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Convergence Results for Ant Routing Algorithms via Stochastic Approximation and Optimization

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***Abstract*— "Ant algorithms" have been proposed to solve a variety of problems arising in optimization and distributed control. They form a subset of the larger class of "Swarm Intelligence" algorithms. The central idea is that a 'swarm' of relatively simple agents can interact through simple mechanisms and collectively solve complex problems. Instances that exemplify the above idea abound in nature. The abilities of ant colonies to collectively accomplish complex tasks have served as sources of inspiration for the design of "Ant algorithms". Examples of "Ant algorithms" are "Ant Routing" algorithms that have been proposed for communication networks. We analyze in this paper an Ant-Based Routing Algorithm for packet-switched wireline networks. The algorithm is an attractive multiple path probabilistic routing scheme, that is fully adaptive and distributed. Using methods from adaptive algorithms and stochastic approximation, we show that the evolution of the link delay estimates can be closely tracked by a deterministic ODE system. A study of the equilibrium points of the ODE then gives us the equilibrium behavior of the routing algorithm, in particular, the equilibrium routing probabilities, and mean delays in the links under equilibrium. We also show that the fixed-point equations that the equilibrium probabilities satisfy are actually the necessary and sufficient conditions of an appropriate optimization problem. Simulations supporting the analytical results are provided.**

I. INTRODUCTION

"Ant algorithms" constitute a class of algorithms that have been proposed to solve a variety of problems arising in optimization and distributed control. They form a subset of the larger class of "Swarm Intelligence" algorithms, a topic extensively discussed in the book by Bonabeau, Dorigo and Theraulaz [4]. The central idea is that a "swarm" of simple agents can interact through simple mechanisms and collectively solve complex problems. Instances that exemplify the above idea abound in nature. Bonabeau, Dorigo and Theraulaz [4] give examples of insect societies like those of ants, honey bees, and wasps, which accomplish complex tasks of building intricate nests, finding food, responding to external threats etc., even though the individual insects themselves have limited capabilities. The abilities of ant colonies to collectively accomplish complex tasks have served as sources of inspiration for the design of "Ant algorithms".

Examples of "Ant algorithms" are "Ant-Based Routing" algorithms that have been proposed for communication networks. It was observed in an experiment conducted by biologists, called the double bridge experiment [6], that a colony of ants, when presented with two paths to a source of food, is able to collectively converge to the shorter path. Every ant lays a trail of a chemical substance called *pheromone* as it walks along a path; subsequent ants follow and reinforce this trail. This leads progressively to a large accumulation of pheromone on the shorter path, which is how ants discover the shorter path. Most of the Ant-Based Routing Algorithms (called Ant Routing Algorithms, for short) proposed in the literature are inspired by this basic idea. These algorithms employ probe packets, called ant packets (analogues of ants), to explore the network and measure various quantities related to network routing performance, for e.g., link and path delays. These measurements are used to update the routing tables at the network nodes. The update algorithms reinforce those outgoing links which lead to paths with lower delays.

Schoonderwoerd *et. al.* [11], [4] tested an Ant Routing Algorithm on the British Telecom telephone network, and reported superior performance compared to other algorithms including shortest paths based schemes. This generated interest in the study of Ant Routing Algorithms for both connection-oriented networks and for packet-switched networks (both wired and wireless). Ant Routing Algorithms for wireless networks have been proposed and analyzed in papers by Baras and Mehta [1] and Gunes *et. al.* [8].

We consider in this paper packet-switched wired networks. Ant Routing Algorithms for such networks have been proposed, for example, in the works of Di Caro and Dorigo [7] and Bean and Costa [2]. Though numerous Ant Routing Algorithms have been proposed, very few analytical studies are available in the literature. Examples of such analytical studies are [12], [9], and [2]. Yoo, La, and Makowski [12] and Gutjahr [9] consider very simple network cases where the delays in the network links are deterministic quantities, independent of the offered traffic loads. They show that their algorithms converge to the shortest path solutions in the network. Bean and Costa [2] propose a multiple-path routing scheme which tries to form estimates of the delays and use them to update routing tables. The link delays can be stochastically varying. The authors conduct a numerical study based on their model and find that the results agree well with simulations. However, they do not provide any convergence results. Borkar and Kumar [5], too, have a delay estimation scheme and a probability update scheme which utilizes the delay estimates. The probability update scheme moves on a slower time-scale than the delay estimation scheme. By two time-scale stochastic approximation methods they prove convergence of their algorithm to the routing solution called Wardrop equilibrium.

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We consider the algorithm proposed by Bean and Costa [2] in this paper. The scheme is fully adaptive and distributed. We consider in this paper a simple routing scenario where data traffic entering a single source node has to be routed to a single destination node, and there are N available parallel paths between them. We model the arrival processes of both the ant and the data streams that arrive at the source node, and argue, using methods from adaptive algorithms and stochastic approximation, that the evolution of the link delay estimates can be closely tracked by a deterministic ODE system when the step size of the estimation scheme is small. Then a study of the equilibrium points of the ODE gives us the equilibrium behavior of the routing algorithm; in particular, the equilibrium routing probabilities, and mean delays in the N paths under equilibrium can be obtained. We also find that the fixed-point equations that the equilibrium routing probabilities satisfy are actually the necessary and sufficient conditions of a convex minimization problem.

Our paper is organized as follows. Section II provides the general framework of ant routing and the routing scheme we consider. Section III provides a description of our N parallel paths model. Section IV contains an analysis of the routing scheme. Section V provides illustrative simulation results, and in Section VI we provide a few conclusive remarks.

II. GENERAL FRAMEWORK OF ANT ROUTING ALGORITHMS AND OUR SCHEME

We provide, in this section, a brief description of the general framework of ant routing for a wired communication network. The framework that we follow is the one described in Di Caro and Dorigo [6], [7]. Alongside, we describe the Bean and Costa [2] scheme, that we analyse in this paper.

Every node i in the network maintains two key data structures - a matrix of routing probabilities, the *routing table* $\mathcal{R}(i)$, and a matrix of various kinds of statistics, called the *local network information table*, $\mathcal{L}(i)$. For a particular node i, let $\mathcal{N}(i,k)$ denote the set of neighbors of i through which packets are routed towards destination k. The entries of $\mathcal{R}(i)$ are the probabilities ϕ_j^{ik}. ϕ_j^{ik} denotes the probability of routing an incoming packet bound for destination k via the neighbor $j \in \mathcal{N}(i,k)$. The (j,k)-th entry of $\mathcal{L}(i)$ contains various statistics pertaining to the route $(i,j,\dots,k)$. Examples of such statistics could be mean delay and delay variance estimates of route $(i,j,\dots,k)$. These statistics are updated based on the information the ant packets collect about the route. The matrix $\mathcal{L}(i)$ represents the characteristics of the network that are learned by the nodes through the ant packets, and based on which the local routing table $\mathcal{R}(i)$ is updated. The iterative algorithms used to update $\mathcal{L}(i)$ and $\mathcal{R}(i)$ will be referred to as the *learning algorithms*.

We now describe the mechanism of operation of ant-based routing algorithms. For ease of exposition, we restrict attention to a particular destination node, and consider the problem of routing from every other node to this node, which we label as D.

Forward ant generation and routing. At certain intervals, forward ant (FA) packets are launched from a node i towards the destination D to discover low delay paths to it. The FA packets sample walks on the graph representing the network, based on the current routing probabilities at the nodes. FA packets share the same queues as data packets and so experience similar delay characteristics as data packets. Every FA packet maintains a stack of data structures containing the IDs of nodes in its path and the per hop delays (or other relevant information) encountered. The per hop delay measurements are obtained through time stamping of the packets as they pass through the various nodes.

Backward ant generation and routing. Upon arrival of an FA at destination D, a backward ant (BA) packet is generated. The FA packet transfers its stack to the BA. The BA packet then retraces back to the source the path traversed by the FA packet. BA packets travel back in high priority queues, so as to minimize the effects of outdated measurements. At each node that the BA packet traverses through, it transfers the information that was gathered by the corresponding FA packet. This information is used to update the tables $\mathcal{L}$ and $\mathcal{R}$ at the nodes. Thus the arrival of the BA packet at the nodes triggers the iterative learning algorithms. Of the various learning algorithms in the literature, we consider the one proposed by Bean and Costa [2].

Bean and Costa suggest the following scheme for the learning algorithms. Suppose that an FA packet measures the delay Δ_j^{iD} associated with a walk $(i,j,\dots,D)$. When the corresponding BA packet arrives at node i, the delay information is used to update the estimate of the mean delay X_j^{iD} using the simple exponential estimator

$$X_j^{iD} := X_j^{iD} + \eta(\Delta_j^{iD} - X_j^{iD}), \tag{1}$$

where $\eta > 0$ is a small constant. The mean delay estimates X_m^{iD}, corresponding to the other neighbors m of node i, are left unchanged.

Simultaneously, the routing probabilities at the nodes are updated using the scheme:

$$\phi_j^{iD} = \frac{\left(\frac{1}{X_j^{iD}}\right)^{\beta}}{\sum_{k\in\mathcal{N}(i,D)}\left(\frac{1}{X_k^{iD}}\right)^{\beta}}, \ \forall\ j \in \mathcal{N}(i,D), \tag{2}$$

where β is a constant positive integer. β influences the extent to which outgoing links with lower delay estimates are favored compared to the ones with higher delay estimates.

We can interpret the quantity $\frac{1}{X_j^{iD}}$ as analogous to the pheromone content on the outgoing link (i,j). Equation (2) shows that the outgoing link (i,j) is more desirable when X_j^{iD}, the delay in routing through j, is smaller (i.e., when the pheromone content is higher).

III. THE N PARALLEL PATHS MODEL

The model that we consider pertains to the routing scenario where arriving traffic at a single source node S has to be routed to a single destination node D. There are N available parallel paths between the source and the destination node through which the traffic could be routed. The network and its equivalent queueing theoretic model are shown in Figures

1 and 2 respectively. The queues represent the output buffers (assumed to be infinite) at the source and are associated with the N outgoing links. We assume in our model that the queueing delays dominate the propagation and the packet processing delays in the N branches. Two traffic streams, an ant and a data stream, arrive at the node S. At S, every packet of the combined stream is routed with probabilities $\phi_1, \ldots, \phi_N$ (the *current* values) towards the queues $Q_1, \ldots, Q_N$, respectively. These probabilities are updated dynamically based on running estimates of the means of the delays (waiting times) in the N queues. Samples of the delays in the N queues are collected by the ant packets (these are forward ant packets) as they traverse through the queues. These samples are then used to construct the running estimates of the mean delays in the N queues. We now describe our model in detail.

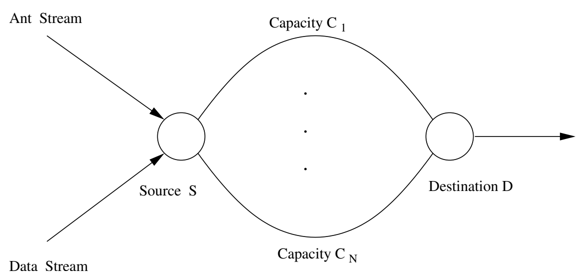

Fig. 1. The network with N parallel paths

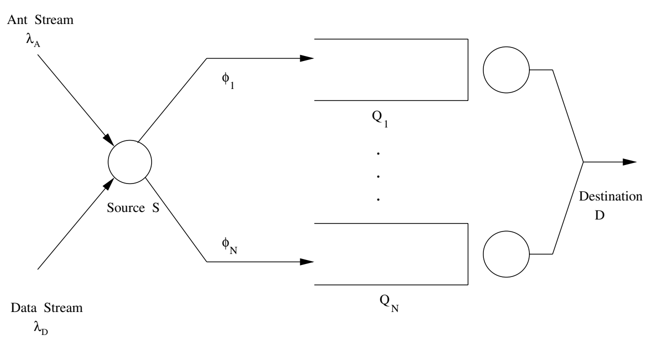

Fig. 2. N parallel paths : The queueing theoretic model

We model the arrival processes of ant and data stream packets at node S as independent Poisson processes of rates λ_A and λ_D packets/sec, respectively. The lengths of the packets of the combined stream constitute an i.i.d. sequence, which is also statistically independent of the packet arrival processes. The capacity of link i is C_i bits/sec ($i = 1, \ldots, N$). We assume that the length of an ant packet is generally distributed with mean L_A bits, and that of a data packet is generally distributed with mean L_D bits. If we denote the service times of an ant and a data packet in queue Q_i by the generic random variables S_i^A and S_i^D, then S_i^A and S_i^D are generally distributed (according to some c.d.f.'s, say G_i^A and G_i^D) with means $E[S_i^A] = \frac{L_A}{C_i}$ and $E[S_i^D] = \frac{L_D}{C_i}$, respectively. The packets of the ant stream, which is essentially a probing stream in our system, would be much smaller in size compared to the data packets.

Let $\{\Delta_i(m)\}$ denote the sequence of delays experienced by successive ant packets traversing Q_i. Here delay refers to the total waiting time in the system Q_i (waiting time in the queue plus packet service time). Also, let $\{\delta(n)\}$ denote the sequence of successive arrival times of ant packets at the destination node D. Then the n-th arrival of an ant packet at D occurs at $\delta(n)$. Suppose that this ant packet has arrived via Q_i. We denote the decision variable for routing by $R(n)$; that is, for $i = 1, \ldots, N$, we say that the event $\{R(n) = i\}$ has occurred if the n-th ant packet that arrives at D has been routed via Q_i. $\psi_i(n) = \sum_{k=1}^{n} I_{\{R(k)=i\}}$, thus, gives the number of ant packets that have been routed via Q_i among a total of n ant arrivals at destination D. Once the ant packet arrives, the estimate $X_i(n)$ of the mean of the delay through queue Q_i is immediately updated using a simple exponential averaging estimator

$$X_i(n) = X_i(n-1) + \epsilon \left(\Delta_i(\psi_i(n)) - X_i(n-1)\right), \quad (3)$$

$0 < \epsilon < 1$, being a small constant.

The delay estimates for the other queues are left unchanged, i.e.,

$$X_j(n) = X_j(n-1), \quad j \in \{1, \ldots, N\},\ j \neq i. \quad (4)$$

Thus, in general, the evolution of the delay estimates in the N queues can be described by the following set of stochastic iterative equations

$$\begin{aligned} X_i(n) \quad &= \quad X_i(n-1) + \epsilon\ I_{\{R(n)=i\}} \Big(\Delta_i(\psi_i(n)) - \\ & \qquad X_i(n-1)\Big),\ i = 1, \ldots, N, \end{aligned} \quad (5)$$

along with a set of initial conditions $X_1(0) = x_1, \ldots, X_N(0) = x_N$.

At time $\delta(n)$, besides the delay estimates, the routing probabilities $\phi_i(n), i = 1, \ldots, N$, are also updated simultaneously according to the equations

$$\phi_i(n) = \frac{(X_i(n))^{-\beta}}{\sum_{j=1}^{N} (X_j(n))^{-\beta}}, \quad i = 1, \ldots, N, \quad (6)$$

β being a constant positive integer. The initial values of the probabilities are $\phi_i(0) = \frac{(x_i)^{-\beta}}{\sum_{j=1}^{N} (x_j)^{-\beta}}, i = 1, \ldots, N$.

In the above model, we assume that the mean delay estimates $X_i(.)$ and the probabilities $\phi_i(.)$ are updated as soon as the (forward) ant packets arrive at the destination D, and this information is available instantaneously thereafter at the source node S. We do not consider thus the additional delay involved as the backward ant packets travel back carrying the delay information to the source. Because backward ants are expected to travel back to the source through priority queues, the effects of delayed information might not be very significant, except for large-sized networks. On the other hand, incorporating the effect of delays in our model introduces additional asynchrony, making the problem harder.

IV. ANALYSIS OF THE ALGORITHM

We view the routing algorithm, consisting of equations (5) and (6), as a set of discrete stochastic iterations of the type usually considered in the literature on stochastic approximation methods [3]. We provide below the main convergence result which states that, when ϵ is small enough, the discrete iterations are closely tracked by a system of Ordinary Differential Equations (ODEs).

A. The ODE Approximation

An analysis of the dynamics of the system, as given by equations (5) and (6), is fairly complicated. However, when $\epsilon > 0$ is small, a time-scale decomposition simplifies matters considerably. The key observation is that, when ϵ is small, the delay estimates X_i evolve much more slowly compared to the waiting time (delay) processes Δ_i. Also, because the probabilities ϕ_i are (memoryless) functions of the delay estimates X_i, they, too, evolve at the same time-scale as the delay estimates. Consequently, with the vector $(X_1(n), \ldots, X_N(n))$ fixed at $(z_1, ..., z_N)$ (equivalently, $\phi_i(n), i = 1, \ldots, N$, fixed at $\phi_i = \frac{(z_i)^{-\beta}}{\sum_{j=1}^{N}(z_j)^{-\beta}}$, $i = 1, \ldots, N$), the delay processes $\Delta_i(.), i = 1, \ldots, N$, can be considered as converged to a stationary distribution, which depends on $(z_1, \ldots, z_N)$. Also, when ϵ is small, the evolution of delay estimates can be tracked by a system of ODEs. A heuristic analysis of the algorithm, as outlined in [10], shows that the ODE system for our case is given by

$$\begin{aligned} \frac{dz_1(t)}{dt} &= \frac{(z_1(t))^{-\beta}\Big(D_1(z_1(t), \ldots, z_N(t)) - z_1(t)\Big)}{\sum_{j=1}^{N}(z_k(t))^{-\beta}}, \\ &\vdots \qquad \vdots \\ \frac{dz_N(t)}{dt} &= \frac{(z_N(t))^{-\beta}\Big(D_N(z_1(t), \ldots, z_N(t)) - z_N(t)\Big)}{\sum_{j=1}^{N}(z_k(t))^{-\beta}}, \end{aligned} \tag{7}$$

with the set of initial conditions $z_1(0) = x_1, \ldots, z_N(0) = x_N$. $D_i(z_1, \ldots, z_N), i = 1, \ldots, N$, are the mean waiting times in the queues under stationarity (as seen by arriving ant packets) with the delay estimates considered fixed at $z_1, \ldots, z_N$.

Formally, the ODE approximation result can be stated as follows (see Benveniste, Metivier, and Priouret [3]). For any fixed $\epsilon > 0$ and for $i = 1, \ldots, N$, consider the piecewise constant interpolation of $X_i(n)$ given by the equations : $z_i^\epsilon(t) = X_i(n)$ for $t \in [n\epsilon, (n+1)\epsilon\ )$, $n = 0, 1, 2, \ldots$, with the initial value $z_i^\epsilon(0) = X_i(0)$. Then the processes $\{z_i^\epsilon(t), t \geq 0\}, i = 1, \ldots, N$, converge to the solution of the ODE system (7) in the following sense : as $\epsilon \downarrow 0$, for any $0 \leq T < \infty$,

$$\sup_{0 \leq t \leq T} |z_i^\epsilon(t) - z_i(t)| \xrightarrow{P} 0, \tag{8}$$

where $\xrightarrow{P}$ denotes convergence in probability.

In order to obtain the evolution of the ODE, we need to compute the quantities $D_i(z_1, \ldots, z_N)$ for our queueing system. With the delay estimates considered fixed at $z_1, \ldots, z_N$, the routing probabilities to the N queues then are fixed at $\phi_i = \frac{(z_i)^{-\beta}}{\sum_{j=1}^{N}(z_j)^{-\beta}}, i = 1, \ldots, N$. We now discuss how to compute the quantities $D_i(z_1, \ldots, z_N)$ given our assumptions on the statistics of the arrival processes and the packet lengths of the arrival streams.

Under such conditions, every incoming arrival at S from either of the Poisson streams (ant or data) is routed independently with probability ϕ_i towards queue Q_i. Thus the incoming arrival process in queue Q_i (for each i) is a superposition of two independent Poisson processes with rates $\lambda_A\phi_i$ and $\lambda_D\phi_i$. Consequently, every incoming packet into Q_i is, with probability $\frac{\lambda_A}{\lambda_A+\lambda_D}$, an ant packet, and with probability $\frac{\lambda_D}{\lambda_A+\lambda_D}$, a data packet. The cumulative incoming stream into Q_i is Poisson with rate $(\lambda_A + \lambda_D)\phi_i$, and every incoming packet's service time is distributed according to the c.d.f G_i^A with probability $\frac{\lambda_A}{\lambda_A+\lambda_D}$ and according to the c.d.f. G_i^D with probability $\frac{\lambda_D}{\lambda_A+\lambda_D}$. We further assume that the queues are within the stability region of operation given by the inequalities : $(\lambda_A+\lambda_D)\phi_i E[S_i] < 1, i = 1, \ldots, N$, where $E[S_i]$, the mean packet service time in Q_i, is given by $E[S_i] = \frac{\lambda_A E[S_i^A] + \lambda_D E[S_i^D]}{\lambda_A+\lambda_D}$. We note that the average waiting time in the system as experienced by successive ant arrivals to Q_i, is the same as the average waiting time in Q_i by the PASTA (Poisson Arrivals See Time Averages) property. Thus, using the Pollaczek-Khinchin formula for the average waiting time, we finally obtain the expression for $D_i(z_1, \ldots, z_N)$ $(i = 1, \ldots, N)$:

$$D_i(z_1, \ldots, z_N) = E[S_i] + \frac{(\lambda_A + \lambda_D)\phi_i E[S_i^2]}{2(1 - (\lambda_A + \lambda_D)\phi_i E[S_i])}, \tag{9}$$

where $E[S_i^2]$ is given by $E[S_i^2] = \frac{\lambda_A E[(S_i^A)^2] + \lambda_D E[(S_i^D)^2]}{\lambda_A+\lambda_D}$, and $\phi_i = \frac{(z_i)^{-\beta}}{\sum_{j=1}^{N}(z_j)^{-\beta}}$.

Once expressions for $D_i(z_1, \ldots, z_N)$ are available, we can numerically solve (7), starting with initial conditions $z_1(0), \ldots, z_N(0)$. We observe in our simulations that if we start the system with initial conditions such that we are inside the stability region, the system stays stable thereafter.

B. Equilibrium behavior of the routing algorithm

We now obtain the equilibrium points of the ODE system (7) which would, in turn, enable us to obtain the equilibrium routing behavior . In particular, we can obtain the equilibrium routing probabilities and the mean delays. For ϵ small, the steady state values of the mean delay estimates in the N queues are approximately given by the components of the equilibrium points z^* of the ODE system (7). The points z^* must satisfy the set of equations given by

$$\begin{aligned} \frac{(z_1^*)^{-\beta}}{\sum_{j=1}^{N}(z_j^*)^{-\beta}} \cdot \Big[D_1(z_1^*, \ldots, z_N^*) - z_1^*\Big] &= 0, \\ \vdots \qquad & \vdots \end{aligned}$$

$$\frac{(z_N^*)^{-\beta}}{\sum_{j=1}^{N}(z_j^*)^{-\beta}}\cdot\Big[D_N(z_1^*,\ldots,z_N^*)-z_N^*\Big] \;=\; 0. \quad (10)$$

The steady state routing probabilities, $\phi_1^*,\ldots,\phi_N^*$, and delay estimates are related through the equations: $\phi_i^* = \frac{(z_i^*)^{-\beta}}{\sum_{j=1}^N (z_j^*)^{-\beta}}$, $i = 1,\ldots,N$. Because we have assumed stable queues, the mean delay estimates, i.e., $z_1^*,\ldots,z_N^*$, must be finite. Then the steady state routing probabilities, $\phi_i^* = \frac{(z_i^*)^{-\beta}}{\sum_{j=1}^N (z_j^*)^{-\beta}}$, $i = 1, \ldots, N$, are all strictly positive. Equations (10) then reduce to : $z_i^* = D_i(z_1^*,\ldots,z_N^*), i = 1,\ldots,N$. We also notice, from equation (9), that for each i, $D_i(z_1^*,\ldots,z_N^*)$ is a function solely of ϕ_i^*, and so, with a slight abuse of notation, we denote it by $D_i(\phi_i^*)$. Then, using the fact that $\phi_i^* = \frac{(z_i^*)^{-\beta}}{\sum_{j=1}^N (z_j^*)^{-\beta}}$, we find that equilibrium routing probabilities must satisfy the following fixed-point system of equations:

$$\begin{aligned}\phi_1^* &= \frac{(D_1(\phi_1^*))^{-\beta}}{\sum_{j=1}^{N}(D_j(\phi_j^*))^{-\beta}},\\ &\vdots\\ \phi_N^* &= \frac{(D_N(\phi_N^*))^{-\beta}}{\sum_{j=1}^{N}(D_j(\phi_j^*))^{-\beta}}.\end{aligned} \quad (11)$$

Besides satisfying (11) the probabilities $\phi_1^*,\ldots,\phi_N^*$ must also satisfy the following system stability conditions:

$$(\lambda_A+\lambda_D)\phi_i^* E[S_i] < 1, i = 1,\ldots,N. \quad (12)$$

We now show that the equations (11) are also the necessary and sufficient optimality conditions for a problem involving the minimization of a convex objective function of $(\phi_1,\ldots,\phi_N)$ subject to the above mentioned constraints (12). We show consequently that, if there exists a solution to the set of equations (11) satisfying the constraints (12), then such a solution is unique.

Consider the optimization problem:

$$\begin{aligned}&\text{Minimize } F(\phi_1,\ldots,\phi_N) = \textstyle\sum_{i=1}^N \int_0^{\phi_i} x[D_i(x)]^{\beta}dx,\\ &\text{subject to } \phi_1+\cdots+\phi_N = 1, \quad (13)\\ &0<\phi_1<a_1,\\ &\vdots\\ &0<\phi_N<a_N,\end{aligned}$$

where $a_i = \frac{1}{(\lambda_A+\lambda_D)E[S_i]}, i = 1,\ldots,N$.

Before we attempt to solve the optimization problem (13), we make certain natural assumptions on the delay functions $D_i(x), i = 1,\ldots,N$. We assume that the functions $D_i(x)$ are positive, differentiable and monotonically increasing on their domains of definition. This holds true in most cases of interest, because when the routing probability for an outgoing link increases, the amount of traffic flow into that link also increases, resulting in an increase of the delay. Let C be the feasible set (set defined by the constraints) of the optimization problem (13). The following proposition characterizes the optimal solutions ϕ^* of the above optimization problem. Its proof can be found in [10].

Proposition 1: Given the above assumptions on the delay functions $D_i(x), i = 1,\ldots,N$, a probability vector ϕ^* is a local minimum of F over C if and only if ϕ^* satisfies the set of fixed-point equations (11). ϕ^* is then also the unique global minimum of F over C.

For our model, it is easy to check that the functions $D_i(x)$, as given by (9), are positive, differentiable and monotonically increasing on their domains of definition. Thus, there is a unique equilibrium probability vector ϕ^* which satisfies (11).

V. SIMULATION RESULTS AND DISCUSSION

We describe in this section an illustrative example. The queueing system as described in Section III was implemented using a discrete event simulator. We consider a case when the number of parallel paths N is 3, the step size ϵ is 0.002, and β is 1.

The ant and data traffic arrival processes are Poisson with rates $\lambda_A = 1$ and $\lambda_D = 1$, respectively. For the ant packets, the service times in the three queues are exponential with means $E[S_1^A] = 1/3.0$, $E[S_2^A] = 1/4.0$ and $E[S_3^A] = 1/5.0$, respectively. For the data packets also, the service times in the three queues are exponential with means $E[S_1^D] = 1/3.0$, $E[S_2^D] = 1/4.0$ and $E[S_3^D] = 1/5.0$, respectively. The initial values of the delay estimates in the three queues were set at $X_1(0) = 0.8$, $X_2(0) = 2.8$, and $X_3(0) = 5.6$. Then, the initial routing probabilities are $\phi_1(0) = 0.7$, $\phi_2(0) = 0.2$, and $\phi_3(0) = 0.1$, which ensures that, initially, we are inside the stability region of the queueing sytem. We observed in our simulations that as the queueing system evolved over time, never did the system become unstable.

Figures 3, 4 and 5 provide plots of the interpolated delay estimates $(z_i^\epsilon(t), i = 1,2,3)$ in the three queues, averaged over ten sample paths, versus the ODE approximation, $z_1(t), z_2(t), z_3(t)$, obtained by numerically solving (7). We see that the theoretical ODE tracks the simulated delay estimates fairly well. Figures 6, 7 provide plots of routing probabilities $\phi_1(n), \phi_2(n)$. (Because $\phi_3(n) = 1-\phi_1(n)-\phi_2(n)$, its plot is not provided.) The routing probabilities converge to the equilibrium values $\phi_1^* = 3/12, \phi_2^* = 4/12, \phi_3^* = 5/12$, which is actually the unique solution to equations (11). These probabilities are in the reverse order as packet service times in the three queues, with link 3 having the highest equilibrium probability, and link 1 the lowest.

VI. CONCLUSIONS

We have provided convergence results for an Ant Routing Algorithm for a simple network consisting of N parallel paths between a source-destination pair. We have modelled the link delays using a stochastic queueing model, and we have studied a routing scheme where routing probabilities are updated based on estimates of path delays. We have shown that the equilibrium routing probabilities are solutions of a fixed-point system of equations, which in turn, form the necessary and sufficient conditions for a convex optimization problem. We aim to extend the analysis to the network case, where multiple traffic streams with different destinations share a network of links.

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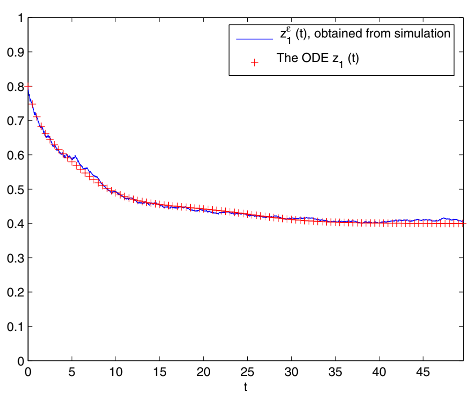

Fig. 3. The ODE approximation for $X_1(n)$

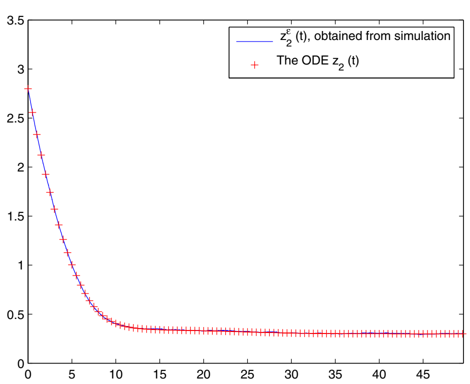

Fig. 4. The ODE approximation for $X_2(n)$

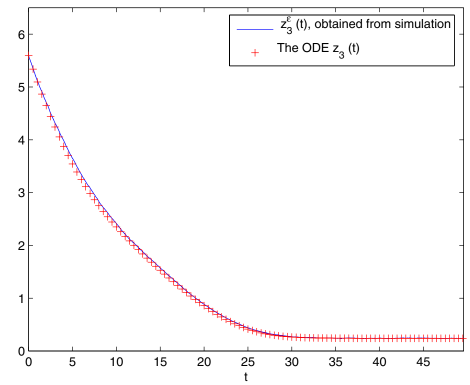

Fig. 5. The ODE approximation for $X_3(n)$

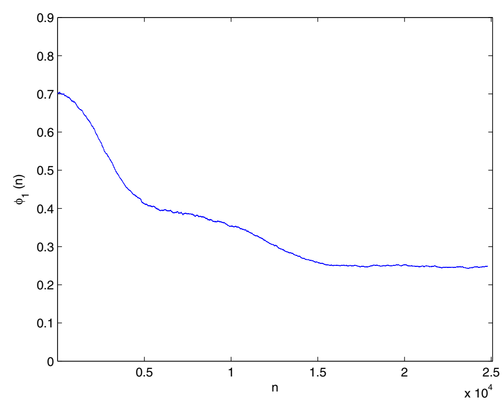

Fig. 6. The plot for $\phi_1(n)$

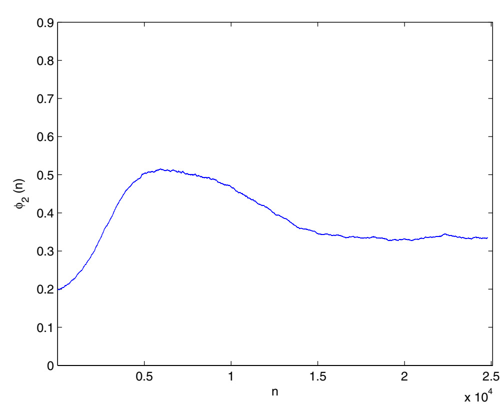

Fig. 7. The plot for $\phi_2(n)$