given by proposition (3.2) indicates that for finding bounds on the convergence rate of probabilistic consensus algorithms on a set of matrices $\mathbb{F}$, we should

1) Find the exact value or bounds for $\lambda_{n-1}(E[L])$,
2) Find the exact value or bounds for $\lambda_1(E[L \otimes L])$.

As the examples in the next section will show $\lambda_{n-1}(E[L])$ can be computed for many different classes of graphs. To find bounds on $\lambda_1(E[L \otimes L])$, we use the fact that all the matrices L that we consider are symmetric and positive semi-definite. For such matrices, the spectral radius $\lambda_1(.)$ is a convex function. Thus, Jensen's inequality can be applied to obtain

$$\lambda_1(E[L \otimes L]) \leq E(\lambda_1([L \otimes L])) = E[(\lambda_1(L))^2].$$

Since all the eigenvalues of a graph Laplacian are bounded by twice the maximum degree of the graph, we obtain

$$\lambda_1(E[L \otimes L]) \leq 4E[d_{max}^2].$$

[2] Usually the i^{th} smallest eigenvalue of the Laplacian is denoted by λ_i in the literature. We have not followed the convention, to be consistent with our choice of ordering of eigenvalues of the weight matrices F.

As mentioned, the upper bound on the spectral gap is $h\lambda_{n-1}(\bar{L})$. We may need to change the amount of h when we use the switching scheme to remain consistent with the change in the degree of graph nodes due to switching. This change is however not very significant, when the switching probability is low enough, as seen in the examples of the next sections. We develop a necessary condition for a graph to improve convergence rate as a result of uniform probabilistic switching.

Consider a given graph $G(V, E)$, with Laplacian L_0. Let its complement graph be denoted by $G^c = (V, E^c)$, where $E^c = \{e | e \notin E\}$. Consider a uniform switching in which all the edges of E^c can be used with a small probability $0 < \epsilon < 1$. Then the expected Laplacian is:

$$\bar{L} = L_0 + \epsilon L_0^c$$

Notice that $L_0 + L_0^c = nI - \mathbf{1}\mathbf{1}^T$, and that the vector $\lambda_n(\bar{L}) = \mathbf{1}$ is an eigenvector of L_0^c with the corresponding eigenvalue 0. Taking a set of orthogonal eigenvectors, it can be easily verified that for $1 \leq i \leq n-1$, $\lambda_i(L_0^c) = n - \lambda_{n-i}(L_0)$. Therefore, we have the following result:

Corollary 3.3: A necessary condition for getting significant rate improvement by uniform switching is that $\lambda_1(L_0) \ll n$, where $\lambda_1(L_0)$ is the spectral radius of L_0.

IV. Small World Graphs

We now return to our analysis of small world graphs. We consider the nominal graph to be a ring and the so-called ϕ-model as in [1] based on the models of [28]. Similar models have been considered in [22], [27]. To model the existence of a few long range links, we assume that, at each time, each agent can establish a link with non-adjacent nodes with a small probability ϵ. We analyze the effect of such low-probability long range links, on the convergence rate. In particular, we assume that $\epsilon \propto n^{-\alpha}$ where as before n is the number of nodes and α is a natural number.

The consensus matrix of the nominal ring graph is:

$$I - hL_{fix} = \begin{bmatrix} 1-2h & h & 0 & ... & h \\ h & 1-2h & h & ... & 0 \\ . & . & . & . & . \\ h & 0 & ... & h & 1-2h \end{bmatrix}.$$

Thus the spectral gap of the nominal graph is:

$$S.G.(fixed) = 1 - \lambda_2(F_0) = 2h(1 - \cos(\frac{2\pi}{n})).$$

Now we analyze the effect of the additional links. To calculate the expected Laplacian matrix, we realize that

$$\bar{L}_{ij} = \begin{cases} 2 + (n-3)\epsilon & i = j \\ -1 & |i-j| = 1 \\ -1 & (i,j) = (1,n) \text{ or } (i,j) = (n,1) \\ -\epsilon & \text{otherwise.} \end{cases}$$

This graph has a circulant structure. Thus,

$$\lambda_{n-1}(\bar{L}) = 2 + (n-3)\epsilon - 2(1-\epsilon)\cos(2\pi/n)$$

Using Proposition 3.2 we thus obtain the following bounds when long range links are added with a small probability ϵ.

$$\lambda_2(E[(I-hL)\otimes(I-hL)]) \in$$
$$[1-h(2+(n-3)\epsilon-2(1-\epsilon)\cos(\frac{2\pi}{n})),$$
$$1-h(2+(n-3)\epsilon-2(1-\epsilon)\cos(\frac{2\pi}{n}))+4h^2E[d_{max}^2]].$$

The spectral graph for this case evaluates to

$$S.G.(Switching) \in$$
$$[h(2+(n-3)\epsilon-2(1-\epsilon)\cos(\frac{2\pi}{n}))-4h^2E[d_{max}^2],$$
$$h(2+(n-3)\epsilon-2(1-\epsilon)\cos(\frac{2\pi}{n}))]$$

For $\epsilon = n^{-\alpha}$ and $\alpha = 1, 2, 3, \ldots$ we can use a Chernoff bound argument similar to that outlined in Lemma 3 of [27] to show that in the limit of large n, $d_{max} < \log n$ almost surely. Note that d_{max} can never exceed n.

We are now ready to compare the spectral gaps of the nominal topology and of the one with long range links. We assume $h \propto n^{-1}$ in keeping with our assumption relating h to the maximum degree. We note the following observations, which are similar in form to some of the results in [1].

1) As the number of nodes n increases, the spectral gap for the fixed graph varies as n^{-3}.
2) The upper bound for the spectral gap for the graph with large range links evaluates to
$$h[n^{1-\alpha}-n^{-\alpha}+2\pi^2n^{-2}-2n^{-\alpha-2}] = O(n^{-\alpha}).$$
Thus, for $\alpha \geq 3$ the dominant term is also the n^{-3} term. Thus even if we consider the upper bound, the presence of long range links cannot increase the spectral gap if the links are added with too small a probability.
3) The lower bound on the spectral gap of the case with long range links, for large n is approximately
$$h[n^{1-\alpha}-n^{-\alpha}+2\pi^2n^{-2}-2n^{-\alpha-2}]-\frac{(\log n)^2}{n^2}.$$
For $\alpha = 1$, this bound evaluates to $O(n^{-1})$, which is an order of magnitude better than the nominal ring case. This shows the huge impact of long range links, even if they are added with a vanishingly small probability.
4) If we take $\alpha = 2$ then the lower bound is not tight enough to make any statement about the comparison of the two regimes.

From the above observations, we can conclude that long range links can improve the convergence rate of the consensus protocols enormously. As we increase the probability, we get a sharp increase in the convergence rate at $\alpha = 2$. This improvement in performance can be viewed as the consequence of the onset of small world phenomena.

Similar expressions for rings in higher dimensions can be obtained. For general graphs, it may not be possible to prove the increase in rate due to extra edges analytically. However, some graphs of practical concern can be well-approximated by rings, as the number of agents increases. As an example,

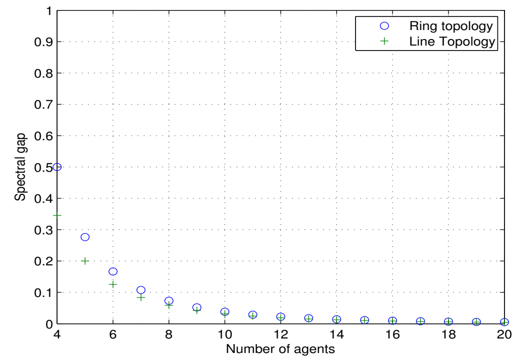

Fig. 1. Spectral gap for a ring and a line topology.

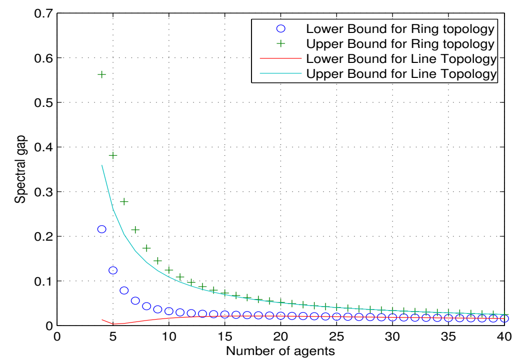

Fig. 2. Upper and lower bounds for spectral gap for a ring and a line topology.

Figure 1 shows the spectral gap for a ring and a line topology. Spectral gaps are quite similar for a fairly small number of agents. Similarly, Figure 2 shows the lower and upper bounds for the spectral gap when edges are added with $\alpha = 1$. It can be seen that the bounds for the ring and the line topologies match for a fairly small number of agents. Also, the bounds seem quite tight, at least for this example.

V. Some Other Scenarios

The present framework, methods and tools can be used to analyze performance of the average consensus protocol when probabilistic switching occurs due to any reason such as farilures in communication. In this section, we demonstrate this using various simple scenarios. The effect of communication failures and constraints on consensus has also been recently studied in [5] and [25].

A. Topology Switch due to Changing Neighbors

Consider n agents placed on a ring with n empty slots. Being on a ring constrains each node's neighbors to the agent to its left and the agent to its right. We consider a protocol in which at every time step an agent chooses two neighbors randomly. This can be viewed as a variation of the Gossip Algorithm proposed in [4], [10]. We model the selection of neighbors by assuming that at every time step, each agent chooses a slot at random and with equal probability among all the possibilities. Moreover, every slot contains only one node at each time. This is equivalent to the assumption that at each time agents randomly choose their neighbors bi-directionally, while constraining the total number of neighbors to two.

We wish to compare the rate of convergence of this scheme with the nominal case in which there is a fixed ring topology. For the fixed ring topology G_0, the consensus matrix is again circulant and the spectral gap is equal to

$$S.G.(fixed) = 2h[1 - \cos(\frac{2\pi}{n})] = 4h\sin^2(\frac{\pi}{n}).$$

For large n this is approximately $\frac{4\pi^2 h}{n^2}$ and varies as n^{-2}.

For the case when topology switch occurs, we need to compute the expected value of the Laplacian matrix. We use the fact that $\bar{L}_{ii}$ equals the expected number of agent i's neighbors, which is 2 in this case. Moreover, for the non-diagonal terms, $-\bar{L}_{ij}$ is equal to the probability that the agents i and j are neighbors. Therefore, $\bar{L}$ has the structure

$$\bar{L} = \begin{bmatrix} 2 & -\frac{2}{n-1} & \cdots & -\frac{2}{n-1} \\ -\frac{2}{n-1} & 2 & \cdots & -\frac{2}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{2}{n-1} & -\frac{2}{n-1} & \cdots & 2 \end{bmatrix}. \quad (4)$$

By exploiting the circulant structure, it is seen that $\lambda_{n-1}(\bar{L}) = 2 + \frac{2}{n-1}$. Therefore,

$$\lambda_2(I - h\bar{L}) = 1 - 2h - \frac{2h}{n-1}.$$

Finally, to calculate the upper bound, notice that the degree of each node is 2 with probability 1. Thus, using Proposition 3.2, the bounds on the spectral gap are

$$h(2 + \frac{2}{n-1}) - 16h^2 \le S.G.(switching) \le h(2 + \frac{2}{n-1})$$

Thus, it can be seen that

$$\frac{S.G.(fixed)}{S.G.(switching)} \le \frac{4h\sin^2(\frac{\pi}{n})}{2h + \frac{2h}{n-1} - 16h^2}$$

For the limit of large n, assuming a constant $h < \frac{1}{8}$, the numerator varies as n^{-2} whereas the denominator varies as n^{-1}. Thus, the ratio approaches zero with increasing n. This shows that even the lower bound of the spectral gap of the switching case shows order of magnitude improvement compared to the spectral gap of the fixed topology.

This remarkable increase in rate of convergence by switching may yield the conjecture that switching to far away neighbors always increases the rate. This conjecture, as stated above, is, however, false, as shown in the next subsection.

B. Erdos-Renyi Random graphs

In this sub-section we consider a case where switching to distant neighbors does not increase the rate of convergence. This case also yields the rate of convergence for the class of random graphs known as Erdos-Renyi random graphs. Consider n nodes to be present. Suppose that a link exists between any two nodes with probability $q \in (0, 1]$. The existence of a link between any two nodes is therefore random and independent of the other connections. We compare the convergence rates between two cases.

1) Fixed random graph: The links are selected randomly at time 0 and the graph stays constant after that.
2) Switched random graph: The link selection is done at every time step. We assume that the selection is done independently with respect to time. Thus, the random graph at each time is independent of the choice of the random graph at other time instants.

For the fixed random graph case, we use a high probability bound due to Fiedler, reported in [19], [12] on the second smallest eigenvalue of the Laplacian of a random graph. For the limit of large n and for $\epsilon \in (0, 2)$, we obtain

$$\lim_{n\to\infty} Pr\{qn - \sqrt{(2+\epsilon)q(1-q)n\log n} <$$
$$\lambda_{n-1}(L(G(n,q))) < qn - \sqrt{(2-\epsilon)q(1-q)n\log n}\} = 1.$$

Therefore if a fixed random topology is used at all times, then, with high probability

$$1 - hqn + h\sqrt{(2-\epsilon)q(1-q)n\log n} < \lambda_2[I - hL_{fixed}] <$$
$$1 - hqn + h\sqrt{(2+\epsilon)q(1-q)n\log n}.$$

For the switched random graph case, we calculate the bounds given by Proposition 3.2. The expected value of the Laplacian matrix is given by

$$\bar{L} = q \begin{bmatrix} n-1 & -1 & ... & -1 \\ -1 & n-1 & ... & -1 \\ . & . & ... & . \\ -1 & -1 & ... & n-1 \end{bmatrix}$$

This is again circulant and thus $\lambda_{n-1}(\bar{L}) = qn$. Using Proposition 3.2, we obtain the lower bound as

$$\lambda_2(E[(I - hL) \otimes (I - hL)]) > 1 - hnq.$$

One interesting regime to consider is when $q = \Theta(\frac{\log n}{n})$. In this regime, with high probability, a random graph is connected [3]. For a given large n, taking $q = k\frac{\log n}{n}$ with $k \ge 2$, yields

$$S.G.(fixed) < hqn - h\sqrt{(2+\epsilon)q(1-q)n\log}$$
$$\sim (k - \sqrt{2k})h\log n$$

On the other hand for the switching case we have:

$$S.G.(Switching) > hnq = kh\log(n)$$

Both bounds are of the same order. This indicates that there is no large improvement in the rate of convergence through switching. We conjecture that for switching to help increase the convergence rate, it is not enough to connect long range neighbors. Instead, it seems that the expected diameter of the graph through switching should be much smaller than that of the fixed graph, for switching to be useful.

C. IID Link Losses due to Communication Failures

Another important case that can lead to unintended switching between topologies is when the communication links can be modelled according to an analog erasure model. In this model, at each time step, a link is functional with a certain probability. We assume that the failures for any particular link occur independently across time and with respect to other link failures. Moreover, we assume that the link failures are bi-directional. Finally, for ease of presentation, we also assume here that each link fails with the same probability p at any time step.

We illustrate the use of our results on a 1-D lattice with a periodic boundary condition, i.e., a ring. The spectral gap of a ring when no link losses occur is given by

$$S.G.(fixed) = 2h(1 - \cos \frac{2\pi}{n}).$$

If link losses occur, the average Laplacian $\bar{L}$ is given by

$$L_{ij} = \begin{cases} 2(1-p) & i = j \\ -(1-p) & |i-j| = 1 \\ -(1-p) & (i,j) = (1,n) \text{ or } (i,j) = (n,1) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the second smallest eigenvalue is given by

$$\lambda_{n-1}(\bar{L}) = 2(1-p)(1 - \cos \frac{2\pi}{n}).$$

Finally, the spectral gap due to link losses decreases at least by

$$\frac{S.G.(fixed)}{S.G.(losses)} \geq \frac{1}{1-p}.$$

VI. CONCLUSIONS AND FUTURE WORK

In this paper we used and built further the probabilistic framework developed in [1] for studying the effects of small world topologies and of switching, on the convergence rate of consensus algorithms. We obtained bounds for the convergence rate and provided a probabilistic interpretation for the increase in rate of convergence provided by small world graphs. We also showed how the models, methods and tools developed could be used for analyzing other situations in which probabilistic switching occurs.

This work represents a step towards a full understanding of probabilistic switching in consensus and other distributed algorithms. A full understanding of the variation of performance metrics such as rate of convergence with intentional or unintentional variations in the topology of the agents will be very useful. As an immediate next step, the tightness of the bounds presented needs to be characterized. From a long term perspective, it will be interesting to relate the spectral gap of the relevant time-varying graphs to that of some suitable 'mean' graph that is time-invariant.

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