Stabilization Using Multiple Sensors over Analog Erasure Channels

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***Abstract*— Consider a discrete-time, linear time-invariant process, two sensors and one controller. The process is observed by the sensors, which are connected to the controller via links that can be modeled as erasure channels. If a link transmits successfully then a finite-dimensional vector of real numbers is conveyed from the sensor to the controller. If an erasure event occurs, then any information conveyed over the link is lost. This paper addresses the problem of designing the maps that specify the processing at the controller and at the sensors for stabilizing the process in the bounded second moment sense. When the information is lost over the links either in an independent and identically distributed (i.i.d.) or Markovian fashion over time, we derive necessary and sufficient conditions for the existence of maps such that the process is stabilized. Such conditions are expressed as inequalities involving the parameters of the plant and the probabilities of link fading. and provide the least conservative stabilization conditions. We also indicate how our approach can be used if more than two sensors are available, if the sensors can cooperate and if the acknowledgment signals are also transmitted over erasure channels. The analysis also carries over to the case when the single channels are replaced by networks of erasure channels.**

I. INTRODUCTION

Recently a lot of attention has been directed towards networked control systems in which components communicate over wireless links or communication networks that may also be used for transmitting other unrelated data (see, e.g., [1], [5], [21] and the references therein). The estimation and control performance in such systems is severely affected by the properties of the communication channels. Communication links introduce many potentially detrimental phenomena, such as quantization error, random delays, data loss and data corruption to name a few, that may lead to performance degradation or even stability loss.

In this work, we are specifically interested in the problem of estimation and control across communication links that exhibit data loss. We consider a dynamical process evolving in time that is being observed by two sensors. The sensors need to transmit the data over communication links to a remote node, which can either be an estimator or a controller. However information transmitted over the links is erased stochastically. Preliminary work in this area has largely concentrated on the case when only one sensor is present. Within the one-sensor framework, both stability [34], [40] and performance [26], [34] have been analyzed. Approaches to compensate for the data loss to counteract the degradation in performance have also been proposed. As some representative examples, Hadjicostis and Touri [17] proposed applying zero control if sensor data is lost, Nilsson [29] proposed using the previous control input or time-updating the previous estimate in case of data loss, Ling and Lemmon [26] posed the problem as a filter-design through a non-linear optimization for scalar observations and Smith and Seiler [36] proposed a sub-optimal but computationally efficient estimator for packet drops occurring according to a Markov chain. Also relevant are the works of Azimi-Sadjadi [2], Schenato et al. [33] and Imer et al. [22] who looked at controller structures to minimize quadratic costs for systems in which both sensor-controller and controller-actuator channels are present. The related problem of optimal estimation across a packet-dropping link was considered by Sinopoli et al. in [35] for the case of one sensor and packet drops occurring in an i.i.d. fashion, while Gupta et al. [13] considered multiple sensors and more general packet drop models.

Most of the above designs aimed at designing a packet-loss compensator. The compensator accepts those packets that the link successfully transmits and comes up with an estimate for the time steps when data is lost. If the estimator is used inside a control loop, the estimate is then used by the controller. We take a more general approach to the control of networked control systems. It has often been recognized that typical network / communication data packets have much more space for carrying information than required inside a traditional control loop. For instance, the minimum size of an ethernet data packet is 72 bytes, while a typical data point will only consume 2 bytes [10]. Many other examples are given in Lian et al. [25]. Moreover, many of the devices used in networked control systems have processing and memory capabilities on account of being equipped to communicate across wireless channels or networks. Thus the question arises if we can use this possibility of pre-processing information prior to transmission and transmission of extra data to combat the effects of packet delays, loss and so on and improve the performance of a networked control system. In Gupta et al. [15] it was shown that pre-processing (or encoding) information before transmission over the communication link can indeed yield significant improvements in terms of stability and performance. Moreover, for a given performance level, it can also lead to a reduced amount of communication. This effect can also be seen in the recent works on receding horizon networked control, in which a few future control inputs are transmitted at every time step by the controller and buffered at the actuator to be used in case subsequent control updates are dropped by the network and do not arrive at the actuator(s),

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see, e.g., [11], [12], [23], [27], [28]. The benefits incurred become even more apparent when the communication link is replaced by a *network* of communication links [14].

In this work, we extend the principle to the case when multiple sensors are present. Suppose a process is observed using two sensors that transmit the data over packet-dropping links to a controller. If the sensors can share their measurements, there is effectively only one sensor. We look at the case when cooperation between the sensors is either not permitted, or occurs over a stochastic communication link. We solve for the conditions on the links and the dynamics of the process that allow the process to be stabilized.

The problem involving the presence of multiple sensors transmitting data in an aperiodic fashion is much more complicated than the problem involving only a single sensor. The problem of finding optimal encoding algorithms for the multi-sensor case and analyzing their performance is similar to the problems of fusion of data from multiple sensors and track-to-track fusion that have long been open. A usual starting point for such works is an attempt to decentralize the Kalman filter as, e.g., in [38]. However this approach requires that data about the global estimate be sent from the fusion node to the local sensors. This difficulty was overcome in [8], [37] and further in [18] where both the measurement and time update steps of the Kalman filter were decentralized. Alternative approaches for data fusion from many nodes include using the Federated filter [6], Bayesian methods [9], a scattering framework [24], algorithms based on decomposition of the information form of the Kalman filter [30] and so on. A recent addition to the literature is [32].

However these approaches assume a fixed communication topology among the nodes with a link, if present, being perfect. In our case, information is erased randomly by the communication channels. This random loss of information reintroduces the problem of correlation between the estimation errors of various nodes [3] and renders the approaches proposed in the literature as sub-optimal. An approach to solve this problem was proposed in [4] in the context of track-to-track fusion through exchange of state estimates based on each sensor's own local measurements but the specific scheme that was used was not proved to be optimal. It was subsequently proved in [7] that the technique was based on an assumption that was not met in general. There are special cases for which the solution is known, e.g., when the process noise is absent [39] or one of the sensors transmits data over a channel that does not erase information [15]. However, as stated earlier, in general, the problem is still open. Owing to a separation principle that we present, our results also carry over to the multi-sensor fusion problem.

The paper is organized as follows. We begin in the next section by describing the problem set-up and our notation. We present the conditions in Section III-A that are necessary for stabilizability for any causal encoding algorithm at the sensors. In Section III-B, we then prove that the conditions are sufficient as well, by presenting a sub-optimal algorithm that stabilizes the system even when acknowledgements from the controller are not available. Section IV generalizes the results in various directions. The case of more than two sensors being present is treated in Section IV-A. Section IV-B considers the case when the sensors transmit information to the controller over networks of erasure links. This also allows us to treat the case when the sensors can co-operate over erasure channels. Finally, in Section IV-C we analyze the case when the channel erasures are correlated in time and can be described by a Markov chain. We finish with some possible directions for future work. The proofs of the results are omitted for space constraints, and can be found in [16].

II. Framework Description and Problem Formulation

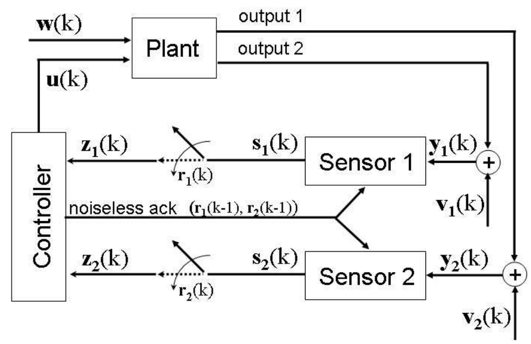

Fig. 1. Basic framework for output feedback using two remote sensors, in the presence of erasure channels. The process and measurement noises at the plant are presented by $\mathbf{w}(k)$ and $\mathbf{v}_{1 \text{ or } 2}(k)$, respectively. Erasure in the links between the sensors and the controller, is governed by $\mathbf{r}_{1 \text{ or } 2}(k)$.

Consider the set-up of Fig 1, and the following associated assumptions regarding the plant, the external sources of randomness and the erasure links that connect the sensors to the controller:

The plant is described by a discrete-time state-space representation of the following type:

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k) + \mathbf{w}(k), \;\; k \geq 1 \tag{1}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is the process state, $\mathbf{u}(k) \in \mathbb{R}^l$ is the control input and $\mathbf{w}(k)$ is the process noise assumed to be white, Gaussian, zero mean with covariance $\Sigma_w > 0$. The initial state $\mathbf{x}(0)$ is a zero mean and Gaussian random variable with covariance matrix Σ_0. The process state is observed using two sensors that generate measurements of the form

$$\mathbf{y}_1(k) = C_1\mathbf{x}(k) + \mathbf{v}_1(k), \;\; k \geq 0 \tag{2}$$

$$\mathbf{y}_2(k) = C_2\mathbf{x}(k) + \mathbf{v}_2(k), \;\; k \geq 0 \tag{3}$$

where $\mathbf{y}_1(k) \in \mathbb{R}^{m_1}$ and $\mathbf{y}_2(k) \in \mathbb{R}^{m_2}$. The measurement noises $\mathbf{v}_1(k)$ and $\mathbf{v}_2(k)$ are also assumed to be white, Gaussian, zero mean with positive definite covariance matrices $\Sigma_{v,1}$ and $\Sigma_{v,2}$ respectively. For ease of notation, we adopt the concatenation $\mathbf{v}(k)' \stackrel{def}{=} [\mathbf{v}_1(k)' \; \mathbf{v}_2(k)']'$ and denote the covariance matrix of $\mathbf{v}(k)$ by $\Sigma_{\mathbf{v}}$. Similarly, we define

$$\begin{aligned} C &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \\ y(k) &= \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix}. \end{aligned}$$

Throughout this work, we adopt the following assumption:

Assumption 1: For simplicity, we assume that the pairs (A, C_1) and (A, C_2) are not observable. In addition, we assume that the overall system is observable, i. e., that (A, C) is observable, where $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$.

Assumption 1 corresponds to the more difficult scenario where the controller might have to combine the information gathered from $\mathbf{y}_1$ and $\mathbf{y}_2$. Later we show that the stability analysis, for the case where (A, C_1) and (or) (A, C_2) are observable, constitutes a particular case of our analysis, indicating that our Assumption 1 comes at no loss of generality.

Definition 2.1: (**Erasure Link Model**) Consider that $\{\mathbf{r}_1(k)\}_{k=0}^{\infty}$ and $\{\mathbf{r}_2(k)\}_{k=0}^{\infty}$ represent Bernoulli stochastic processes taking values in the set $\{\mathbf{1}, \emptyset\}$ and characterized by a probability mass function of the following type:

$$p_{i,j} \stackrel{def}{=} Pr\left(\mathbf{r}(k) = (i,j)\right),\ (i,j) \in \{\mathbf{1}, \emptyset\}^2$$

where $\mathbf{r}(k) \stackrel{def}{=} (\mathbf{r}_1(k), \mathbf{r}_2(k))$. The process $\mathbf{r}(k)$ governs the state of the links that connect the sensors to the controller. More specifically, the relationship between sensor i's output $\mathbf{s}_i(k)$ and the controller's input $\mathbf{z}_i(k)$ is described by:

$$\mathbf{z}_i(k) = \begin{cases} \emptyset & \text{if } \mathbf{r}_i(k) = \emptyset \\ \mathbf{s}_i(k) & \text{if } \mathbf{r}_i(k) = \mathbf{1} \end{cases},\ i \in \{1,2\} \quad (4)$$

where we adopt the symbol $\emptyset$ to represent erasure, i.e., it indicates that the information sent from sensor i to the controller was lost.

Note that, in general, we do *not* assume that the erasure events in the channels are uncorrelated. However, we presuppose that the sources of randomness $\mathbf{x}(0)$, $\{\mathbf{r}(k)\}_{k=0}^{\infty}$, $\{\mathbf{v}(k)\}_{k=0}^{\infty}$ and $\{\mathbf{w}(k)\}_{k=0}^{\infty}$ are mutually independent.

We consider sensors with the following functional structure:

Definition 2.2: (**Sensor map classes** $\mathbb{S}_q$ **and** $\mathbb{S}_q^{NAK}$) For any given positive integer q, we define $\mathbb{S}_q$ as the set containing all sensor maps with the following structure:

$$\mathbf{s}_i(k) = \begin{cases} \mathcal{S}\left(k, \mathbf{y}_i(0), \ldots, \mathbf{y}_i(k), \mathbf{r}(0), \ldots, \mathbf{r}(k-1)\right) & k \geq 1 \\ \mathcal{S}\left(0, \mathbf{y}_i(0)\right) & k = 0 \end{cases} \quad (5)$$

where i is in the set $\{1,2\}$ and $\mathbf{s}_i(k)$ takes values in $\mathbb{R}^q$. Notice that we assume that $\{\mathbf{r}(i)\}_{i=0}^{i=k-1}$ is made available to the sensor via noiseless acknowledgements. In addition, we consider another set $\mathbb{S}_q^{NAK}$ of sensor maps with the following structure:

$$\mathbf{s}_i(k) = \begin{cases} \mathcal{S}^{NAK}\left(k, \mathbf{y}_i(0), \ldots, \mathbf{y}_i(k)\right) & k \geq 1 \\ \mathcal{S}^{NAK}\left(0, \mathbf{y}_i(0)\right) & k = 0 \end{cases} \quad (6)$$

where i is in the set $\{1,2\}$ and $\mathbf{s}_i(k)$ takes values in $\mathbb{R}^q$. Notice that $\mathbb{S}_q^{NAK}$ is the subset of $\mathbb{S}_q$ consisting of the sensor structures that do not rely on the knowledge of past values of the erasure process $\{\mathbf{r}(i)\}_{i=0}^{i=k-1}$. In other words, $\mathbb{S}_q^{NAK}$ is the set of sensor maps when the sensor does not have access to noiseless acknowledgment signals.

In the sequel, we will also refer to the sensor maps as encoding algorithms or information processing algorithms and to the sensors as encoders.

Definition 2.3: (**Controller class**) Consider stochastic processes $\mathbf{z}_1(k)$ and $\mathbf{z}_2(k)$ taking values in $\mathbb{R}^q \bigcup \{\mathbf{1}, \emptyset\}$. We define the controller class $\mathbb{K}$ as the set of all controllers with the following structure:

$$\mathbf{u}(k) = \mathcal{K}(k, \mathbf{z}_1(0), \mathbf{z}_2(0), \ldots, \mathbf{z}_1(k), \mathbf{z}_2(k)) \quad (7)$$

where $\mathbf{u}(k)$ takes values in $\mathbb{R}^l$ and l is the dimension of the control input to the plant specified in (1).

Given the description of the plant and the erasure link statistics, specified by the probability mass function $p_{i,j}$, we want to investigate conditions for the existence of controllers and sensor maps, in the classes $\mathbb{S}_q$ or $\mathbb{S}_q^{NAK}$, that stabilize the plant in the following sense.

Definition 2.4: (**Stability criterion**) Consider the set-up of Figure 1 and assume that the matrices A, B, C_1, C_2 and the erasure link statistics $p_{i,j}$ are given. A selection of controller $\mathcal{K}$, integer q and sensor maps $\mathcal{S}_1$ and $\mathcal{S}_2$, in the set $\mathbb{S}_q$, is stabilizing if and only if the following holds:

$$\sup_{k \geq 0} E_{\sigma(k), \mathbf{x}(0)}[\mathbf{x}(k)' Q \mathbf{x}(k) + \mathbf{u}(k)' R \mathbf{u}(k)] < \infty \quad (8)$$

where Q and R are positive definite matrices, $\mathbf{x}(k)$ is the state of the plant and $\sigma(k) \stackrel{def}{=} \{\mathbf{r}(i), \mathbf{v}(i), \mathbf{w}(i)\}_{i=0}^{i=k}$ is used to indicate that the expectation is taken with respect to all independent sources of randomness.

III. Stability Analysis

We will rely on the following result about representation of linear systems of the form (1)-(3).

Proposition 3.1: Consider an n dimensional linear and time-invariant system satisfying Assumption 1 and let $\mathbf{y}_1(k)$ and $\mathbf{y}_2(k)$, taking values in $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, constitute a bi-partition of the system's output. We can always construct a state-space representation with the structure (1)-(3), where the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times l}$, $C_1 \in \mathbb{R}^{m_1 \times n}$ and $C_2 \in \mathbb{R}^{m_2 \times n}$ are written in one and only one of the following forms, which we refer to as **type I** and **type II**. The first possibility, denoted as **type I**, is given by:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ \mathbf{0}^{n_2 \times n_1} & A_{2,2} \end{bmatrix} \quad (9)$$

$$C_1 = \begin{bmatrix} \mathbf{0}^{m_1 \times n_1} & C_{1,2} \end{bmatrix} \quad (10)$$

$$C_2 = \begin{bmatrix} C_{2,1} & \mathbf{0}^{m_2 \times n_2} \end{bmatrix} \quad (11)$$

where $A_{i,i} \in \mathbb{R}^{n_i \times n_i}$, $C_{i,j} \in \mathbb{R}^{m_i \times n_i}$ and $n_1 + n_2 = n$.

The following is the second possibility (**type II**) :

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ \mathbf{0}^{n_2 \times n_1} & A_{2,2} & A_{2,3} \\ \mathbf{0}^{n_3 \times n_1} & \mathbf{0}^{n_3 \times n_2} & A_{3,3} \end{bmatrix} \quad (12)$$

$$C_1 = \begin{bmatrix} \mathbf{0}^{m_1 \times n_1} & C_{1,2} & C_{1,3} \end{bmatrix} \quad (13)$$

$$C_2 = \begin{bmatrix} C_{2,1} & \mathbf{0}^{m_2 \times n_2} & C_{2,3} \end{bmatrix} \quad (14)$$

where $A_{i,i} \in \mathbb{R}^{n_i \times n_i}$, $C_{i,j} \in \mathbb{R}^{m_i \times n_i}$ and $n_1 + n_2 + n_3 = n$.

Remark 3.1: In the above representations (of types I or II), $A_{1,1}$ describes the dynamics of the state subspace that is not observable from $\mathbf{y}_1(k)$, while the modes that are not observable by $\mathbf{y}_2(k)$ follow the dynamics of $A_{2,2}$. If the representation is of type II, then $A_{3,3}$ specifies the dynamics of the modes that are observable by both $\mathbf{y}_1(k)$ and $\mathbf{y}_2(k)$.

A. Necessary Conditions for Stabilizability

Using the representation outlined in the above Proposition, we can present the necessary conditions for stabilizability of the process. The necessity of the conditions can be proven by considering the stability of the process for an algorithm $\bar{\mathcal{A}}$ in which at every time step k, the sensors transmit all the measurements that they have access to till time step k. Even though this algorithm is not a valid algorithm in the class $\mathbb{S}_q$ since it involves unbounded amount of transmission, if the system is not stabilizable with this algorithm, it will not be stabilizable with any other algorithm. The details of the proof are in [16], where additionally, an algorithm in the class $\mathbb{S}_q$ is provided that achieves the same performance as obtained using $\bar{\mathcal{A}}$.

Theorem 3.2: (**Necessary Conditions for Stabilizability**) Consider the scheme of Fig 1 and let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C_1 \in \mathbb{R}^{m_1\times n}$ and $C_2 \in \mathbb{R}^{m_2\times n}$ be given matrices specifying the state-space representation for the plant. In addition, assume that the plant satisfies Assumption 1 and that the statistics of the erasure links is specified by a given probability mass function $Pr(\mathbf{r}(k) = (i,j))$, with $(i,j) \in \{\mathbf{1},\emptyset\}^2$ that is independent of the time index k. If the state-space representation can be written as in (9)-(11) (type I) then there exists a controller in the class $\mathbb{K}$, a positive integer q and sensors in the class $\mathbb{S}_q$ such that the closed loop system in stable only if the following inequalities hold:

$$\varrho(A_{1,1})^2 Pr(\mathbf{r}_2(k) = \emptyset) < 1 \tag{15}$$

$$\varrho(A_{2,2})^2 Pr(\mathbf{r}_1(k) = \emptyset) < 1 \tag{16}$$

where $\varrho(A_{i,i})$ represents the spectral radius of the matrix $A_{i,i}$. If, instead, the state-space representation is of type II, i. e. of the form (12)-(14), then necessary conditions for stabilization also include the following additional inequality:

$$\varrho(A_{3,3})^2 Pr\left(\mathbf{r}(k) = (\emptyset,\emptyset)\right) < 1 \tag{17}$$

Remark 3.2: The case when Assumption 1 does not hold and the system is observable using only one sensor has already been considered in the literature [15]. Our results can be applied to this case if we adopt the convention that the spectral radius of an empty matrix is 0. Thus, e.g., if the entire state is observable from $\mathbf{y}_1(k)$, then the spectral radius of $A_{1,1}$ is assumed to be 0. A similar statement can be said about the sufficiency conditions given below as well. Thus we will assume that Assumption 1 holds in our analysis from now on.

Remark 3.3: The stabilizability conditions make intuitive sense. The quantity $\varrho(A_{1,1})^2$ measures the rate of increase of the second moment of the modes that are observable using only sensor 2. To keep the estimate error covariance of these modes bounded, we need the information from sensor 2 to arrive at a large enough rate. Equation (15) formalizes this relation. Similarly, the inequality in (16) places a constraint on the drop rate of information from sensor 1 in terms of the rate of increase of the modes that are observable solely through sensor 1. Finally, the relation in (17) places a constraint on the arrival rate of information from at least one of the sensors in terms of the modes that are observable from either sensor.

B. Sufficient Conditions for Stabilizability

It turns out that the above conditions are also sufficient for stabilizability for sensors in the class $\mathbb{S}_q^{NAK}$ (and hence in the class $\mathbb{S}_q$). We have the following result.

Theorem 3.3: (**Sufficient conditions for stabilizability**) Consider the set-up of Figure 1 and let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C_1 \in \mathbb{R}^{m_1\times n}$ and $C_2 \in \mathbb{R}^{m_2\times n}$ be given matrices specifying the state-space representation for the plant. In addition, assume that the plant is controllable and that it satisfies Assumption 1. In addition, let the statistics of the erasure link, given by the probability mass function $Pr(\mathbf{r}(k) = (i,j))$, $(i,j) \in \{\mathbf{1},\emptyset\}^2$, be given. If the state space representation can be written as in (9)-(11) (type I), then there exists a controller of class $\mathbb{K}$, a positive integer q and sensors of class $\mathbb{S}_q^{NAK}$ such that the feedback system is stable, if the following two inequalities hold:

$$\varrho(A_{1,1})^2 Pr(\mathbf{r}_2(k) = \emptyset) < 1 \tag{18}$$

$$\varrho(A_{2,2})^2 Pr(\mathbf{r}_1(k) = \emptyset) < 1 \tag{19}$$

where $\varrho(A_{i,i})$ represents the spectral radius of the matrix $A_{i,i}$. If the state-space representation is of type II, i.e. it is of the form (12)-(14), then stability is assured by requiring that the following additional inequality also holds:

$$\varrho(A_{3,3})^2 Pr\left(\mathbf{r}(k) = (\emptyset,\emptyset)\right) < 1 \tag{20}$$

Remark 3.4: The inequalities in Theorems 3.2 and 3.3 are identical. However, notice that Theorem 3.3 states that if such inequalities hold then stabilization is achievable by using sensors of class $\mathbb{S}_q^{NAK}$, while Theorem 3.2 characterizes the necessary condition for stabilization by allowing sensors of class $\mathbb{S}_q$. This subtle difference, and the fact that $\mathbb{S}_q^{NAK} \subset \mathbb{S}_q$, lead to the interesting conclusion that the use of acknowledge signals at the sensors, or equivalently $\{\mathbf{r}(i)\}_{i=0}^{i=k-1}$, does not impact stabilizability. The use of $\{\mathbf{r}(i)\}_{i=0}^{i=k-1}$ is crucial, however, in the optimal control strategy as identified in [16].

Theorem 3.3 can be proven by considering an algorithm in which the sensors transmit estimates formed by their local measurements, for the modes of the system that are observable from that sensor. As indicated in [16], the sensors do *not* require the knowledge of the control input applied to the process for this purpose.

In the next section, we consider some generalizations of the results that we have presented above.

IV. Extensions and Generalizations

A. Case of Multiple Sensors

Theorems 3.2 and 3.3 can be generalized to the case when N sensors are present. We present the following stability result while omitting the proof.

Proposition 4.1: Consider the process in (1) being observed by N sensors, such that the i-th sensor generates measurements according to the model

$$y_i(k) = C_i x(k) + v_i(k), \qquad 1 \leq i \leq N.$$

The sensors transmit data over erasure channels, with the packet erasure in the i-th channel being denoted by $\mathbf{r}_i = \emptyset$. Consider the 2^N possible ways of choosing m out of the N sensors, for all values of m between 0 and N. For the j-th such way, let the sensors chosen be denoted by n_1, n_2, $\cdots$, n_j and sensors not chosen by m_1, m_2, $\cdots$, m_{N-j}. Denote by $\mathcal{C}^j$ the matrix formed by stacking the matrices C_{m_1}, C_{m_2}, $\cdots$, $C_{m_{N-j}}$. Finally, denote by ϱ^j the spectral radius of the unobservable part of matrix A when the pair $(A, \mathcal{C}^j)$ is put in the observer canonical form. A necessary and sufficient condition for the existence of a positive integer q, an encoding algorithm of either the type $\mathbb{S}_q$ or $\mathbb{S}_q^{NAK}$ and a controller that stabilize the process is that the following 2^N inequalities be satisfied:

$$Pr\left(\mathbf{r}_{n_1} = \emptyset, \mathbf{r}_{n_2} = \emptyset, \cdots, \mathbf{r}_{n_j} = \emptyset\right) \mid \varrho^j \mid^2 < 1, \qquad 1 \leq j \leq 2^N.$$

B. Communication over Networks of Erasure Channels

We can also consider the case when sensors transmit information not over erasure channels, but over networks, in which each link is modeled using the erasure model described above. We require an additional provision for time-stamping the packet. We can use the techniques used in [14] for the case when only one sensor is present and extend the stability conditions to this case. We state the following result without proof.

Proposition 4.2: Consider the set-up of Figure 1 with the erasure links being replaced by networks in which each link is modeled as a erasure link with given probability of packet erasure. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C_1 \in \mathbb{R}^{m_1\times n}$ and $C_2 \in \mathbb{R}^{m_2\times n}$ be given matrices specifying the state-space representation for the plant. In addition, assume that the plant is observable and controllable and that its state-space representation is of type I or type II. If the state space representation is of type I, then there exists a controller of class $\mathbb{K}$, a positive integer q and sensors of class $\mathbb{S}_q$ or $\mathbb{S}_q^{NAK}$ such that the feedback system is stable if and only if the following inequalities hold

$$\varrho(A_{1,1})^2 p_{maxcut,2} < 1 \tag{21}$$

$$\varrho(A_{2,2})^2 p_{maxcut,1} < 1, \tag{22}$$

where $\varrho(A_{i,i})$ represents the spectral radius of the matrix $A_{i,i}$. If the state-space representation is of type II then the necessary and sufficient conditions for stabilizability include the following additional inequality:

$$\varrho(A_{3,3})^2 p_{maxcut,12} < 1. \tag{23}$$

In the above inequalities, the terms $p_{maxcut,i}$ denote the max-cut probabilities of the network. For the case when the packet drops over distinct links are uncorrelated events, they can be calculated as follows:

1) To calculate $p_{maxcut,1}$, form a cut by partitioning the node set of the network connecting sensor 1 and the network into two sets: the source set containing the sensor 1 and the sink set containing the controller. For this cut, consider the edges going from the source set to the sink set and calculate the cut-probability by multiplying the erasure probabilities for these edges. The maximum such cut-probability yields $p_{maxcut,1}$.
2) To calculate $p_{maxcut,2}$, proceed as above. However, the source set now contains sensor 2 instead of sensor 1.
3) To calculate $p_{maxcut,12}$, proceed as above. However, the source set now contains both sensor 1 and sensor 2.

A special case of the network arises when each sensor transmits data over a single link to the controllers. However, in addition, the sensors can cooperate by communicating with each other over a link. If the link is perfect (i.e., does not exhibit erasure), then the two sensors, in effect, form one sensor and the results of [15] apply. However, if this link also exhibits erasure, then we obtain the following stability conditions:

Corollary 4.3 (Sensors cooperating over an erasure link): Consider the set-up of Figure 1 with an additional bi-directional link connecting the two sensors. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C_1 \in \mathbb{R}^{m_1\times n}$ and $C_2 \in \mathbb{R}^{m_2\times n}$ be given matrices specifying the state-space representation for the plant. In addition, assume that the plant is observable and controllable and that its state-space representation is of type I or type II. Let the event of packet erasure over the link connecting sensor 1 to the controller be denoted as before by $\mathbf{r}_1(k) = \emptyset$, over the link connecting sensor 2 to the controller by $\mathbf{r}_2(k) = \emptyset$, and over the link connecting the two sensors by $\mathbf{r}_3(k) = \emptyset$. If the state space representation is of type I, then there exists a controller of class $\mathbb{K}$, a positive integer q and sensors of class $\mathbb{S}_q$ or $\mathbb{S}_q^{NAK}$ such that the feedback system is stable if and only if the following inequalities hold

$$\varrho(A_{2,2})^2 \max\left(Pr(\mathbf{r}_1(k) = \emptyset), Pr(\mathbf{r}_2(k) = \emptyset, \mathbf{r}_3(k) = \emptyset)\right) < 1 \tag{24}$$

$$\varrho(A_{1,1})^2 \max\left(Pr(\mathbf{r}_2(k) = \emptyset), Pr(\mathbf{r}_1(k) = \emptyset, \mathbf{r}_3(k) = \emptyset)\right) < 1, \tag{25}$$

where $\varrho(A_{i,i})$ represents the spectral radius of the matrix $A_{i,i}$. If the state-space representation is of type II then the necessary and sufficient conditions for stabilizability include the following additional inequality:

$$\varrho(A_{3,3})^2 Pr(\mathbf{r}_1(k) = \emptyset, \mathbf{r}_2(k) = \emptyset) < 1. \tag{26}$$

C. Markov Drops

A popular model for the bursty nature of packet drops in a wireless channel is according to a Markov chain. The simplest such model is the classical Gilbert-Elliot channel model. In this model, the channel is assumed to exist in one of two possible modes: state 0 corresponding to a packet drop and state 1 corresponding to no packet drop. The channel transitions between the two states according to a Markov chain. Suppose that the packet drops in the each of the two links in our model be described by such a Markov chain. Let the variable $\mathbf{r}_1(k)$ be governed by a Markov chain with transition probability matrix Q_1 and the variable $\mathbf{r}_2(k)$ by a Markov chain with transition probability matrix Q_2. Further, for simplicity, let the packet drops over the two channels be uncorrelated events. We have the following result.

Proposition 4.4: (**Necessary and sufficient conditions for stabilizability for Markovian packet drops**) Consider the set-up of Figure 1 and let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C_1 \in \mathbb{R}^{m_1\times n}$ and $C_2 \in \mathbb{R}^{m_2\times n}$ be given matrices specifying the state-space representation for the plant. In addition, assume that the plant is observable and controllable and that its state-space representation is of type I or type II. In addition, let the statistics of the erasure links 1 and 2 be described by Markov chains with transition probability matrices Q_1 and Q_2 respectively, with the element $q_{00,i}$ denoting the probability of two consecutive packet drops in the i-th link. Finally, let the packet drops over the two channels be uncorrelated. If the state space representation is of type I, then there exists a controller of class $\mathbb{K}$, a positive integer q and sensors of class $\mathbb{S}_q$ or $\mathbb{S}_q^{NAK}$ such that the feedback system is stable if and only if the following inequalities hold

$$\varrho(A_{1,1})^2 q_{00,2} < 1 \tag{27}$$

$$\varrho(A_{2,2})^2 q_{00,1} < 1 \tag{28}$$

where $\varrho(A_{i,i})$ represents the spectral radius of the matrix $A_{i,i}$. If the state-space representation is of type II then stability is assured if and only if the following additional inequality also holds:

$$\varrho(A_{3,3})^2 q_{00,1} q_{00,2} < 1 \tag{29}$$

V. Conclusions and Future Work

In this paper, we considered the problem of controlling a plant using measurements from multiple sensors. The information from the sensors to the controller is transmitted over erasure channels and dropped stochastically. We identified necessary and sufficient conditions that allow for the plant to be stabilized in a mean square sense using transmission of a vector of constant dimension from the sensors to the controller. We also considered various extensions such as sensors being able to co-operate over an erasure channel and data being transmitted over networks. The results are also relevant to the multi-sensor fusion problem.

There are various directions in which the present work may be extended. We are currently looking at finding the optimal encoding algorithms for other channels such as AWGN or discrete memoryless channels. Another possibility is to analyze similar results for the case when the measurement noises at different sensors are correlated.

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